Modular Cut-Elimination: Finding Proofs or Counterexamples

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Abstract. Modular cut-elimination is a particular notion of "cut-elimination in the presence of non-logical axioms" that is preserved under the addition of suitable rules. We introduce syntactic necessary and sufficient conditions for modular cut-elimination for standard calculi, a wide class of (possibly) multiple-conclusion sequent calculi with generalized quantifiers. We provide a "universal" modular cut-elimination procedure that works uniformly for any standard calculus satisfying our conditions. The failure of these conditions generates counterexamples for modular cut-elimination and, in certain cases, for cut-elimination.

1 Introduction

Cut-elimination is one of the most important techniques in proof theory. The removal of cuts corresponds to the elimination of intermediate statements (lemmas) from proofs, resulting in calculi in which proofs are *analytic* in the sense that all statements in the proofs are subformulae of the result.

A great many different cut-elimination proofs for various sequent calculi have been published since Gentzen's proofs for **LK** and **LJ** (sequent calculi for classical and intuitionistic first-order logic, respectively), most using heavy syntactic arguments and based on case distinctions, usually written without filling in the details[1]. However since it is often the case that "the devil is in the details" (this also explains why so many wrong cut-elimination proofs appear in the literature, e.g. [5]), it is natural to investigate *general criteria* that a sequent calculus should satisfy in order to admit cut-elimination. Such criteria should support a *modular view* of cut-elimination in sequent calculi (i.e. decomposability of the whole calculus into local components when proving cut-elimination), and also provide useful information in the negative case, where a particular cut-elimination method cannot be applied or a cut-elimination proof cannot be found at all.

Necessary and sufficient conditions for cut-elimination were defined in [14] for *canonical calculi*, which are sequent calculi containing identity axioms, the usual structural rules (weakening, exchange and contraction) and possibly "standard" rules for connectives and quantifiers. Canonical calculi extended with (k, n)-ary connectives

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[1] Notable exceptions are the cut-elimination proofs for classical and intuitionistic logic of [7].

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which bind k variables and connect n formulas were investigated in [15] where sufficient conditions for cut-elimination have been introduced in the case $k = 0, 1$. In the context of substructural logics, syntactic and semantic criteria for (additive) structural rules to preserve cut-elimination when added to full Lambek calculus were introduced in [12]. Terui's work was generalized in [3] to provide necessary and sufficient conditions for a large class of propositional single-conclusion sequent calculi to admit reductive cut-elimination, a naturally strengthened version of Buss' free-cut elimination [1] which additionally aims to shift non-eliminable cuts upwards as much as possible. The proposed criteria have two equivalent forms: syntactic (*reductivity* and *weak substitutivity*) and semantic (coherence and propagation). The former arises by weakening the sufficient conditions in [2] while the latter generalize the results in [12].

In this paper we focus on the syntactic aspects of cut-elimination. We refine and extend the (syntactic) results of [3] to *standard calculi*, i.e. commutative (not necessarily single-conclusion) sequent calculi possibly containing (fancy) structural rules and rules for (k, n)-ary connectives, for all k and n. Examples of standard calculi are Maehara's calculus **LJ'** for intuitionistic predicate logic, the calculus **GD** for the logic of constant domains [5], the multiplicative additive fragment of linear logic extended with any structural rule, or the calculi investigated in [15]. We investigate *modular cut-elimination* in standard calculi, a particular notion of "cut-elimination in the presence of non-logical axioms," that is preserved under the addition of suitable rules. *Weak substitutivity* and *reductivity*, the syntactic conditions of [3], are adapted to standard sequent calculi (Section 4), and shown to be necessary and sufficient for modular cut-elimination (the former holds when logical rules satisfy some additional properties, see Section 5). The necessity result is used for *counterexamples generation*: given a standard sequent calculus for which our criteria fail, counterexamples for modular cut-elimination are automatically generated and, in certain cases, lead to counterexamples for cut-elimination. The sufficient result is shown by providing a constructive proof of modular cut-elimination, from which a concrete cut-elimination procedure can be read off (Section 6). Remarkably enough this procedure is "universal" in the sense that when a standard sequent calculus admits modular cut-elimination, then our procedure always transforms derivations with cuts into cut-free derivations (Corollary 3).

Our results also support a modular view of cut-elimination. Indeed when adding a new connective and/or a new structural rule to a standard calculus for which modular cut-elimination has been already established, it is enough to show that the newly added rules are reductive and weakly substitutive. Moreover the task of proving modular cut-elimination for a standard calculus can be decomposed into the sub-tasks of proving cut-elimination for appropriate sub-calculi. In particular, in analogy with *Toyama's Lemma*[2] in term rewriting theory, modular cut-elimination is preserved by taking the disjoint union of two (sets of rules of) standard sequent calculi (Corollary 2).

2 Standard Calculi

We start by formalizing the notion of a standard sequent calculus. In the following we consider formulas built over a *vocabulary* $\mathcal{V}$ consisting of (countably many): (term)

[2] It states that the disjoint union of two confluent term rewriting systems is also confluent.

variables $x, y, z, \ldots$, for each $n \geq 0$, n-ary function and predicate symbols, as well as (m, n)-ary connectives $\star_1, \star_2, \ldots$ for each $m, n \geq 0$. As usual, *terms* $t, u, v, \ldots$ (in the vocabulary $\mathcal{V}$) are built up from variables using function symbols while atomic formulae are built up from terms using predicate symbols. A *formula* (in the vocabulary $\mathcal{V}$) is either an atomic formula or a compound formula of the form $\star_i \boldsymbol{x}(\boldsymbol{A})$ with $\star_i$ an (m, n)-ary connective, which binds $\boldsymbol{x} \equiv x_1, \ldots, x_m$ distinct variables, and connect formulas $\boldsymbol{A} \equiv A_1, \ldots, A_n$. Given a formula, its free and bound variables are defined in the standard way. As usual, we identify formulas only differing in the names of bound variables (i.e. formulas are considered up to α-equivalence).

Example 1.

1. The standard quantifiers $\forall$ and $\exists$ can be seen as $(1, 1)$-ary connectives, while propositional connectives as $(0, n)$-ary connectives, for some $n \geq 1$.
2. The Henkin quantifier Q_H (see e.g. [15]) can be seen as a $(4, 1)$-ary connective.
3. Bounded quantified formulae $\forall x \leq t.A$, $\exists x \leq t.A$ can be built with $(1, 2)$-ary connectives $\forall^b x(X, Y)$, $\exists^b x(X, Y)$ with the proviso that the meta-variable X is always instantiated by an inequation of the form $x \leq t$.

We indicate with $\Gamma, \Delta, \Pi, \Sigma, \ldots$ multisets of formulae. When $\lambda \geq 0$, Γ^λ denotes $\Gamma, \ldots, \Gamma$ (λ times). A *sequent* $\Gamma \Rightarrow \Delta$ (Γ said to be *antecedent* and Δ *consequent*) is *atomic* if all formulae in Γ and Δ are atomic. $\Gamma \Rightarrow \Delta$ is *single-conclusion* if Δ contains at most one formula, otherwise it is *multiple-conclusion*.

To specify inference rules we use *meta-variables* $X, Y, Z, X[{}^{\boldsymbol{t}}/_{\boldsymbol{x}}] \ldots$ ($\boldsymbol{t} \equiv t_1, \ldots, t_m$ and $\boldsymbol{x} \equiv x_1, \ldots, x_m$) standing for arbitrary formulae and $\Theta, \Xi, \Phi, \Psi, \Upsilon, \ldots$ for (possibly empty) multisets of meta-variables.

