On Colored Heap Games of Sumbers*

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Abstract. A sumber is a sum of ups, downs and star. Sumbers can describe the positions of many partisan infinitesimal games. Earlier, we provided a simplification rule [6] that can determine whether a game G is a sumber or not, and if it is, determine the exact sumber of G from its left and right options, G^L and G^R. This article extends the previous result and presents three variations of colored heap games; each of them can be solved by sumbers.

1 Introduction

We are concerned with combinatorial games and follow the notations and conventions of *Winning Ways* [1]. We also assume the readers are familiar with *numbers* [3] and *nimbers* [2,4]. Numbers and nimbers are well-known game subgroups with the following two properties.

1. There exists a simple rule to determine the outcome of any game in the subgroup.
2. There exists a simplification rule that can simplify games in the subgroup.

In March 2005, we presented another subgroup, *sumbers* [6], having the above properties. This section briefly reviews the definitions and major results about sumbers.

Definition 1. *For any number d, define*

$$\uparrow(d) = \{\uparrow(d^L), *|*, \uparrow(d^R)\}\,, \tag{1}$$

where $* = \{0|0\}$ (pronounced star). $\uparrow$ followed by the empty set is the empty set.

Each $\uparrow(d), d > 0$, is called an *up*. The negation of an up is called a *down*. We use the notation $n{\cdot}\uparrow(d)$ to denote the sum of n copies of $\uparrow(d)$.

Definition 2. *A sum of ups, downs, and star is called a sumber. A sumber S can be written in the standard form:*

$$S = s_0 \cdot * + \sum_{k=1,n} s_k{\cdot}\uparrow(d_k)\,, \tag{2}$$

where $s_0 = 0$ or 1, $s_k \neq 0, 0 < k \leq n$, and $0 < d_k < d_{k+1}, 0 < k < n$. $\sum_{k=1,n} s_k$ is called the net weight of S.

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The outcome of a sumber can be determined by theorem 1 [6].

Theorem 1. *Let S be a sumber in the above standard form. Then,*

- $S > 0$ *if and only if* $(\sum_{k=1,n} s_k > s_0)$ *or* $(\sum_{k=1,n} s_k = s_0$ *and* $s_1 < 0)$*;*
- $S < 0$ *if and only if* $-S > 0$*;*
- $S = 0$ *if and only if* $n = 0$ *and* $s_0 = 0$*;*
- $S \mid 0$ *, otherwise.* □

Definition 3. *Let S be a sumber in the above standard form. For each $m \in \{0, d_k : 1 \le k \le n\}$, define*

$$S^m = \sum_{k=1,n;\ d_k \ge m} s_k{\cdot}\!\uparrow\!(d_k) \quad - \sum_{k=1,n;\ d_k \ge m} s_k{\cdot}\!\uparrow\!(m)\,. \tag{3}$$

We say S has a cut at m if $S^m \le 0$.

Each sumber has at least one cut. We are only concerned with the *minimum cut*: the smallest number in $\{0, d_k : 1 \le k \le n\}$ which is a cut. When S is a sumber, and m is the minimum cut of S, we call S^m the *upper section* and

$$S_m = S - S^m \tag{4}$$

the *lower section* of S.

Sums of ups and downs (excluding $*$) are totally ordered. If G^L and G^R are sets of sumbers, then G has at most two non-dominated options, one contains $*$ and the other does not, in each of G^L and G^R. In other words, G can be simplified as

$$G = \{A, B | C, D\}\,, \tag{5}$$

where A, B are the non-dominated options in G^L and C, D are the non-dominated options in G^R.

Definition 4. *Let $G = \{A, B|C, D\}$, where A, B, C, D are sumbers and the net weight of C is less than or equal to the net weight of D. The critical section $X(G)$ of $G = \{A, B|C, D\}$ is defined as the set of numbers $x \ge m$ satisfying all the following inequalities:*

$$\begin{aligned}
\uparrow\!(x) &\ |\!> A - C_m + *\,,\\
\uparrow\!(x) &\ |\!> B - C_m + *\,,\\
\uparrow\!(x) &\ <\!| \ C - C_m + *\,,\\
\uparrow\!(x) &\ <\!| \ D - C_m + *\,,
\end{aligned}$$

where m is the minimum cut of C.

Theorem 2 [6] can simplify games with sumber options.

