Regularization for Super-Resolution Image Reconstruction

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Abstract. Super-resolution image reconstruction estimates a high-resolution image from a sequence of low-resolution, aliased images. The estimation is an inverse problem and is known to be ill-conditioned, in the sense that small errors in the observed images can cause large changes in the reconstruction. The paper discusses application of existing regularization techniques to super-resolution as an intelligent means of stabilizing the reconstruction process. Some most common approaches are reviewed and experimental results for iterative reconstruction are presented.

1 Introduction

Under sampling of images occurs in many imaging sensors. It results in aliased imagery and, consequently, in partial loss of scene information. Super-resolution image reconstruction refers to image processing technique that attempts to reconstruct high quality, high-resolution images by utilising incomplete and degraded scene information contained in a sequence of aliased, low-resolution images. Super-resolution makes use of the fact that due to relative motion between the sensor and the scene each low-resolution image carries slightly different information about the scene. By fusing the partial information from many frames it is possible to reconstruct an image of higher spatial resolution [1, 2].

The problem of image reconstruction from noisy, aliased or otherwise degraded imagery occurs in a wide variety of scientific and engineering areas including civilian and military applications. Examples of these applications include: medical imaging, computer vision, target detection and recognition, radar imaging as well as surveillance applications. Many of these applications may also involve a related technique of image restoration. This technique, in contrast to super-resolution, does not attempt to increase pixel resolution but produces improved image from a degraded image at the same resolution scale.

In this paper we consider the problem of reconstructing a single high-resolution image $\underline{X}$ from N number of low-resolution, observed images $\underline{Y}_k$ (k = 1….N) of the same scene. It is convenient to represent the images as vectors (as shown by an underscore) that are ordered column-wise lexicographically. Each observed image is the result of sampling, camera and atmosphere blur, motion effects, geometric

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warping and decimation performed on the ideal high-resolution real scene. It is usually assumed that this imaging process can be represented by a linear operator H_k :

$$\underline{Y}_K = H_K \underline{X} + \underline{E} \qquad \text{for } 1 \le k \le \text{N} . \tag{1}$$

where E is the additive noise present in any imaging system. The process of super-resolution is an inverse problem of estimating a high-resolution image from a sequence of observed, low-resolution images and it is now widely known to be intrinsically unstable or "ill-conditioned". The common feature of such ill-conditioned problems is that small variations in the observed images Y_k can cause (arbitrary) large changes in the reconstruction. This sensitivity of the reconstruction process on the input data errors may lead to the restoration errors that are practically unbounded.

The important part of super-resolution process is thus to modify the original problem in such a way that the solution is a meaningful and close approximation of the true scene but, at the same time, it is less sensitive to errors in the observed images. The procedure of achieving this goal and to stabilize the reconstruction process is known as *Regularization*. The field of regularization has grown extensively [3-6] since the seminal paper by Tikhonov [7, 8] in 1963.

This paper is aimed at giving an overview of some most common regularization techniques and parameter estimations and their significance and application to the problem of super-resolution image reconstruction. We also show reconstruction results from a test sequence of images to illustrate our regularization procedure based on iterative approach. The paper is organized as follows: Section 2 describes the differences between well-posed and ill-posed problems and how regularization solves the problem of ill-conditioning. A few of the most common approaches to regularization parameter estimation and techniques are reviewed in section 3 and 4 respectively. Finally, we give our concluding remarks in section 5 along with our results.

2 Regularization

There are many ways of explaining well-posed and ill-posed problems. For example,

$$H x = y . \tag{2}$$

where H is known. If y is determined by x, this is a well-posed problem whereas if x has to be determined from y, it's an inverse or ill-posed problem. The latter relates to super-resolution as explained in the introduction.

A problem whose solution exists, is unique and depends on the data continuously is known as a *well-posed* problem as defined by Hadamard [9] in 1902. On the contrary, the *ill-posed* problem is the one which disobeys the above given rules by Hadamard. In addition, as the solution of the ill-posed problem depends in a discontinuous fashion on the data, small errors such as round-off and measurement errors, may lead to a highly erroneous solution. The solution for an ill-posed problem is unstable and extremely sensitive to fluctuations in the data and other parameters. The classical example of an inverse and ill-posed problem is the Fredholm integral equation of the first kind, where, k is the kernel and g is the right-hand side.

$$\int_a^b k(t,s)\, f(s)\, dx = g(t) \ . \tag{3}$$

Both of these parameters are known, while *f* is the unknown function to be computed [10]. The theory on ill-posed problems is quite extensive and well developed. Engl [11] conducted a survey on a number of practical inverse problems in various applications such as computerised tomography, heat conduction, inverse scattering problems. Inverse problems are seen in various different fields, for example, medical imaging, astronomy, tomography, and many more. Ill-conditioning of inverse problems has always attracted a great deal of interest and research.