Definition 1. *A* standard *sequent calculus* $\mathcal{L}$ *consists of:*

– identity axiom *of the form* $X \Rightarrow X$
– *the multiplicative version of the* cut rule*, i.e.*

$$\frac{\Theta \Rightarrow \Xi, X \quad X, \Theta' \Rightarrow \Xi'}{\Theta, \Theta' \Rightarrow \Xi', \Xi}\ (CUT)$$

– structural inference rules *of the form* ($n > 0$)*:*

$$\frac{\Theta_1 \Rightarrow \Xi_1 \quad \cdots \quad \Theta_n \Rightarrow \Xi_n}{\Theta \Rightarrow \Xi}\ (R_i)$$

satisfying the conditions

(str0) Θ *and* Ξ *are disjoint.*

(str1) *any meta-variable occurring in* $\Theta_1, \ldots, \Theta_n$ *occurs in* Θ *and any meta-variable occurring in* $\Xi_1, \ldots, \Xi_n$ *occurs in* Ξ.

(Note that since $\Theta, \Xi, \ldots$ *are multisets, we implicitly assume that permutation rule(s) always belong to* $\mathcal{L}$*)*

– left logical rules $\{(\star, l, \boldsymbol{y})_i\}_{i \in \Lambda}$ *and* right logical rules $\{(\star, r, \boldsymbol{z})_j\}_{j \in \Lambda'}$ *(Λ and Λ' could be empty) for each (k,l)-ary connective $\star$, with $k, l, m, n \geq 0$:*

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\star\boldsymbol{x}(\boldsymbol{X}), \Theta \Rightarrow \Xi}\ (\star, l, \boldsymbol{y})_i \qquad \frac{\Upsilon'_1 \Rightarrow \Psi'_1 \quad \cdots \quad \Upsilon'_m \Rightarrow \Psi'_m}{\Theta \Rightarrow \Xi, \star\boldsymbol{x}(\boldsymbol{X})}\ (\star, r, \boldsymbol{z})_j$$

where $\boldsymbol{x} \equiv x_1, \ldots x_k$, $\boldsymbol{X} \equiv X_1, \ldots, X_l$ and for each $i = 1, \ldots l$, $X_i[^{\boldsymbol{t}}/_{\boldsymbol{x}}]$ ($\boldsymbol{t} \equiv t_1, \ldots, t_k$, where each t_i is a term) may appear in $\Upsilon_j \Rightarrow \Psi_j, \Upsilon'_{j'} \Rightarrow \Psi_{j'}$ with $j = 1, \ldots, n$ and $j' = 1, \ldots, m$. $\boldsymbol{y}$ and $\boldsymbol{z}$ are the eigenvariables *of the rules.*
$(\star, l, \boldsymbol{y})_i$ must satisfy the following conditions
(log0) *Θ, Ξ and $\{\boldsymbol{X}\}$ are mutually disjoint.*
(log1) *Any meta-variable occurring in $\Upsilon_1, \ldots, \Upsilon_n$ occurs in Θ or it is of the form $X_i[^{\boldsymbol{t}}/_{\boldsymbol{x}}]$ where $X_i \in \boldsymbol{X}$. Any meta-variable occurring in $\Psi_1, \ldots, \Psi_n$ occurs in Ξ or it is of the form $X_i[^{\boldsymbol{t}}/_{\boldsymbol{x}}]$ where $X_i \in \boldsymbol{X}$.*
The corresponding conditions hold for $(\star, r, \boldsymbol{z})_j$.

Remark 1. Conditions **(str1)** and **(log1)** ensure that rules satisfy the subformula property and do not allow meta-variables in Θ and Ξ to move from antecedent to consequent of sequents and vice versa.

We identify rules up to the renaming of meta-variables and logical rules up to the renaming of (term) variables.

Definition 2. Instances *(resp.* atomic instances*) of identity axiom, (CUT), and structural rules are obtained by substituting arbitrary formulae (resp. atomic formulae) for meta-variables. An* instance *(resp.* atomic instance*) of a logical rule $(\star, l, \boldsymbol{y})_i$ or $(\star, r, \boldsymbol{y})_j$ is obtained*

1. *by replacing each meta-variable Y with a formula (resp. atomic formula)* that does not contain $\boldsymbol{y}$ as free variables.
2. *when a meta-variable X_i ($\in \boldsymbol{X}$) in its conclusion is replaced by a formula (resp. atomic formula) A (that does not contain $\boldsymbol{y}$ as free variables), then each meta-variable $X_i[^{\boldsymbol{t}}/_{\boldsymbol{x}}]$ in its premises is replaced with the formula (resp. atomic formula) A in which all free occurrences of the variable x_j (if any) are replaced by the term t_j, for $j = 1, \ldots, k$.*

A derivation *in $\mathcal{L}$ is obtained by composing instances of axioms and rules of $\mathcal{L}$.*

Condition 1. above ensures that the eigenvariable condition is satisfied.

Definition 3. *In logical and structural rules (or their instances) the meta-variables (formulae) in Θ are called* left context meta-variables *(*left context formulae*), those in Ξ* right context meta-variables *(*right context formulae*), and (in the former rules) the meta-variables (formulae) of the form $X_i, X_i[^{\boldsymbol{t}}/_{\boldsymbol{x}}]$* active meta-variables *(*active formulae*).*

In a logical rule (or its instance) the introduced $\star\boldsymbol{x}(\boldsymbol{X})$ (or the formula of the form $\star\boldsymbol{x}(A_1, \ldots, A_l)$) is called principal formula*. Moreover, the two occurrences of the formula instantiating the meta-variable X in (CUT) are called left and right* cut formulae *(and the corresponding premises of (CUT) are called left and right premises).*

Example 2.

1. Simple sequent calculi with permutation (see [3]) are particular standard calculi in which each sequent is single-conclusion and whose connectives are of type $(0, n)$.
2. The ordinary rules for quantifiers fit into our framework. For instance, the left and right rules for $\forall$ are represented by the following rules:

$$\frac{X[^t/_x], \Theta \Rightarrow \Xi}{\forall x(X), \Theta \Rightarrow \Xi}\ (\forall, l, \emptyset) \qquad \frac{\Theta \Rightarrow \Xi, X[^y/_x]}{\Theta \Rightarrow \Xi, \forall x(X)}\ (\forall, r, y)$$

   where t is an arbitrary term and Θ, Ξ are arbitrary multisets of meta-variables.
3. Canonical calculi with (n, k)-ary connectives (see [15]) are particular standard calculi that contain all the structural rules (weakening, contraction and exchange).

3 Modular Cut-Elimination

Generalizations of cut-elimination with extra (non-logical) axioms have been considered e.g. in [13,1,11]. They play an important role in the proof theory of formalized mathematical theories such as fragments of arithmetic. Given a deduction in **LK** of a sequent S_0 from a set $\mathcal{S}$ of non-logical axioms closed under substitutions, *free-cut elimination* described in [1] aims at finding a deduction of S_0 containing only *anchored-cuts*, i.e. cuts whose premises (at least one, for cuts with compound cut-formulas) derive from sequents in $\mathcal{S}$. If $\mathcal{S}$ consist of atomic sequents closed under mix (and substitutions) then Gentzen's cut-elimination method generates a cut-free **LK**-derivation of S_0, see e.g. [13]. To characterize the "stepwise process of local transformations to eliminate cuts" in a large class of propositional single-conclusion sequent calculi we introduced in [3] reductive cut-elimination, a naturally strengthened version of free-cut elimination which in addition aims to shift upward anchored-cuts in these calculi *as much as possible*.

