Monads Can Be Rough

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Abstract. Traditionally, rough sets build upon relations based on ordinary sets, i.e. relations on X as subsets of $X \times X$. A starting point of this paper is the equivalent view on relations as mappings from X to the (ordinary) power set PX. Categorically, P is a set functor, and even more so, it can in fact be extended to a monad (P, η, μ). This is still not enough and we need to consider the partial order $(PX, \leq)$. Given this partial order, the ordinary power set monad can be extended to a *partially ordered monad*. The partially ordered ordinary power set monad turns out to contain sufficient structure in order to provide rough set operations. However, the motivation of this paper goes far beyond ordinary relations as we show how more general power sets, i.e. partially ordered monads built upon a wide range of set functors, can be used to provide what we call *rough monads*.

1 Introduction

Partially ordered monads are monads [9], where the underlying endofunctor is equipped with an order structure. Some additional structure is imposed. Partially ordered monads are useful for various generalized topologies and convergence spaces [3,4], and have also been used for generalisation of Kleene algebras [12,7,2].

Partially ordered monads over the category Set of sets are defined by means of functors from Set to the category acSLAT of almost complete semilattices[1]. A partially ordered monad is a quadruple $(\varphi, \leq, \eta, \mu)$, where $(\varphi, \leq, \eta)$ is a basic triple[2], (φ, η, μ) is a monad[3] (over Set), and further, for all mappings $f, g : Y \to \varphi X$, $f \leq g$ implies $\mu_X \circ \varphi f \leq \mu_X \circ \varphi g$, where $\leq$ is defined argumentwise with respect to the partial ordering of φX. We also require that for each set X, $\mu_X : (\varphi\varphi X, \leq) \to (\varphi X, \leq)$ preserves non-empty suprema.

The classical example of a partially ordered monad is the power set partially ordered monad $(P, \leq, \eta, \mu)$, where PX is the ordinary power set of X and $\leq$ its set inclusion $\subseteq$

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[1] An almost complete semilattice is a partially ordered sets $(X, \leq)$ such that the suprema $\sup \mathcal{M}$ of all non-empty subsets $\mathcal{M}$ of X exists.

[2] A *basic triple* ([3]) is a triple $(\varphi, \leq, \eta)$, where $(\varphi, \leq) : \mathsf{Set} \to \mathsf{acSLAT}$, $X \mapsto (\varphi X, \leq)$ is a covariant functor, with $\varphi : \mathsf{Set} \to \mathsf{Set}$ as the underlying set functor, and $\eta : \mathrm{id} \to \varphi$ is a natural transformation.

[3] A *monad* (φ, η, μ) over a category C consists of a covariant functor $\varphi : \mathsf{C} \to \mathsf{C}$, together with natural transformations $\eta : id \to \varphi$ and $\mu : \varphi \circ \varphi \to \varphi$ fulfilling the conditions $\mu \circ \varphi\mu = \mu \circ \mu\varphi$ and $\mu \circ \varphi\eta = \mu \circ \eta\varphi = id_\varphi$.

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making $(PX, \leq)$ a partially ordered set. The unit $\eta : X \to PX$ is given by $\eta(x) = \{x\}$ and the multiplication $\mu : PPX \to PX$ by $\mu(\mathcal{B}) = \cup\mathcal{B}$.

In this paper we will show that partially ordered monads contain sufficient structure for modelling rough sets [10] in a generalized setting with set functors. Even for the ordinary relations, the adaptations through partially ordered monads open up avenues towards an understanding of rough sets in a basic many-valued logic [5] setting. However, the motivation of this paper goes far beyond ordinary relations, and indeed we show how various set functors extendable to partially ordered monads establish the notion of rough monads.

2 Ordinary Relations and Rough Sets

Let R be a relation on X, i.e. $R \subseteq X \times X$. We represent the relation as a mapping $\rho_X : X \to PX$, where $\rho_X(x) = \{y \in X | xRy\}$. The corresponding inverse relation R^{-1} is represented as $\rho_X^{-1}(x) = \{y \in X | xR^{-1}y\}$.

Based on indistinguishable relations, *rough sets* are introduced by defining the upper and lower approximation of sets. These approximations represent uncertain or imprecise knowledge. To be more formal, given a subset A of X, the lower approximation of A correspond to the objects that surely (with respect to an indistinguishable relation) are in A.

The lower approximation of A is obtained by

$$A^{\downarrow} = \{x \in X | \rho(x) \subseteq A\}$$

and the upper approximation by

$$A^{\uparrow} = \{x \in X | \rho(x) \cap A \neq \emptyset\}.$$

In what follows we will assume that the underlying almost complete semilattice has finite infima, i.e. is a join complete lattice.