For many decades, it has been known that the best way to analyse a scientific problem is through its mathematical analysis. The most common analytical tool used in the case of ill-posed problems is *Singular Value Decomposition (SVD)*. This tool helps in diagnosing whether or not the singular values of a matrix are zero or decaying slowly towards zero (a number is so numerically small that due to the round off error it is rounded to zero). The SVD for a matrix *A* of dimension *m* by *n* where $m \geq n$, is given by:

$$A = \mathrm{U\, S\, V^T} \; => \; A = \sum_{i=1}^{n} u_i s_i v_i^{\,T} \ . \tag{4}$$

For the above decomposition, U ($u_1.....u_n$) is an *m* by *m* and $\mathrm{V^T}$ is the transpose of matrix V ($v_1.....v_n$) which is *n* by *n*. The matrix S is a diagonal matrix containing the non-negative singular values of *A* arranged in descending order. The matrix U and V are orthogonal and their columns are orthonormal. The columns u_i and v_i of U and V are known as the left and right singular vectors. Also, for certain applications, as the dimension of matrix *A* increases, the numerical value of the singular values in S gradually decreases to zero which causes more oscillations in the left and right singular vectors. The greater the number of singular values in S tending to zero, the more singular is matrix *A* making it more ill-conditioned. Thus, SVD gives a good approximation on the ill-conditioning of the system.

Another easier way of testing a system for ill-conditioning is by computing the *condition number* of the matrix. The condition number can be defined as a ratio of the maximum and minimum singular values of the matrix in consideration, in our case, *H* (2). A high condition number points to an ill-posed problem, whereas a low condition number points to a well-posed problem. If *H* is an *m* by *n* matrix:

$$condition(H) = \left| \frac{\sigma_{\max}(H)}{\sigma_{\min}(H)} \right|_{euclidean-norm} \ . \tag{5}$$

where, $\sigma_{\max}$ and $\sigma_{\min}$ represent the maximum and minimum singular values of matrix *H*. With ill-posed problems, the challenge is not of computing a solution, but computing a unique and stabilized solution. Thus, an ill-conditioned system requires an intelligent method of mathematical computation to generate a meaningful solution, rather than the usual computational methods.

Referring to (1), the minimum norm solution for the estimation of high-resolution image would be:

$$\min \left\| Y_k - H_k X \right\|_2^2 \quad \text{for } 1 \le k \le \mathrm{N} . \tag{6}$$

The matrix H is singular in nature and highly ill-conditioned. There is no uniqueness and stability in the solution for (6). Thus, to make the solution unique and stable, i.e. to make the above equation well-conditioned (as per Hadamard criteria), another term is added to (6) known as the *Regularization Term.* Most of the inverse problems (like super-resolution) are ill-posed and the solution is tremendously sensitive to the data. The solution can vary tremendously in an arbitrary manner with very small changes in the data. The solution to (6) would be highly sensitive and noise contaminated. The regularization term takes control of the ill-conditioned nature of the problem. The aim of this term is to make the solution more stable and less noise contaminated. The term also tries to converge the approximate solution as close as possible to the true solution. The modified version of (6) is:

$$\min \left\| Y_k - H_k X \right\|_2^2 + \lambda \left\| LX \right\|_2^2 \quad \text{for } 1 \le k \le \mathrm{N} . \tag{7}$$

In (7), the parameter $\lambda > 0$, is known as the regularization parameter and L is a regularization / stabilization matrix. In [12], the stabilization matrix is referred to as the regularization operator. The regularization operator if given by an identity matrix ($L = \mathrm{I}$), the regularization term is of *standard form* whereas when $L \neq \mathrm{I}$, the term is in the *general form*. When treating problems numerically, it is easier to use the standard form rather than the general form as only one matrix, H, needs to be handled. In practical applications, however, it is recommended that the general form of the regularization term should be used.