Here below we rework the above notions of cut-elimination in the presence of axioms to define a "modular" cut-elimination for standard calculi, namely if such calculi enjoy it, they also do when extended by any rule satisfying suitable conditions (weak substitutivity and reductivity, see Section 4).

Definition 4. *A set $\mathcal{S}$ of sequents (*non-logical axioms*) is called* elementary *if*

1. *all formulae in $\mathcal{S}$ are atomic.*
2. *$\mathcal{S}$ is* closed under substitutions*: whenever $S(x) \in \mathcal{S}$ and t is any term, the sequent $S(t)$, obtained by substituting in S the term t for all free occurrences of x, is in $\mathcal{S}$.*
3. *$\mathcal{S}$ is* closed under cuts*: whenever $\Gamma_1 \Rightarrow \Delta_1, A$ and $A, \Gamma_2 \Rightarrow \Delta_2$ belong to $\mathcal{S}$, so does $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.*
4. *it is not the case that sequents of the forms $\Gamma \Rightarrow \Delta, A, A$ and $A, A, \Sigma \Rightarrow \Pi$, both belong to $\mathcal{S}$.*

Definition 5. *A standard sequent calculus $\mathcal{L}$ admits* modular cut-elimination *if whenever a sequent S_0 is derivable in $\mathcal{L}$ from an elementary set $\mathcal{S}$ of sequents in $\mathcal{L}$, S_0 has a cut-free derivation in $\mathcal{L}$ from $\mathcal{S}$.*

Remark 2. Modular cut-elimination implies the ordinary cut-elimination (set $\mathcal{S} = \emptyset$).

Notice that if we remove condition 4 from Def. 4, the resulting notion of cut-elimination is not admitted e.g. by **LK**: indeed $\mathcal{S} \equiv \{A, A \Rightarrow\ ;\ \Rightarrow A, A\ ;\ A \Rightarrow A\}$ with A atomic satisfies the conditions 1-3 of Def. 4. It is easy to check that the empty sequent $\Rightarrow$ is derivable from $\mathcal{S}$ in **LK** only using (CUT).

4 Syntactic Criteria

In this section we introduce the notions of reductive logical rules and weakly substitutive rules for standard calculi. Intuitively, a logical rule is reductive if it allows the replacement of cuts by "smaller" cuts, and a rule is weakly substitutive when any cut can be permuted upward. Reductivity and weak substitutivity are obtained by suitably modifying the homonymous conditions of [3] defined for simple calculi (see Ex. 2.1).

Let S be a sequent, A a formula, $T_1 \equiv A, \Sigma \Rightarrow \Pi$ and $T_2 \equiv \Sigma \Rightarrow \Pi, A$. We define

$$[S \hookleftarrow^r_A T_1] = \{\Gamma, \Sigma^\lambda \Rightarrow \Delta, \Pi^\lambda \mid S \equiv \Gamma \Rightarrow \Delta, A^\lambda \text{ with } \lambda \geq 0\}$$
$$[S \hookleftarrow^l_A T_2] = \{\Gamma, \Sigma^\lambda \Rightarrow \Delta, \Pi^\lambda \mid S \equiv A^\lambda, \Gamma \Rightarrow \Delta \text{ with } \lambda \geq 0\}$$

Namely, each $U \in [S \hookleftarrow^r_A T_1]$ is obtained by applying (CUT) possibly several times between S and (several copies of) T_1 with cut formula A. $[S \hookleftarrow^l_A T_2]$ is dually defined. In case T does not contain any occurrence of A in the antecedent (resp. consequent), we define $[S \hookleftarrow^r_A T] = \{S\}$ (resp. $[S \hookleftarrow^l_A T] = \{S\}$).

Definition 6. *Let $\mathcal{L}$ be a standard sequent calculus. A rule (R) is said to be* weakly substitutive *in $\mathcal{L}$ if for each instance of (R) with premises $S_1, \ldots, S_n$ and conclusion S_0 the following condition holds:*

(*) for any $c \in \{r, l\}$, context formula A and any sequent T of $\mathcal{L}$ (which does not contain any eigenvariable of (R)), every $U \in [S_0 \hookleftarrow^c_A T]$ has a derivation from $\bigcup_{i=1}^n [S_i \hookleftarrow^c_A T]$ only using structural rules and, when (R) is a left (resp. right) logical rule with principal formula B, left (resp. right) logical rules with principal formula B.

Remark 3. The above condition was defined (in fact, using rule *schemas* instead of rule *instances*) in [3] only for structural rules. Indeed, the logical rules considered there satisfy a condition stronger than (*), namely: for any $c \in \{r, l\}$, context formula A (right or left context formula, depending on c) and single-conclusion sequent T, every $U \in [S_0 \hookleftarrow^c_A T]$ is derivable from $\bigcup_{i=1}^n [S_i \hookleftarrow^c_A T]$ with an application of (R).

Example 3. The rules of **LJ** (resp. **LK**) are weakly substitutive in **LJ** (resp. **LK**). Consider now:

1. Maehara's calculus **LJ**' for intuitionistic logic, that is an equivalent version of Gentzen's **LJ** where the intuitionistic restriction (i.e. consequent of sequents contain at most one formula) applies not generally but only in the case of the right rules for $\rightarrow$, $\neg$ and $\forall$, see e.g. [11].

2. The calculus **GD** for the logic of constant domains[3]. **GD** was defined in [5] by modifying **LK** as follows: (1) the sequents of **GD** have at most two formulas in their consequents and (2) the rules $(\rightarrow, r, \emptyset)$ and $(\neg, r, \emptyset)$ obey the intuitionistic restriction.

It is easy to see that e.g. the rule $(\rightarrow, r, \emptyset)$ is weakly substitutive neither in **LJ**' nor in **GD**. Indeed, take any instance of $(\rightarrow, r, \emptyset)$, say

$$\frac{S_1}{S_0} \equiv \frac{\Gamma, C, A \Rightarrow B}{\Gamma, C \Rightarrow A \rightarrow B}\ (\rightarrow, r, \emptyset)$$

and $T \equiv \Sigma \Rightarrow \Pi, C$, where Π contains at least one formula. Then $\Gamma, \Sigma \Rightarrow A \rightarrow B, \Pi \in [S_0 \hookleftarrow^l_C T]$ is in general not cut-free derivable from $[S_1 \hookleftarrow^l_C T]$ in **LJ**' or **GD**.

Although Definition 6 refers to *all* instances of any rule, in practice to check that a particular rule is weakly substitutive it is enough to consider *certain atomic* instances.