Theorem 2. *Let $G = \{A, B|C, D\}$, where A, B, C, D are sumbers and the net weight of C is less than or equal to the net weight of D.*

– *If* $A <| 0$, $B <| 0$ *and* $C |> 0$, $D |> 0$ *then* $G = 0$.
– *If* $A || *$, $B || *$, *and* $C || *$, $D || *$ *then* $G = *$.
– *Otherwise (without loss of generality, we may assume either* $G > 0$ *or* $G > *$*),* G *is a sumber if and only if* $X(G)$ *is not empty. Moreover, when* G *is a sumber,*

$$G = C_m + * + \uparrow(p) \tag{6}$$

where m *is the minimum cut of* C *and* p *is the simplest number in* $X(G)$. □

Sumbers can describe the positions of many partisan infinitesimal games. In each of the next three sections, we study one variation of a colored heap game.

2 Up-Down Game

The Up-Down game was first proposed by K. Y. Kao [5]. It is played on a number of heaps of counters. Each counter is colored either black or white. Left and Right move alternatively and their legal moves are different.

– When it is L's turn to move, he[1] can choose any one of the heaps and repeatedly removes the top counter until either he removed a white counter or the heap has become empty.
– When it is R's turn to move, he can choose any one of the heaps and repeatedly removes the top counter until either he removed a black counter or the heap has become empty.

The player who removes the last counter is the winner.

Let S be a heap, we use the notation $S : B$ (or $S : W$) to denote the heap by adding a black (or white) counter on top of S.

Proposition 1. *Each colored heap in Up-Down game is a sumber (or the negation of a sumber) of the form:*

$$S = s_0.* + \sum_{k=1,n} s_k{\cdot}\uparrow(d_k)\,, \tag{7}$$

where $s_0 = 0$ *or* 1, $s_k > 0, 0 < k \le n$ *and* $0 < d_k < d_{k+1}, 0 < k < n$.
Moreover, a heap with one single counter has the value $*$ *(in this case, we can assume* $n = 0$ *and* $d_0 = 0$.*)*

When S *has at least two counters, and the color of the counter next to the bottom counter is black,*

$$S : B = \{S^L|S\} = S + \uparrow(d_n) + *\,, \tag{8}$$

$$S : W = \{S|S^R\} = S - \uparrow(d_n) + \uparrow(d_n + 1)\,. \tag{9}$$

[1] In this paper, we use 'he' and 'him' wherever 'he or she' and 'his or her' are meant.

When S has at least two counters, and the color of the counter next to the bottom counter is white,

$$S : B = \{S^L|S\} = S + \uparrow(d_n) - \uparrow(d_n + 1)\,, \tag{10}$$

$$S : W = \{S|S^R\} = S - \uparrow(d_n) + *\,. \tag{11}$$

Proof. The proof is by induction. We assume S is a sumber in the standard form. Consider the case where S has at least two counters, and the color of the counter next to the bottom counter is black. According to the rule of the game, we have $S : B = \{S^L|S\}$. Since d_n is the minimum cut of S, the critical section of $S : B$ is

$$\begin{aligned} X(S\ :\ B) &= \{x \geq d_n : S^L - S + * <|\uparrow(x)\} \\ &= \{x \geq d_n : \uparrow(d_n - 1) - \uparrow(d_n) + * <|\uparrow(x)\} = \{x \geq d_n\}\,. \end{aligned} \tag{12}$$

The simplest number in $X(S : B)$ is d_n. Thus, according to theorem 2,

$$S : B = S + \uparrow(d_n) + *\,. \tag{13}$$

According to the rule, $S : W = \{S|S^R\}$. The critical section of $S : W$ is

$$\begin{aligned} X(S : W) &= \{x : S - S^R + * <|\uparrow(x)\} \\ &= \{\uparrow(d_n) <|\uparrow(x)\} = \{x > d_n\}\,. \end{aligned} \tag{14}$$

The simplest number in $X(S : W)$ is $d_n + 1$. Thus,

$$S : W = S^R + \uparrow(d_n + 1) + * = S - \uparrow(d_n) + \uparrow(d_n + 1)\,. \tag{15}$$

The case where the color of the counter next to the bottom counter is white can be proven in a similar way. □

Proposition 1 can help us figuring out the exact sumber of any Up-Down heap. The color of the counter next to the bottom counter determines whether the sumber is positive or not. If it is black then the sumber is positive, otherwise (white) the sumber is negative. When the sumber of a heap is positive, the number of black counters (other than the bottom counter) equals the net weight of the sumber; the number of white counters plus 1 equals the highest order of the up terms in the sumber.