The regularization term aims at filtering out the noise that contaminates the image and also makes it smoother. The regularization term can also include *a priori* information of the true solution which facilitates the minimization process to converge as close as possible. The regularization parameter controls the measure of smoothness in the final solution of (7). It is critical to choose the regularization parameter best suited to the particular application in which it is involved. If the regularization parameter is too small, the regularization term will have no effect on the solution and the noise will not be filtered out, thus leaving the approximate solution far from converging with the true solution. On the other hand, if the parameter is too large, the regularization term will have a dominating effect on the solution making it too smooth and there is a risk of loosing important information from the solution. Hence, there needs to be a proper balance of smoothness and preservation of information when regularization is implemented.

There exist many techniques of regularization and parameter estimation in the literature. To discuss each of them is outside the scope of this paper. Thus, only those most commonly used will be discussed.

3 Estimating λ, the Regularization Parameter

Being the most critical part of regularization term, one has to carefully choose the appropriate technique based on their applications and expected results. The regularization parameter also depends on the properties of *Y, H, X* and noise (7). The parameter should balance the regularization and perturbation error in the computed solution. Over the years, many techniques have been proposed and discussed in relation to estimating the regularization parameter [12-14]. The techniques that will be discussed fall into two categories – one which require knowledge of error and the ones which do not require knowledge of error.

3.1 Method Which Require Error Knowledge – The Discrepancy Principle

In practical scenarios, considering (2) and (6), the right –hand side, *Y*, is never free from errors and contains various types of errors. Thus, *Y* can be written as $Y = Y_{true} + e$, where e is the errors and Y_{true} is the actual unperturbed right-hand side. Now, as per the discrepancy principle [15], the regularization parameter is chosen such that the residual norm of the regularized solution is equal to the norm of the errors.

$$\left\| Y - HX_{reg} \right\|^2 = \left\| e \right\|^2 \ . \tag{8}$$

If there is a rough estimate of the error norm, the discrepancy principle can be used to estimate a good regularization parameter. Unfortunately, in the practical world the knowledge about the error norm is not available and can be erroneous. Such data can lead to wrong estimations of the regularization parameter, thereby generating an unstable final solution.

3.2 Methods Which Do Not Require Error Knowledge – GCV and L-Curve

Generalized Cross-Validation (GCV)

GCV is one of the most popular methods used for estimating the regularization parameter [16]. It is based on the statistical cross-validation technique. In GCV, if a random element, Y_k, is left out of *Y*, then the estimated regularized solution should be able to predict the missing element, Y_k. The regularization parameter is chosen as the one which minimizes the prediction error and is independent of the orthogonal transformation of *Y* [17]. In this technique, no knowledge of the error norm is required. The GCV function is given as:

$$GCV = \frac{\left\| Y - HX_{reg} \right\|^2}{\tau^2} \ . \tag{9}$$

where, the numerator is the squared residual norm and the denominator is the squared effective number of degrees of freedom. For further details on this refer to chapter 7 from [18]. Although computation of regularization parameter using GCV technique works for many applications, it should also be noted that GCV may have a very flat minimum, making it difficult to locate numerically [19].

L-Curve Criterion

The L-curve criterion proposed in [20, 21] was inspired from graphical analysis discussed in [22]. The L-curve is a plot of term 2 in (7) $\| L X \|^2$ or $\| X \|^2$ versus term 1 of (7) $\| Y - H X \|^2$ (which is the corresponding residual norm). This curve, when plotted on a log-log scale, takes the shape which resembles the alphabet 'L' and hence the name, L-Curve (see Figure 1. for illustration). This is the most powerful graphical tool for analysis as it shows the relationship between both the terms 1 and 2. The 'corner' of L-curve is the optimum point of balance between both the errors (one caused by regularization and the other by errors in *Y*). The value at this point (corner of L-curve) is chosen as the optimal regularization parameter. This is the L-curve criterion. The curve is continuous when the regularization parameter is continuous, but in the case when the parameter is discrete the curve is plotted as a set of points.

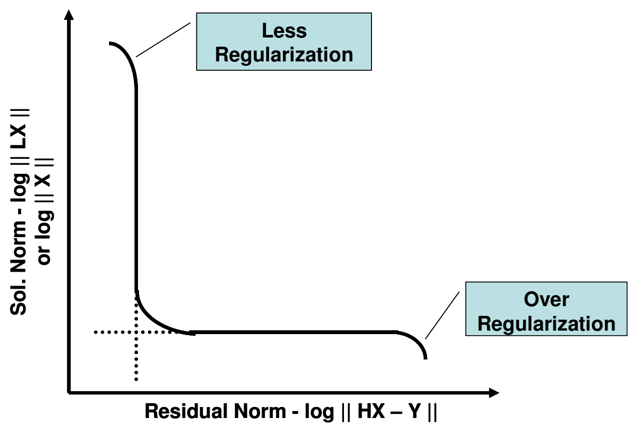

Fig. 1. A general graph of L-curve and its corner. The corner is the optimum point of balance between the regularization errors and errors in the right-hand side data, (*Y*). This corner can be taken as the regularization parameter.