Considering P as the functor in its corresponding partially ordered monad we then immediately have

Proposition 1. *The upper and lower approximations of a subset A of X are given by*

$$A^{\uparrow} = \bigvee_{\rho_X(x) \wedge A > 0} \eta_X(x) = \mu_X \circ P\rho_X^{-1}(A)$$

and

$$A^{\downarrow} = \bigvee_{\rho_X(x) \leq A} \eta_X(x),$$

respectively.

Proof. For the upper approximation,

$$\mu_X \circ P\rho_X^{-1}(A) = \bigvee P\rho_X^{-1}(A)$$

$$= \bigvee \{\rho_X^{-1}(y) \,|\, y \in A\}$$
$$= \{x \in X \,|\, xRy,\, y \in A\}$$
$$= \bigvee_{\rho_X(x) \wedge A > 0} \eta_X(x) = A^{\uparrow}.$$

And for the lower approximation, since $\eta_X(x) = \{x\}$, we immediately obtain:

$$A^{\downarrow} = \{x \in X | \rho(x) \subseteq A\}$$
$$= \bigvee_{\rho_X(x) \leq A} \eta_X(x).$$

The corresponding R-weakened and R-substantiated sets of a subset A of X are given by

$$A^{\Downarrow} = \{x \in X | \rho^{-1}(x) \subseteq A\}$$

and

$$A^{\Uparrow} = \{x \in X | \rho_X^{-1}(x) \cap A \neq \emptyset\}.$$

Proposition 2. *The R-weakened and R-substantiated sets of a subset A of X are given by*

$$A^{\Uparrow} = \mu_X \circ P\rho_X(A)$$

and

$$A^{\Downarrow} = \bigvee_{\rho_X^{-1}(x) \leq A} \eta_X(x),$$

respectively.

Proof. Similarly as Proposition 1.

The upper and lower approximations, as well as the R-weakened and R-substantiated sets, can be viewed as $\uparrow_X, \downarrow_X, \Uparrow_X, \Downarrow_X \colon PX \to PX$ with $\uparrow_X (A) = A^{\uparrow}$, $\downarrow_X (A) = A^{\downarrow}$, $\Uparrow_X (A) = A^{\Uparrow}$ and $\Downarrow_X (A) = A^{\Downarrow}$.

3 Inverse Relations

Inverse relations in the ordinary case means to mirror pairs around the diagonal. The following propositions relate inverses to the multiplication of the corresponding monads.

Proposition 3. *In the case of P,*

$$\bigvee_{\rho_X(x) \wedge A > 0} \eta_X(x) = \mu_X \circ P\rho_X^{-1}(A)$$

if and only if

$$\rho_X^{-1}(x) = \bigcup_{\eta_X(x) \leq \rho_X(y)} \eta_X(y).$$

Proof. To see $\Longrightarrow$, let us consider the one element set, $A = \{x\}$. Renaming the variables, by hypothesis we have that $\rho_X(y) \wedge A > 0$, e.g. $x \in \rho_X(y)$, therefore,

$$\bigvee_{\rho_X(y)\wedge A>0} \eta_X(y) = \bigcup_{x\in\rho_X(y)} \eta_X(y) = \bigcup_{\eta_X(x)\leq\rho_X(y)} \eta_X(y).$$

On the other hand, since A contains only one element, $\mu_X \circ P\rho_X^{-1}(A) = \rho_X^{-1}(x)$. The other implication, $\Longleftarrow$, holds by Proposition 1.

The many-valued extension of P is as follows. Let L be a completely distributive lattice. For $L = \{0,1\}$ we write $L = 2$. The functor L_{id} is obtained by $L_{id}X = L^X$, i.e. the set of mappings $A : X \to L$. These mappings are usually called *fuzzy sets* (over L). The partial order $\leq$ on $L_{id}X$ is given pointwise. Morphism $f\colon X \to Y$ in `Set` are mapped according to

$$L_{id}f(A)(y) = \bigvee_{f(x)=y} A(x).$$

Finally $\eta_X : X \to L_{id}X$ is given by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x' \leq x \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_X : L_{id}X \circ L_{id}X \to L_{id}X$ by

$$\mu_X(\mathcal{M})(x) = \bigvee_{A\in L_{id}X} A(x) \wedge \mathcal{M}(A).$$

Concerning inverse relations, in the case of $\varphi = L_{id}$ we would accordingly define $\rho_X^{-1}(x)(x') = \rho_X(x')(x)$.