Definition 7. *Let* (R_0) *be any instance of a structural rule. The* associated atomic instance $\langle R_0 \rangle$ *is defined by replacing each context formula occurrence* A *with a new atomic formula* $\langle A, c \rangle$ *with no free variables (c is either l or r according to whether the formula occurrence appears in the antecedent or consequent of sequents in* (R_0)*).*

When (R_0) *is an instance of a logical rule with the principal formula* $\star\boldsymbol{x}(\boldsymbol{A})$ *with* $\boldsymbol{x} \equiv x_1, \ldots, x_k$ *and* $\boldsymbol{A} \equiv A_1, \ldots, A_l$*, the* associated atomic *instance* $\langle R_0 \rangle$ *is defined by replacing*

- *each context formula* A *with* $\langle A, c \rangle$ *as above,*
- *its principal formula* $\star\boldsymbol{x}(\boldsymbol{A})$ *with* $\star\boldsymbol{x}(\langle A_1, 1\rangle(\boldsymbol{x}), \ldots \langle A_l, l\rangle(\boldsymbol{x}))$*, where for each* $i = 1, \ldots, l$ $\langle A_i, i \rangle$ *is a new k-ary predicate symbol*
- *each* $A_i[^{\boldsymbol{t}}/\boldsymbol{x}]$ *with* $\langle A_i, i \rangle(\boldsymbol{t})$*.*

Note that $\langle R_0 \rangle$ strictly distinguishes active, left and right context formulae.

Lemma 1. *(1) If* (R_0) *is an instance of a rule* (R)*, so is* $\langle R_0 \rangle$*. (2) If condition (*) of Def. 6 holds for* $\langle R_0 \rangle$ *then the same condition holds for* (R_0)*.*

Proof. (1) Follows by conditions **(str0)**, **(str1)**, **(log0)** and **(log1)**. (2) Easy.

To introduce reductivity we need some additional notation and terminology. Given a set $\mathcal{S}$ of sequents (resp. a set $\mathcal{A}$ of formulae), we denote by $\mathcal{S}^s$ (resp. $\mathcal{A}^s$) the least set containing $\mathcal{S}$ (resp. $\mathcal{A}$) and closed under substitutions. We call any instance of (CUT) with cut-formula in $\mathcal{A}$ an $\mathcal{A}$*-cut*.

Definition 8. *Let* $\mathcal{L}$ *be a standard sequent calculus. We call its logical rules* $\{(\star, r, \boldsymbol{y})_j\}_{j \in \Lambda}$ *and* $\{(\star, l, \boldsymbol{z})_k\}_{k \in \Lambda'}$ *for introducing a* (k, l)*-ary connective* $\star$ reductive *in* $\mathcal{L}$ *if*

1. *either* Λ *or* Λ' *is empty or*
2. *for any pair of instances of left and right logical rules with principal formula* $\star\boldsymbol{x}(\boldsymbol{A})$*:*

[3] A Hilbert calculus for this logic is obtained by adding to that of intuitionistic logic the shifting law of universal quantifiers w.r.t. $\vee$, i.e. $\forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B$, where x does not appear free in B.

$$\frac{S_1 \quad \ldots \quad S_n}{\Gamma \Rightarrow \Delta, \star\boldsymbol{x}(\boldsymbol{A})} \qquad \frac{T_1 \quad \cdots \quad T_m}{\star\boldsymbol{x}(\boldsymbol{A}), \Sigma \Rightarrow \Pi}$$

($\star$) *$\Gamma, \Sigma \Rightarrow \Delta, \Pi$ is derivable from $\{S_1, \ldots, S_n, T_1, \ldots, T_m\}^s$ only using $\{\boldsymbol{A}\}^s$-cuts and structural rules.*

Remark 4. The above definition generalizes the reductivity condition of [3] and the principal formula condition of [8], both defined for propositional calculi (single-conclusion, in case of the former). Reductivity is also related to the coherence criterion of [6].

Lemma 2. *If condition ($\star$) of Def. 8 holds for $\langle R_0 \rangle$ then it holds for (R_0).*

Example 4. Consider the $(1,1)$-ary logical connectives $\flat, \natural$ defined by the following rules:

$$\frac{X[^t/_x], \Theta \Rightarrow \Xi}{\flat x(X), \Theta \Rightarrow \Xi}\ (\flat, l, \emptyset) \qquad \frac{\Theta \Rightarrow \Xi, X[^t/_x]}{\Theta \Rightarrow \Xi, \flat x(X)}\ (\flat, r, \emptyset)$$

$$\frac{X[^y/_x], \Theta \Rightarrow \Xi}{\natural x(X), \Theta \Rightarrow \Xi}\ (\natural, l, y) \qquad \frac{\Theta \Rightarrow \Xi, X[^y/_x]}{\Theta \Rightarrow \Xi, \natural x(X)}\ (\natural, r, y)$$

The rules for $\natural$ are reductive in **LK** while those for $\flat$ are not.

Example 5. Let $\mathcal{L}_1$ be the standard calculus that consists of the following rules introducing the $(0,2)$-ary connective $\sqcap$ (together with permutation rules and identity axioms)

$$\frac{\Theta \Rightarrow X, \Xi \quad \Theta \Rightarrow Y, \Xi}{\Theta \Rightarrow X \sqcap Y, \Xi}\ (\sqcap, r, \emptyset) \qquad \frac{\Theta, X, Y \Rightarrow \Xi}{\Theta, X \sqcap Y \Rightarrow \Xi}\ (\sqcap, l, \emptyset)$$

$(\sqcap, r, \emptyset)$ and $(\sqcap, l, \emptyset)$ are not reductive in $\mathcal{L}_1$.

5 Necessary Conditions

We show that reductivity and weak substitutivity are necessary conditions for modular cut-elimination in standard sequent calculi whose logical rules satisfy certain additional conditions. Specifically, for each logical rule $(\star, l, \boldsymbol{y})_i$ and $(\star, r, \boldsymbol{z})_j$ we define the following conditions:

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\star\boldsymbol{x}(\boldsymbol{X}), \Theta \Rightarrow \Xi}\ (\star, l, \boldsymbol{y})_i \qquad \frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta \Rightarrow \Xi, \star\boldsymbol{x}(\boldsymbol{X})}\ (\star, r, \boldsymbol{z})_j$$

(log2) if any active meta-variable $X[^{\boldsymbol{t}}/_{\boldsymbol{x}}]$ occurs in $\Upsilon_1, \ldots, \Upsilon_n$, then no $X[^{\boldsymbol{t}'}/_{\boldsymbol{x}'}]$ (for any $\boldsymbol{t}', \boldsymbol{x}'$) occurs in $\Psi_1, \ldots, \Psi_n$, and *vice versa*.

(log3) each active meta-variable X_i ($1 \leq i \leq l$) occurs at most once in each premise $\Upsilon_j \Rightarrow \Psi_j$ ($1 \leq j \leq n$).

Theorem 1. *Let $\mathcal{L}$ be a standard sequent calculus. If $\mathcal{L}$ admits modular cut-elimination, (i) its structural rules are weakly substitutive and (if in addition each logical rule of $\mathcal{L}$ satisfies* **(log2)***) (ii) its logical rules are weakly substitutive.*

Proof. We prove (ii) since (i) is similar. Let (R_0) be any instance of a logical rule with principal formula B. By Lemma 1 it is enough to prove condition (*) of Definition 6 for the associated atomic instance $\langle R_0 \rangle$ with premises $S_1, \dots S_n$ and conclusion S_0. Let $c \in \{l, r\}$, T an atomic sequent without free variables and A any atomic formula. W.l.o.g. we may assume that T does not share any atomic formula other than A with S_0. Let $\mathcal{S}$ be the least set that contains $\{S_1, \dots, S_n, T\}$ and is closed under substitutions and cuts. By condition **(log2)** and the definition of $\langle R_0 \rangle$ and T, $\mathcal{S}$ is elementary and is equivalent to $\bigcup_{i=1,\dots,n}[S_i \hookleftarrow^c_A T]$.