Example 1. Consider the heap $BBBWWB$ (from bottom up).
The color of the counter next to the bottom counter is black. By repeatedly applying proposition 1, we have

$$\begin{aligned} B : B &= * : B = \uparrow(1)\,, \\ BB : B &= BB + \uparrow(1) + * = 2{\cdot}\uparrow(1) + *\,, \\ BBB : W &= BBB - \uparrow(1) + \uparrow(2) = \uparrow(1) + \uparrow(2) + *\,, \\ BBBW : W &= BBBW - \uparrow(2) + \uparrow(3) = \uparrow(1) + \uparrow(3) + *\,, \\ BBBWW : B &= BBBWW + \uparrow(3) + * = \uparrow(1) + 2{\cdot}\uparrow(3)\,. \end{aligned} \tag{16}$$

Thus,

$$BBBWWB = \uparrow(1) + 2{\cdot}\uparrow(3)\,.$$

□

3 Up-Down Game II

The setup of Up-Down II is the same as the Up-Down game, but the legal moves are different.

- When it is L's turn to move, he can choose any one of the heaps and
  1. repeatedly removes the top counter until a white counter is removed, or
  2. removes all the counters from the heap.
- When it is R's turn to move, he can choose any one of the heaps and
  1. repeatedly removes the top counter until a black counter is removed, or
  2. removes all the counters from the heap.

The player who removes the last counter is the winner.

Each colored heap in Up-Down II corresponds to a Hackenbush number [1].

1. First, *translate the heap into a binary string.* From bottom up and ignoring the bottom counter, each black counter is translated into digit 0; each white counter is translated into digit 1. If the string starts with digit 1, add digit 1 to the end of the above string; if the string starts with digit 0, add digit 0 to the end of the above string.
2. Next, *translate the binary string into a Hackenbush number.* A string starting with digit 1 is translated into a positive Hackenbush number; a string starting with digit 0 is translated into a negative Hackenbush number. Without loss of generality, we may assume the string starting with digit 1. From left to right, the first place where the digits changes from 1 to 0 is translated into a decimal point. Let n the number of 1s to the left of the decimal point. Then $n-1$ is the integer part of the Hackenbush number. The digits to the left of the decimal point is the dyadic rational part of the Hackenbush number.

The next proposition gives the solution of Up-Down II.

Proposition 2. *Each colored heap in Up-Down II is a sumber of the form:*

$$S_d = \uparrow(d) + * , \tag{17}$$

where d equals the Hackenbush number of the heap.

Proof. We prove by induction. Assume the claim is true for numbers simpler than d. By induction hypothesis,

$$S_{d^L} = \uparrow(d^L) + * \qquad \text{and} \qquad S_{d^R} = \uparrow(d^R) + * . \tag{18}$$

According to the rule,

$$\begin{aligned} S_d &= \{S_{d^L}, 0|0, S_{d^R}\} \\ &= \{\uparrow(d^L) + *, 0|0, \uparrow(d^R) + *\} \\ &= \{\uparrow(d^L), *|*, \uparrow(d^R)\} + * \\ &= \uparrow(d) + * . \end{aligned} \tag{19}$$

□

Example 2. Consider the heap $WWWBBBWW$ (from bottom up). The corresponding binary string of the heap is 11<u>0</u>0111. The number of 1's to the left of the first 0 is 2. Thus, the integer part of the number is 1 (=2-1). The rational part is .0111 (in binary notation) which equals 0.4375. This string represents the Hackenbush number 1.4375. According to proposition 2, the heap has the value $\uparrow(1.4375) + *$. □

4 Up-Down Game III

The setup of Up-Down III is the same as Up-Down and Up-Down II, but the legal moves are different.

- When it is L's turn to move, he can choose any one of the heaps and
  1. repeatedly removes the top counter until a white counter is removed, or
  2. removes all the counters from the heap, or
  3. splits the heap into two none-empty heaps.
- When it is R's turn to move, he can choose any one of the heaps and
  1. repeatedly removes the top counter until a black counter is removed, or
  2. removes all the counters from the heap, or
  3. splits the heap into two none-empty heaps.

The player who removes the last counter is the winner.

Let S_d denotes the heap whose Hackenbush number is d. With simple induction, one can show that $S_d > *$ iff $d > 0$, and $S_{d_2} > S_{d_1}$ iff $d_2 > d_1$. At the first glance, it may be seen that both players have many splitting options. But, after a detailed analysis, we know that each player has at most one dominant split option.

1. L will never split S_d when $d > 0$; R will never split S_d when $d < 0$.
2. When L splits $S_d (d < 0)$ into two heaps $-S_b + S_u$, u must be a number greater than b and b must be the minimum among all the possible split options.
3. When R splits $S_d (d > 0)$ into two heaps $S_b - S_u$, u must be a number greater than b and b must be the minimum among all the possible split options.