Proposition 4. *[1]* $\mathbf{L}_{id} = (L_{id}, \leq, \eta, \mu)$ *is a partially ordered monad.*

Note that $\mathbf{2}_{id}$ is the usual partially ordered power set monad $(P, \leq, \eta, \mu)$.

Proposition 5. *In the case of* L_{id}*,*

$$\mu_X \circ L_{id}\rho_X^{-1}(A)(x) = \bigvee_{x'\in X} (\rho_X(x) \wedge A)(x').$$

Proof. We have

$$\begin{aligned}
\mu_X \circ L_{id}\rho_X^{-1}(A)(x) &= \bigvee_{B\in L_{id}X} B(x) \wedge L_{id}\rho_X^{-1}(A)(B) \\
&= \bigvee_{B\in L_{id}X} B(x) \wedge \Big(\bigvee_{\rho_X^{-1}(x')=B} A(x') \Big) \\
&= \bigvee_{B\in L_{id}X} \bigvee_{\rho_X^{-1}(x')=B} B(x) \wedge A(x') \\
&= \bigvee_{x'\in X} \rho_X^{-1}(x')(x) \wedge A(x') \\
&= \bigvee_{x'\in X} (\rho_X(x) \wedge A)(x').
\end{aligned}$$

The generalization from the ordinary power set monad to involving a wide range of set functors and their corresponding partially ordered monads requires an appropriate management of relational inverses and complement. Obviously, for more complicated set functors, the corresponding relational views no longer rest upon 'mirroring over the diagonal'. The general representation of inverses is still an open question and for the the purpose of this paper we specify inverses *in casu*. Inverses and complements in the end need to build upon logic operators in particular concerning negation as derived from implication operators used within basic many-valued logic [5].

4 Monadic Relations and Rough Monads

Let $\Phi = (\varphi, \leq, \eta, \mu)$ be a partially ordered monad. We say that $\rho_X : X \to \varphi X$ is a *Φ-relation* on X, and by $\rho_X^{-1} : X \to \varphi X$ we denote its *inverse*. The inverse must be specified for the given set functor φ.

For any $f \colon X \to \varphi X$, the following condition is required:

$$\varphi f(\bigvee_i a_i) = \bigvee_i \varphi f(a_i)$$

This condition is valid both for P as well as for L_{id}.

Remark 1. Let ρ_X and ρ_Y be relations on X and Y, respectively. Then the mapping $f : X \to Y$ is a congruence, i.e. $x' \in \rho_X(x)$ implies $f(x') \in \rho_Y(f(x))$, if and only if $Pf \circ \rho_X \leq \rho_Y \circ f$. Thus, congruence is related to kind of weak naturality.

Let $\rho_X : X \to \varphi X$ be a Φ-relation and let $a \in \varphi X$. The Φ-ρ-upper and Φ-ρ-lower approximations, and further the Φ-ρ-weakened and Φ-ρ-substantiated sets, now define rough monads using the following monadic instrumentation:

$$\begin{aligned} \Uparrow_X (a) &= \mu_X \circ \varphi \rho_X(a) \\ \downarrow_X (a) &= \bigvee_{\rho_X(x) \leq a} \eta_X(x) \\ \uparrow_X (a) &= \mu_X \circ \varphi \rho_X^{-1}(a) \\ \Downarrow_X (a) &= \bigvee_{\rho_X^{-1}(x) \leq a} \eta_X(x) \end{aligned}$$

Proposition 6. *If $a \leq b$, then $\Uparrow_X a \leq \Uparrow_X b$, $\downarrow_X a \leq \downarrow_X b$, $\uparrow_X a \leq \uparrow_X b$, $\Downarrow_X a \leq \Downarrow_X b$.*

Proof. The proof is straighforward as e.g.

$$\downarrow_X (a) = \bigvee_{\rho_X(x) \leq a} \eta_X(x) \leq \bigvee_{\rho_X(x) \leq b} \eta_X(x) = \downarrow_X (b)$$

and

$$\uparrow_X (a) = \mu_X \circ \varphi \rho_X^{-1}(a) \leq \mu_X \circ \varphi \rho_X^{-1}(b) = \uparrow_X (b).$$

Definition 1. *$\rho_X : X \to \varphi X$ is* reflexive *if $\eta_X \leq \rho_X$, and* symmetric *if $\rho = \rho^{-1}$.*