Then, any $U \in [S_0 \hookleftarrow^c_A T]$ is derivable from $\mathcal{S}$ using $\langle R_0 \rangle$ and (CUT). Hence by modular cut-elimination, U has a cut-free derivation d from $\bigcup_{i=1,\dots,n}[S_i \hookleftarrow^c_A T]$. Since B is the only compound formula in U, d uses only structural rules and logical rules introducing B. □

Theorem 2. *Let $\mathcal{L}$ be any standard sequent calculus whose logical rules satisfy* **(log2)** *and* **(log3)**. *If $\mathcal{L}$ admits modular cut-elimination, then its logical rules are reductive.*

Proof. Let $(\star, r, \boldsymbol{y})_k$ and $(\star, l, \boldsymbol{z})_j$ be a pair of instances of right and left logical rules for $\star$ in $\mathcal{L}$ and $\langle \star, r, \boldsymbol{y} \rangle_k$ and $\langle \star, l, \boldsymbol{z} \rangle_j$ be the associated atomic instances (see Def. 7):

$$\frac{S_1 \quad \dots \quad S_n}{\Gamma \Rightarrow \Delta, \star[\boldsymbol{x}](\boldsymbol{A})} \langle \star, r, \boldsymbol{y} \rangle_k \qquad \frac{T_1 \quad \cdots \quad T_m}{\star[\boldsymbol{x}](\boldsymbol{A}), \Sigma \Rightarrow \Pi} \langle \star, l, \boldsymbol{z} \rangle_j$$

Without loss of generality, we may assume that (†) the context formulae of $\langle \star, r, \boldsymbol{y} \rangle_k$ are distinct from those of $\langle \star, l, \boldsymbol{z} \rangle_j$. Thus the active formulae (in $\{\boldsymbol{A}\}^s$) are the only formulae that occur both in the antecedent of a premise and in the consequent of another. Let $\mathcal{S}$ be the least set that contains $\{S_1, \dots, S_n, T_1, \dots, T_m\}$ and is closed under substitutions and cuts. $\mathcal{S}$ is elementary due to conditions **(log2)** and **(log3)** and the definition of $\langle \star, r, \boldsymbol{y} \rangle_k$ and $\langle \star, l, \boldsymbol{z} \rangle_j$. By modular cut-elimination $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ is cut-free derivable from $\mathcal{S}$. Hence it is derivable from $\{S_1, \dots, S_n, T_1, \dots, T_m\}^s$ only using $\{\boldsymbol{A}\}^s$-cuts and structural rules. The claim follows by Lemma 2. □

6 Sufficient Conditions

Weak substitutivity and reductivity are sufficient conditions for a standard sequent calculus to admit *modular cut-elimination* (and hence cut-elimination). Here below we give a constructive proof of this result.

In the sequel, $\mathcal{L}$ denotes a standard calculus whose rules are weakly substitutive and whose logical rules are reductive while $\mathcal{S}_0$ any elementary set of non-logical axioms.

Definition 9. *The* length $|d|$ *of a derivation d is the maximal number of inference rules + 1 occurring on any branch of d. The* complexity $|A|$ *of a formula A is defined as the number of occurrences of its (n, k)-ary connectives. The* cut rank $\rho(d)$ *of d is (the maximal complexity of the cut-formulae in d) + 1 ($\rho(d) = 0$ if d has no cuts). Given a compound formula B and $c \in \{l, r\}$, $\sharp^c_B(d)$ is the maximal number of c-side (left or right) logical rules with principal formula B on any branch of d.*

To prove modular cut-elimination for $\mathcal{L}$, we proceed by removing cuts which are topmost among all cuts with cut rank equal to the rank of the whole deduction. Let, e.g.

$$\dfrac{\begin{array}{c}\mathcal{S}_0\\ \vdots\, d_1\\ \Gamma \Rightarrow \Delta, A\end{array} \quad \begin{array}{c}\mathcal{S}_0\\ \vdots\, d_2\\ A, \Sigma \Rightarrow \Pi\end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{CUT})$$

be a subderivation ending in such a cut. Roughly speaking our strategy is as follows: If the cut-formula A is a compound formula, using the fact that rules are weakly substitutive, we shift up this cut over d_2 *as much as possible* until we meet (a) an identity axiom or (b) a logical rule introducing the cut formula A (Lemma 5). In the first case the cut is easily eliminated while in case (b) is replaced by cuts with smaller complexity. The latter can be done being logical rules reductive (Lemma 4 and Lemma 5). If A is atomic, the cut is shifted upward over d_2 or d_1 (according to whether the elementary set $\mathcal{S}_0$ contains a sequent of the form $\Phi \Rightarrow \Psi, A, A$ or $\Phi, A, A \Rightarrow \Psi$, respectively) until we meet (a) an identity axiom or (b) a non-logical axiom in $\mathcal{S}_0$ (Lemma 6.(ii)). In both cases the cut can be easily eliminated (for case (b) see Lemma 6.(i)).

Henceforth we write $d, \mathcal{S} \vdash_{\mathcal{L}} S$ if d is a derivation in $\mathcal{L}$ of S from a set $\mathcal{S}$ of sequents.

Lemma 3 (Substitution). *Let $\mathcal{S}$ be any set of sequents closed under substitutions and $d, \mathcal{S} \vdash_{\mathcal{L}} S(x)$. Then for any term t there is a derivation d' with $|d'| = |d|$ and $\rho(d') = \rho(d)$ such that $d', \mathcal{S} \vdash_{\mathcal{L}} S(t)$. Moreover, for any compound formula A which contains neither x nor an eigenvariable of a rule in d and for any $c \in \{l, r\}$, $\sharp^c_A(d') = \sharp^c_A(d)$.*

Proof. By induction on $|d|$. The crucial case is when the last inference (R) in d is a logical rule with eigenvariables $\boldsymbol{y}$ and with premises $S_1(x, \boldsymbol{y}), \ldots, S_n(x, \boldsymbol{y})$. The term t might contain eigenvariables $\boldsymbol{y}$. So, take fresh variables $\boldsymbol{z}$. Then each $S_i(t, \boldsymbol{z})$ $(i = 1, \ldots, n)$ has derivations with the required properties. We can now apply (R) and obtain $S(t)$. Since A contains neither x nor $\boldsymbol{y}$, $\sharp^c_A(d)$ remains unchanged. □

The following lemma shows how to *reduce* a cut on a compound formula B (i.e. replace it by cuts with cut-formula smaller than B) in case one of its premises is the conclusion of a logical rule introducing B on the left hand side and with atomic context formulae. This lemma is needed when proving the general case: reducing any cut on a compound formula (Lemma 5).

Lemma 4. *Let*

$$\dfrac{T_1 \quad \ldots \quad T_m}{T} \equiv B, \Sigma \Rightarrow \Pi$$

be an instance of a left logical rule with principal formula B and in which all context formulae are atomic. If $d_1, \mathcal{S}_0 \cup \{T_1, \ldots, T_m\}^s \vdash_{\mathcal{L}} S$ with $\rho(d_1) < |B|$ then each $U \in [S \hookleftarrow^r_B T]$ has a derivation $d, \mathcal{S}_0 \cup \{T_1, \ldots, T_m\}^s \vdash_{\mathcal{L}} U$ with $\rho(d) < |B|$ and $\sharp^r_B(d) \leq \sharp^r_B(d_1)$.

Of course, one could derive U by applying (CUT), but the resulting derivation would have cut rank $|B| + 1$.