The next proposition gives the solution of Up-Down III.

Proposition 3. *Each colored heap in Up-Down III is a sumber (or the negation of a sumber) of the form:*

$$S_d = \{S_{d^L}, 0 | 0, S_{d^R}\} = \uparrow(d) + * , \tag{20}$$

(when R has no dominant split option)

or the form:

$$S_d = \{S_{d^L}, 0 | S_b - S_u, S_{d^R}\} = S_b - S_u + \uparrow(m) + * , \tag{21}$$

(when R has a dominant split option)

where $d > 0$ is the Hackenbush number of the heap, $S_b - S_u$ is the dominant split, and m is the simplest number greater than u.

Proof. The proof is by induction. Assume the claim is true for games simpler than S_d. When R has no dominant split option, S_d has the same value as in Up-Down II. When R has a dominant split option, there are two possible cases.

– case $d^L = b$:

$$\begin{aligned} X(S_d) &= \{x : S_{d^L} - S_b + S_u + * <| \uparrow(x)\} \\ &= \{x : S_u + * <| \uparrow(x)\} \\ &= \{x > u\} \end{aligned} \tag{22}$$

– case $d^L > b$: In this case, R has a dominant split option on d^L. Moreover, by induction hypothesis,

$$S_{d^L} = S_b - S_{u^R} + \uparrow(m') + * , \tag{23}$$

where $S_b - S_{u^R}$ is the dominant split, and m' is the simplest number greater than u^R.

$$\begin{aligned} X(S_d) &= \{x : S_{d^L} - S_b + S_u + * <| \uparrow(x)\} \\ &= \{x : -S_{u^R} + S_u + \uparrow(m') <| \uparrow(x)\} \\ &= \{x > u\} \end{aligned} \tag{24}$$

In both cases, we have $X(S_d) = \{x > u\}$. By theorem 2,

$$S_d = \{S_{d^L}, 0 | S_b - S_u, S_{d^R}\} = S_b - S_u + \uparrow(m) + * , \tag{25}$$

(when R has no dominant split option)

where m is the simplest number in $X(S_d)$. □

Example 3. Consider the heap $WWWBBWW$ (from bottom up).
The corresponding Hackenbush number is 1.4375. R can split $WWWBBWW$ into $WW + WBBWW$ whose corresponding Hackenbush numbers are 1 and -1.25 respectively. The simplest number greater than 1.25 is 2. Thus,

$$WWWBBWW = S_{1.4375} = S_1 - S_{1.25} + \uparrow(2) + * . \tag{26}$$

Next, consider the heap $WBBWW = -S_{1.25}$. L can split $WBBWW$ into $WB + BWWW$ of which the corresponding Hackenbush numbers are -1 and 3, respectively. The simplest number greater than 3 is 4. Thus,

$$WBBWW = -S_{1.25} = -S_1 + S_3 - \uparrow(4) + * . \tag{27}$$

Since R has no dominant split option at WW and $BWWW$, we have

$$WW = S_1 = \uparrow(1) + * , \tag{28}$$

$$BWWW = S_3 = \uparrow(3) + * . \tag{29}$$

Finally, putting all together,we have

$$WWWBBWW = S_{1.4375} = \uparrow(2) + \uparrow(3) - \uparrow(4) + * . \tag{30}$$

□

5 Conclusion

We presented three variations of colored heap games. The sumber simplification rule is a powerful tool to analyze these games. In the Up-Down game, the sumbers contain only positive (or negative) ups with integer orders. In the Up-Down II game, the sumbers may contain ups with rational orders. In the Up-Down III game, we find sumbers that contains both positive and negative terms. It is interesting to see how a small change of the rule may produce different game values. The most interesting thing is that all these games can be solved by sumbers.

References

1. E.R. Berlekamp, J.H. Conway, and R.K. Guy. *Winning Ways*, Academic Press, New York, 1982.
2. C.L Bouton. Nim: A Game With a Complete Mathematical Theory, *Annals of Mathematics*, 3:35–39, 1902.
3. J.H. Conway, *On Numbers and Games*, Academic Press, New York, 1976.
4. P.M. Grundy, Mathematics and Games, *Eureka*, 2:6–8, 1939. Reprinted in *Eureka*, 27:9–11, 1964.
5. K.Y. Kao. *On Hot and Tepid Combinatorial Games*, PhD Thesis, University of North Carolina, Charlotte, 1997.
6. K.Y. Kao. Sumbers - Sums of Ups and Downs, *Elec. Journal of Combinatorial Number Theory*, 5, G1, 2005.