Proposition 7. *If ρ is reflexive, $a \leq \Uparrow_X (a)$.*

Proof. By one of the monads conditions wrt multiplication and the fact that for all mappings $f, g : Y \to \varphi X$, $f \leq g$ implies $\mu_X \circ \varphi f \leq \mu_X \circ \varphi g$, we have:

$$\begin{aligned} a &= id_\varphi(a) \\ &= \mu_X \circ \varphi\eta_X(a) \\ &\leq \mu_X \circ \varphi\rho_X(a) \\ &= \Uparrow_X (a) \end{aligned}$$

Proposition 8. *ρ is reflexive iff $\downarrow_X (a) \leq a$.*

Proof. If ρ is reflexive, then

$$\begin{aligned} \downarrow_X (a) &= \bigvee_{\rho_X(x) \leq a} \eta_X(x) \\ &\leq \bigvee_{\rho_X(x) \leq a} \rho_X(x) \\ &\leq a \end{aligned}$$

and, conversely, if $\downarrow_X (a) \leq a$, then we have

$$\begin{aligned} \eta_X(x) &\leq \bigvee_{\rho_X(x') \leq \rho_X(x)} \eta_X(x') \\ &= \downarrow_X (\rho_X(x)) \\ &\leq \rho_X(x). \end{aligned}$$

Proposition 9. *ρ_X^{-1} is reflexive iff $a \leq \uparrow_X (a)$.*

Proof. If ρ_X^{-1} is reflexive, then $\eta_X \leq \rho_X^{-1}$. Therefore, by using monads conditions and properties of the underlying lattice, we obtain

$$a = \mu_X \circ \varphi\eta_X(a) \leq \mu_X \circ \varphi\rho_X^{-1}(a) = \uparrow_X (a).$$

Conversely, we have that $\eta_X(x) \leq \uparrow_X (\eta_X(x))$. Further, by naturality of η_X with respect to ρ_X^{-1}, and by using one of the monad conditions, we have

$$\mu_X \circ \varphi\rho_X^{-1}(\eta_X(x)) = \mu_X \circ \eta_{\varphi X}(\rho_X^{-1}(x)) = \rho_X^{-1}(x).$$

Therefore,

$$\eta_X(x) \leq \uparrow_X (\eta_X(x)) = \mu_X \circ \varphi\rho_X^{-1}(\eta_X(x)) = \rho_X^{-1}(x)$$

which yields the reflexivity of ρ_X^{-1}.

Note that in the case of relations for P and L_{id}, if the relations are reflexive, so are their inverses.

Proposition 10. *If ρ is symmetric, then $\uparrow_X (\downarrow_X (a)) \leq a$.*

Proof. We have

$$\begin{aligned}\uparrow_X (\downarrow_X (a)) &= \mu_X \circ \varphi\rho_X^{-1}(\downarrow_X (a)) \\ &= \mu_X \circ \varphi\rho_X^{-1}(\bigvee_{\rho_X(x)\leq a} \eta_X(x)) \\ &= \bigvee_{\rho_X(x)\leq a} \mu_X \circ \varphi\rho_X^{-1}(\eta_X(x)) \\ &= \bigvee_{\rho_X(x)\leq a} \rho_X^{-1}(x) \\ &= \bigvee_{\rho_X(x)\leq a} \rho_X(x) \\ &\leq a.\end{aligned}$$

In the particular case of $a = \eta_X(x)$ we have $a \leq\downarrow_X \circ \uparrow_X (a)$. Indeed, by naturality of η_X, and symmetry, we have

$$\rho_X(x) = \mu_X \circ \varphi\rho_X^{-1}(a).$$

Therefore,

$$a = \eta_X(x) \leq \bigvee_{\rho_X(x')\leq\mu_X\circ\varphi\rho_X^{-1}(a)} \eta_X(x') =\downarrow_X (\uparrow_X (a)).$$

5 Future Work

Algebraic structures of rough sets [6] will be further investigated, both in direction towards topological notions as well as involving logical structures. For instance, relations to topological approaches based on modal-like operators [8] need to be better understood. Concerning algebras, it is important to note that the power set based rough monad, i.e. the ordinary rough sets, fulfill conditions of Boolean algebras where calculi e.g. on inverses are natural and well understood. Going beyond Boolean algebras means dropping complements and the recovery of the notion of complement needs to take other routes, such as those provided by implications in many-valued logic. Further, substructures of partially ordered monads are important for the provision of more examples. It is also interesting to observe how rough sets and their algebraic structures resemble operations on images as found with morphological analysis [11]. Images seen not just as matrices of pixels but, more general, as being placed on a canvas based on rather elaborate set functors which are far more complex than the ordinary power set functor.

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