Proof. Proceeds by a double induction on $(\sharp^r_B(d_1), |d_1|)$. Let $\mathcal{T} = \{T_1, \dots, T_m\}^s$.

Base case: $|d_1| = 1$. Then S is either an identity axiom or belongs to $\mathcal{S}_0 \cup \mathcal{T}$. In the former case $U \in [S \hookleftarrow^r_B T]$ is S or T, while in the latter case U is S (since S does not contain B). Hence the claim is trivial.

Inductive case: $|d_1| > 1$. If $U \equiv S$ the claim is trivial. Otherwise, suppose that d_1 ends in a rule (R) with premises $S_1, \dots, S_n$ and conclusion S. Two cases can arise:

(Case 1) (R) is not a right logical rule with principal formula B. Since (R) is weakly substitutive, (previously applying Lemma 3, if needed) $U \in [S \hookleftarrow^r_B T]$ has a derivation d' from $U_1, \dots, U_k \in \bigcup_{i=1}^n [S_i \hookleftarrow^r_B T]$, in which neither (CUT) nor a rule introducing B in the consequent is used. By the inductive hypothesis, we can find derivations $d'_i, \mathcal{S}_0 \cup \mathcal{T} \vdash_{\mathcal{L}} U_i$ with $\rho(d'_i) < |B|$ and $\sharp^r_B(d'_i) \leq \sharp^r_B(d_1)$ for $1 \leq i \leq k$. Therefore the required derivation for U can be obtained by plugging $d'_1, \dots, d'_k$ into d'.

(Case 2) Otherwise, S can be written as $\Gamma \Rightarrow \Delta, B$. Let U_0 be $\Gamma, \Sigma \Rightarrow \Delta, \Pi$. Then,

(1) $U \in [U_0 \hookleftarrow^r_B T]$,

(2) U_0 has a derivation d'_0 from $U_1, \dots, U_k \in \{S_1, \dots, S_n, T_1, \dots, T_m\}^s$ only using structural rules and $\{\boldsymbol{A}\}^s$-cuts, being (R) reductive. In particular, no rule introducing B in the consequent is used in d'_0.

By hypothesis, each S_i ($i = 1, \dots, n$) has a derivation δ_i from $\mathcal{S}_0 \cup \mathcal{T}$ with cut-rank $< |B|$ and and $\sharp^r_B(\delta_i) < \sharp^r_B(d_1)$. By Lemma 3, each U_i has a derivation $d'_i, \mathcal{S}_0 \cup \mathcal{T} \vdash_{\mathcal{L}} U_i$ with $\rho(d'_i) < |B|$ and $\sharp^r_B(d'_i) < \sharp^r_B(d_1)$ for $1 \leq i \leq k$. Therefore by plugging $d'_1, \dots, d'_k$ into d'_0, we obtain a derivation $d', \mathcal{S}_0 \cup \mathcal{T} \vdash_{\mathcal{L}} U_0$ with $\rho(d') < |B|$ and $\sharp^r_B(d') < \sharp^r_B(d_1)$. The required derivation for U can be obtained by (1) and the inductive hypothesis. □

To reduce any cut on a compound formula we use a similar argument as in the previous lemma. Here we need more care of the parameter on which the induction proceeds. To this aim we consider the *marking* (or *decoration*, see [2]) of some formulae occurring in a derivation. Let us fix a formula $B \equiv \star\boldsymbol{x}(\boldsymbol{A})$. A *marked sequent* is a sequent with some (possibly zero) underlined occurrences of B in the antecedent. A *marked derivation* d consists of marked sequents, with the following proviso:

(!) for any instance of a rule (R) used in d and any occurrence of B in the conclusion of (R) which instantiates a meta-variable X, if that occurrence is marked, so are all occurrences of B in the premises which instantiate X.

Given a not marked sequent $S \equiv \Gamma \Rightarrow \Delta, B$ and a marked sequent T, $[T \hookleftarrow^l_{\underline{B}} S]$ stands for $\{\Gamma^\lambda, \Sigma \Rightarrow \Delta^\lambda, \Pi \mid T \equiv \underline{B}^\lambda, \Sigma \Rightarrow \Pi \text{ with } \lambda \geq 0\}$. (Notice that Σ may contain other occurrences of $\underline{B}$.) Finally, let $\sharp^l_{\underline{B}}(d)$ be the maximal number of logical rules introducing *marked* occurrences of B on the left side on any branch of d.

Lemma 5 (Compound formulae). *Let B be any compound formula, T be a marked sequent in which some occurrences of B in the antecedent are marked and $d_2, \mathcal{S}_0 \vdash_{\mathcal{L}} T$ be a marked derivation. Assume $d_1, \mathcal{S}_0 \vdash_{\mathcal{L}} S$ (d_1 and S are not marked) where $\rho(d_1), \rho(d_2) < |B|$. Then, each $U \in [T \hookleftarrow^l_{\underline{B}} S]$ has a marked derivation $d, \mathcal{S}_0 \vdash_{\mathcal{L}} U$ with $\rho(d) < |B|$ and $\sharp^l_{\underline{B}}(d) \leq \sharp^l_{\underline{B}}(d_2)$.*

Proof. Proceed by a double induction on $(\sharp^l_{\underline{B}}(d_2), |d_2|)$.

Base case: $|d_2| = 1$. T is either an identity axiom or ($B \notin T$ and) $T \in \mathcal{S}_0$. Then U is either S or T, and the required derivation d is either d_1 or just consists of T. In both cases, we have $\rho(d) < |B|$ and $\sharp^l_{\underline{B}}(d_1) = 0$. Hence our claim holds.

Inductive case: $|d_2| > 1$. If $U \equiv T$, the claim is trivial. Otherwise, assume that d_2 ends with an instance of a rule (R) with premises $T_1, \dots, T_m$ and conclusion T. Two cases can arise:

(Case 1) (R) is not a left logical rule introducing a marked occurrence of B. This case is similar to (Case 1) in the proof of Lemma 4.

(Case 2) Otherwise, we may assume that T is of the form $\underline{B}, \Sigma \Rightarrow \Pi$ and S of the form $\Gamma \Rightarrow \Delta, B$. Let U_0 be $\Gamma, \Sigma \Rightarrow \Delta, \Pi$. Then any $U \in [T \hookleftarrow^l_{\underline{B}} S]$ other than T also belongs to $[U_0 \hookleftarrow^l_{\underline{B}} S]$. Hence it is enough to find a derivation $d, \mathcal{S}_0 \vdash_{\mathcal{L}} U_0$ with $\rho(d) < |B|$ and $\sharp^l_{\underline{B}}(d) < \sharp^l_{\underline{B}}(d_2)$. The claim will then be established by the inductive hypothesis.