4 Regularization Techniques

Regularization is an intelligent technique for computing a solution for an ill-posed problem. The main aim of this term is to make sure that the final solution is smooth and regularized with respect to the input data. It also makes sure that the final solution is less contaminated with errors and noise components. In the process of achieving this, the regularization term filters out the high-frequency components, thereby giving a smooth final solution. In the field of image restoration or super-resolution, a smooth approximate solution might not solve the purpose of being an appropriate solution. The high-frequency components filtered out by the regularization technique relate to the edges and discontinuities in the image. These components hold a significant value in image restoration. As seen in section 2, the singular values of the matrix H, are of critical significance. These singular values relate to the high-frequency components. Thus, if there are too many small singular values (which can decay to zero), then the information relating to these is lost and only the information related to the large values is recoverable. There are various techniques for computing a regularized solution for ill-posed problems. The scope of this paper is limited and hence only the most common of these will be discussed.

4.1 Tikhonov Regularization

Tikhonov regularization was first introduced in 1963[7, 8]. It is defined as:

$$X_{reg.} = \min \|Y - HX\|_2^2 + \lambda \|LX\|_2^2 \ . \tag{10}$$

where, $\lambda > 0$, is known as the regularization parameter and L is a regularization/stabilization matrix. The regularization matrix can be $L = \mathrm{I}$ or $L \neq \mathrm{I}$ where I is an identity matrix. It is recommended to consider the regularization matrix as unequal to the identity matrix (see [12]). It should be noted that since the regularization matrix can also contain *a priori* knowledge, greater care must be taken in its selection. The regularization parameter is also of great importance as it is a trade-off between the smoothness and the accuracy of the solution. The above equation (10) can be also written as:

$$(H^T H + \lambda L^T L) X_{approx} = H^T Y \ . \tag{11}$$

From (11), it is evident how the regularization term manages to regularize the solution. It is also evident how the proper or improper selection of λ and L can lead to a good or bad approximation. A high value of λ diverts the solution to be very smooth, suppressing the high-frequency components even though the system has been regularized. Although Tikhonov regularization seems to be a straight forward technique, it has a high-computational cost and requires a lot of storage space when used in large-scale problems. Thus, this technique is more suitable to small-scale problems as compared to large-scale problems.

4.2 Maximum Entropy Method

The Maximum Entropy technique [23] of regularization is often used in astronomical image reconstruction. This technique is also known to preserve point edges in the estimated image, which makes it promising in the field of astronomical image restoration. The maximum entropy regularization term [18] is given as:

$$S(X) = \lambda^2 \sum_{i=1}^{n} x_i \log(w_i x_i) \ . \tag{12}$$

where, x_i are the positive elements of vector X and w_i are weights *(* w_1 ..... w_n*)*. The above given function is negative of the entropy function Therefore, (10) is given as:

$$\left\| Y - HX \right\|_2^2 + S(X) \ . \tag{13}$$

The estimated solution from maximum entropy regularization is quite consistent as it is not related to the missing information of the right-hand side to a great extent. Although solving (12) and (13) is computationally intensive, there exist many iterative algorithms which are significantly less computationally intensive.

4.3 Conjugate Gradients (Iterative Regularization)

The conjugate gradient is one of the most commonly used numerical algorithms for symmetric positive definite systems. It is also known as the oldest and best known non-stationary method. The conjugate gradient can be computed as a direct method much like Tikhonov and maximum entropy but it proves to be much more efficient if it is used as an iterative method. Direct methods fail to perform when it comes to large-scale problems or huge sparse matrices, where only iterative technique comes to the rescue. The iterative conjugate gradient method can successfully compute solutions for large scale problems. Since the iterative method utilizes the property of matrix-vector multiplications between huge sparse matrices and vectors, computational time decreases and storage requirements for such matrices and vectors decreases tremendously. These advantages make iterative conjugate gradient regularization technique more favorable when compared with others. The iterative method generates successive approximations of the solution and their residuals. The conjugate gradient for a set of unregularized normal equations, $H\,X = Y$, is given as:

$$H^T HX = H^T Y \ . \tag{14}$$

It is seen that for (14), the low-frequency components of the estimated solution converge faster than the high-frequency components [12]. The iterative conjugate gradient technique generates X_K estimated solutions and calculates the residuals for each K. The number of iterations assigned is denoted by K. In this iterative technique of generating the regularized solution, K acts as the regularization parameter. It is very important to generate iterations up to an optimal number because the iterative solution can sometimes converge faster and if K is greater than K-optimal, the estimated solution might diverge from the true solution. The equation for the K^{th} iterative CG approach is given by:

$$X_{(K)} = X_{(K-1)} + \alpha_{(K)} p_{(K)} \ . \tag{15}$$

where, $X_{(K)}$ is the K^{th} iterative approximation of X. The conjugate gradient least squares is given by:

$$X_{reg.} = \min \left\| Y - HX \right\|_2^2 \ . \tag{16}$$

Equation (16) is similar to (10) – Tikhonov regularization technique only in (16), $\lambda = 0$, making the regularization term go to zero. Hence, in this technique, like in CG, the number of iterations, K, acts as the regularization parameter.

Computational cost and storage requirements are certainly the prime factors in choosing a particular regularization technique for a particular application. Iterative methods for estimating a regularized solution of an ill-posed problem are fast gaining popularity due to their low computational cost and low storage requirements as compared to direct methods of regularization.

5 Results and Concluding Remarks

It is a fact that super-resolution image reconstruction is an inverse problem which is highly ill-conditioned. If such a system (9) is solved, the image constructed would be highly sensitive and unstable. Thus, the term of regularization is introduced to make the final approximated solution less sensitive and more stable. The choice of technique used for regularization and estimation of the regularization parameter depends upon the application field and the expected output. In the field of super-resolution, the images are of band-limited nature, and hence, to restore the image, the Nyquist criterion needs to be fulfilled. The current regularization techniques are concentrated towards smoothing the final approximated or regularized image. The regularization matrix is taken in such a fashion that it blurs the regularized image by cutting off a major part of the high-frequency component.

We conclude with some experimental results of image reconstruction from simulated imagery. We have implemented an iterative technique for super-resolution that inherently stabilizes the reconstruction process without excessive blurring. In this approach the K+1 approximation to the high-resolution image is given by:

$$X_{K+1} = X_K + R_0(Y - H \cdot X_K), \rightarrow K = 0,1,2..... \qquad (17)$$

where, H is the imaging operator and Y is the set of low-resolution images with X_0 being the first approximation input to the iterations algorithm. R_0 in (17) is an approximate reconstruction operator. In our approach the essential part of this operator is sub-pixel interpolation. In our initial experimentation we used truncated Sinc function for interpolation. From our experiments and figure 2, it can be seen that R_0 acts as a regularizing routine, cutting off the high-frequency components and noise and leaving a smooth approximated image. Truncated Sinc acts as an implicit regularization on the image. The extent of this regularization on the image can be controlled by the extent of the Sinc function. The Sinc function is also known as an ideal reconstruction filter, which in the frequency space, has a rectangular function. Although a true Sinc function cannot be used for reconstruction purposes, the regularization matrix can be chosen such that in frequency space it is like a rectangular function. Even if the rectangular function cuts off the high-frequency components, it doesn't blur the image and tries to preserve as much of the high-frequency components as possible from the band-limited image.

This paper signifies the role of regularization in the field of super-resolution and also reviews the most common regularization techniques. It is also recommended to use iterative regularization techniques rather than direct techniques for applications where computational cost and storage requirements are constraints.

(a) (b)

Fig. 2. (a) – One of the 10 low-resolution images [42 x 42] simulated on the original image [512 x 512] using a sampling ratio of 12. (b) – The 20^{th} final iterative super-resolution image generated [504 x 504] by our algorithm using truncated Sinc as the interpolation technique.

The plan is to take this research to the next step where we would implement the regularization technique with the new idea of considering the regularization matrix such that in frequency space, its response is more like the rectangular function. Such a technique will help to preserve the edges, rather than blurring it, thereby keeping the significant information intact. The problem of super-resolution is rewritten to combine the linear operator H which represents the imaging process along with the regularization term, such that (10) is given as:

$$\min \left\| \begin{bmatrix} H \\ \lambda L \end{bmatrix} X - \begin{bmatrix} Y \\ 0 \end{bmatrix} \right\|_2^2 \quad \Rightarrow \quad \min_X = \| \hat{H} X - \hat{Y} \|_2^2 \ . \tag{18}$$

The above equation is a least squares problem and can be solved using an iterative approach so as to tackle huge and sparse matrices.

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