Let us replace the principal formula $\underline{B}$ by B and each context formula $C(\boldsymbol{y})$ (resp. marked context formula $\underline{C}(\boldsymbol{y})$) in $T, T_1, \dots, T_m$ with free variables $\boldsymbol{y}$ by a fresh atomic formula $\langle C\rangle(\boldsymbol{y})$ (resp. $\langle \underline{C}\rangle(\boldsymbol{y})$) to obtain sequents $\langle T\rangle, \langle T_1\rangle, \dots, \langle T_m\rangle$. In particular, $\langle T\rangle$ is of the form $B, \langle\Sigma\rangle \Rightarrow \langle\Pi\rangle$ and $\langle T_1\rangle, \dots, \langle T_m\rangle / \langle T\rangle$ is an instance of (R) in which context formulas are atomic. Since $\langle U_0\rangle \equiv \Gamma, \langle\Sigma\rangle \Rightarrow \Delta, \langle\Pi\rangle \in [S \hookleftarrow^r_B \langle T\rangle]$, Lemma 4 implies that there is a derivation $d_0, \mathcal{S}_0 \cup \{\langle T_1\rangle, \dots, \langle T_m\rangle\}^s \vdash \langle U_0\rangle$ with $\rho(d_0) < |B|$ and $\sharp^l_{\underline{B}}(d_0) = 0$ (since d_0 does not contain any $\underline{B}$). From this, we can easily obtain a derivation $d_0', \mathcal{S}_0 \cup \{T_1, \dots, T_m\}^s \vdash_{\mathcal{L}} U_0$ with the same property. On the other hand, by hypothesis and Lemma 3 any $U' \in \{T_1, \dots, T_m\}^s$ has a derivation $d', \mathcal{S}_0 \vdash_{\mathcal{L}} U'$ with $\rho(d') < |B|$ and $\sharp^l_{\underline{B}}(d') < \sharp^l_{\underline{B}}(d_2)$. Hence by plugging them into d_0', we obtain the required derivation d for U_0. □

Lemma 6 (Atomic formulae). *(i) Suppose that a sequent S has a cut-free derivation d_1 from $\mathcal{S}_0$ and $T \in \mathcal{S}_0$. Then, for any atomic formula A and any $c \in \{l, r\}$, each $U \in [S \hookleftarrow^c_A T]$ has a cut-free derivation from $\mathcal{S}_0$.*

(ii) Let d_1 and d_2 be cut-free derivations of $d_1, \mathcal{S}_0 \vdash_{\mathcal{L}} S$ and $d_2, \mathcal{S}_0 \vdash_{\mathcal{L}} T$ and A be an atomic formula. Then, each $U \in [T \hookleftarrow^l_A S]$ (resp. each $U \in [S \hookleftarrow^r_A T]$) has a cut-free derivation $d, \mathcal{S}_0 \vdash_{\mathcal{L}} U$ provided that no sequent of the form $A, A, \Sigma \Rightarrow \Pi$ (resp. $\Gamma \Rightarrow \Delta, A, A$) belongs to $\mathcal{S}_0$.

Proof. (i) Proceeds by induction on $|d_1|$, similarly as (Case 1) in the proof of Lemma 4. (ii) Proceeds by induction on $|d_2|$ (resp. $|d_1|$). When $|d_2| = 1$, then T is an identity axiom or $T \in \mathcal{S}_0$. If $U \equiv T$ or $U \equiv S$ the claim is trivial. Otherwise, since T does not contain more than one occurrence of A in the antecedent, $U \in [T \hookleftarrow^l_A S]$ also belongs to $[S \hookleftarrow^r_A T]$. Hence the claim follows by (i). The case $|d_2| > 1$ is as before. □

Theorem 3 (Modular Cut-Elimination). *Any standard sequent calculus $\mathcal{L}$ whose rules are weakly substitutive and whose logical rules are reductive admits modular cut-elimination.*

Proof. Let $\mathcal{S}_0$ be an elementary set of non-logical axioms in $\mathcal{L}$, d a derivation in $\mathcal{L}$ from $\mathcal{S}_0$ with $\rho(d) > 0$. The proof proceeds by a double induction on $(\rho(d), n\rho(d))$, where

$n\rho(d)$ is the number of cuts in d with cut rank $\rho(d)$. Let us take in d an uppermost cut with cut rank $\rho(d)$. Let $d_1, \mathcal{S}_0 \vdash_{\mathcal{L}} \Gamma \Rightarrow \Delta, A$ and $d_2, \mathcal{S}_0 \vdash_{\mathcal{L}} A, \Sigma \Rightarrow \Pi$ its premises.

When A is not atomic, let d_2' be a marking of d_2 in which the indicated A is marked, and apply Lemma 5 to d_1 and d_2'. When A is atomic, apply Lemma 6 (ii) to d_1 and d_2 (by Definition 4, multiple copies of A cannot occur both in the antecedent and consequent positions of any sequent in $\mathcal{S}_0$). In any case, either $\rho(d)$ or $n\rho(d)$ decreases. □

When a standard sequent calculus satisfies some additional properties, weak substitutivity and reductivity *characterize* modular cut-elimination:

Corollary 1. *Let $\mathcal{L}$ be a standard sequent calculus satisfying* **(log2)** *and* **(log3)**. *Then $\mathcal{L}$ admits modular cut-elimination if and only if all rules are weakly substitutive and all logical rules are reductive.*

Theorem 3 allows us to prove cut-elimination for a given standard sequent calculus in an "incremental" way:

Corollary 2 (Modularity). *Let $\mathcal{L}$ and $\mathcal{L}'$ be standard calculi with disjoint sets of logical connectives (and the same cut rule). Suppose that their logical rules satisfy* **(log2)** *and* **(log3)**. *If both $\mathcal{L}$ and $\mathcal{L}'$ admit modular cut elimination, so does $\mathcal{L} \cup \mathcal{L}'$, obtained by taking the union of logical connectives and rules in $\mathcal{L}$ and $\mathcal{L}'$.*

Remark 5. The same result does not hold for cut-elimination. E.g. let $\mathcal{L}_1'$ be the calculus containing exchange and the rules for implication in linear logic. $\mathcal{L}_1'$ admits cut-elimination and so does (trivially) the calculus $\mathcal{L}_1$ of Example 5 (the only sequents provable in $\mathcal{L}_1$ are instances of identity axioms) while $\mathcal{L}_1 \cup \mathcal{L}_1'$ does not anymore.

Our modular cut-elimination procedure is 'universal' for standard sequent calculi with additional conditions in the following sense:

Corollary 3. *Let $\mathcal{L}$ be a standard sequent calculus satisfying* **(log2)** *and* **(log3)**. *If $\mathcal{L}$ admits modular cut-elimination and $\vdash_{\mathcal{L}} S$, the procedure described in this section always provides a cut-free derivation in $\mathcal{L}$ for S.*

Remark 6. The same does not hold for cut-elimination and e.g. the procedures of Gentzen [4] and Schütte-Tait [10,9]. Indeed, Gentzen's cut-elimination method can be applied only when suitable "ad hoc" (derivable) generalizations of the cut rule (e.g. Gentzen's mix) are found. These generalizations, needed to cope with rules duplicating formulas (e.g. contraction), are not needed for the Schütte-Tait method whose applicability relies on the inversion of (at least) one of the premises of the cut. This cannot always be done in calculi that admit cut-elimination. For example let $\mathcal{L}_2$ be the calculus consisting of weakening, exchange and the following rules:

$$\frac{\Theta \Rightarrow X_1 \quad \Theta' \Rightarrow X_2}{\Theta, \Theta' \Rightarrow X_1 \wedge X_2}\ (\wedge, r) \qquad \frac{\Theta, X_i \Rightarrow Y}{\Theta, X_1 \wedge X_2 \Rightarrow Y}\ (\wedge, l)_{i=1,2}$$

$\mathcal{L}_2$ admits cut-elimination (e.g. using our method: it is easy to check that these rules are reductive and weakly substitutive) although neither of the premises of a cut with cut formula $A \wedge B$ can be inverted in the usual way and hence the Schütte-Tait procedure does not apply.

7 Counterexamples to (Modular) Cut-Elimination

We have introduced syntactic criteria (weak substitutivity and reductivity) that when met by a standard sequent calculus $\mathcal{L}$, $\mathcal{L}$ admits modular cut-elimination. If the logical rules of $\mathcal{L}$ satisfy **(log2)** and **(log3)** our conditions are also necessary and hence a *counterexample for modular cut-elimination* (i.e. a derivation in $\mathcal{L}$ from an elementary set of sequents in which cuts cannot be eliminated) can be extracted from their failure.

Now, what can we say about plain cut-elimination? The failure of weak substitutivity or reductivity for a standard calculus $\mathcal{L}$ is not enough to conclude that $\mathcal{L}$ does not admit cut-elimination, being modular cut-elimination a notion strictly stronger than cut-elimination (e.g. both **LJ**' and $\mathcal{L}_1$ admit cut-elimination although they do not admit modular cut-elimination, see Examples 3, 5 and Remark 5).

Our conditions are however useful for pinning down the difficulty of (dis)proving cut-elimination and reduce the search space when finding counterexamples for cut-elimination (or cut-admissibility). Indeed

Definition 10. *Let $\mathcal{L}$ be a standard sequent calculus. The following derivations d in $\mathcal{L}$ are called* candidates of counterexamples *for $\mathcal{L}$.*

- *Let (R) be an instance of a rule in $\mathcal{L}$ which is not weakly substitutive. Let S_0 be its conclusion and $S_1, \ldots, S_n$ its premises. Take a sequent T, a formula A, $c \in \{l, r\}$ and $U \in [S_0 \hookleftarrow^c_A T]$ which violates condition (*) of Def. 6. Then let d be the following:*

$$\dfrac{T \qquad \dfrac{S_1 \quad \cdots \quad S_n}{S_0}}{U}\ (CUT)$$

- *Let $\star$ be a connective in $\mathcal{L}$ whose rules are not reductive. Take a pair of instances of left and right logical rules with conclusions $\Gamma \Rightarrow \Delta, \star\boldsymbol{x}(\boldsymbol{A})$ and $\star\boldsymbol{x}(\boldsymbol{A}), \Sigma \Rightarrow \Pi$ which violates the condition $(\star)$ of Def. 8. Then let d be the following:*

$$\dfrac{\dfrac{S_1 \quad \cdots \quad S_n}{\Gamma \Rightarrow \Delta, \star\boldsymbol{x}(\boldsymbol{A})} \qquad \dfrac{T_1 \quad \cdots \quad T_m}{\star\boldsymbol{x}(\boldsymbol{A}), \Sigma \Rightarrow \Pi}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (CUT)$$

A candidate of counterexamples $d, U_1, \ldots, U_n \vdash_{\mathcal{L}} U_0$ is resolvable *if whenever $U_1, \ldots,$ U_n are provable in $\mathcal{L}$, U_0 is cut-free provable in $\mathcal{L}$.*

Example 6. The rule $(\rightarrow, r, \emptyset)$ is weakly substitutive neither in Maehara's **LJ**' nor in **GD** (see Example 3). A candidate of counterexamples for **LJ**' and **GD**, that is also a counterexample for modular cut-elimination is then provided by any cut-free derivable sequent with one implicative formula on its right end side, e.g. $D \Rightarrow C \rightarrow D$ and any set of non-logical axioms containing the sequent $\Gamma \Rightarrow D, \Delta$, for any Δ that contains at least one formula. This counterexample for modular cut-elimination can be easily turned into a *counterexample for cut-elimination* in **GD** by suitably choosing Γ, Δ and D such that $\vdash_{\mathbf{GD}} \Gamma \Rightarrow D, \Delta$ while $\vdash_{\mathbf{GD}} \Gamma \Rightarrow C \rightarrow D, \Delta$ only using (CUT). E.g. take $\Gamma \equiv \forall x(P(x) \vee B)$, $D \equiv \forall x P(x)$ and $\Delta \equiv B$, it is easy to see that the sequent $\forall x(P(x) \vee B) \Rightarrow C \rightarrow \forall x P(x), B$ is not cut-free derivable in **GD** while a derivation with (CUT) is as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a) \quad B \Rightarrow B}{P(a) \vee B \Rightarrow P(a), B}\,(\vee,l)}{\forall x(P(x) \vee B) \Rightarrow P(a), B}\,(\forall,l)}{\forall x(P(x) \vee B) \Rightarrow \forall x P(x), B}\,(\forall,r) \qquad \dfrac{\dfrac{\forall x P(x) \Rightarrow \forall x P(x)}{\forall x P(x), C \Rightarrow \forall x P(x)}\,(w,l)}{\forall x P(x) \Rightarrow C \rightarrow \forall x P(x)}\,(\rightarrow,r)}{\forall x(P(x) \vee B) \Rightarrow C \rightarrow \forall x P(x), B}\,(\text{CUT})
$$

This proves that **GD** does not admit cut-elimination (in contrast with the claim in [5]).

Notice that all candidates of counterexamples are resolvable in **LJ'**. Indeed, a careful inspection of the modular cut-elimination proof shows:

Theorem 4. *Let $\mathcal{L}$ be a standard sequent calculus for which either weak substitutivity or reductivity fails. Then $\mathcal{L}$ admits cut-elimination if and only if all candidates of counterexamples for $\mathcal{L}$ are resolvable.*

To conclude, although our conditions do not directly yield a counterexample for cut-elimination, they do provide the class of candidates among which, if a standard calculus does not admit cut-elimination, such a counterexample can be found.

References

1. S. Buss. An Introduction to Proof Theory. *Handbook of Proof Theory*, Elsevier Science, pp. 1–78, 1998.
2. A. Ciabattoni. Automated Generation of Analytic Calculi for Logics with Linearity. *Proceedings of CSL'04*, vol. 3210 LNCS, pp. 503–517, 2004.
3. A. Ciabattoni and K. Terui. Towards a semantic characterization of cut-elimination. *Studia Logica*. Vol. 82(1). pp. 95 - 119. 2006.
4. G. Gentzen. Untersuchungen über das logische Schliessen I, II. *Mathematische Zeitschrift*, 39: 176–210, 405–431. 1934.
5. E. G. K. Lopez-Escobar. On the Interpolation Theorem for the Logic of Constant Domains. *J. Symb. Log.*. 46(1). pp. 87-88. 1981.
6. D. Miller and E. Pimentel, Tableaux'02, LNAI, Using Linear Logic to reason about sequent systems, 2-23, 2002,
7. F. Pfenning. Structural Cut Elimination: I. Intuitionistic and Classical Logic. *Inf. Comput*. 157. pp. 84-141. 2000.
8. G. Restall. *An Introduction to Substructural Logics*. Routledge, London, 1999.
9. K. Schütte. *Beweistheorie*. Springer Verlag. 1960.
10. W.W. Tait. Normal derivability in classical logic. In *The Sintax and Semantics of infinitary Languages*, LNM 72, 204–236. 1968.
11. G. Takeuti. *Proof Theory*, 2nd edition, North-Holland, 1987.
12. K. Terui. Which Structural Rules Admit Cut Elimination? — An Algebraic Criterion. To appear in *Journal of Symbolic Logic*.
13. A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory (2nd Edition)*. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2000.
14. A. Zamanski and A. Avron. Cut-Elimination and Quantification in Canonical Systems. *Studia Logica*, Vol. 82(1), pp. 157–176. 2006.
15. A. Zamanski and A. Avron. Canonical Gentzen-type calculi with (n,k)-ary quantifiers. *Proceedings of IJCAR'06*. To appear.