Mappings, Maps and Tables: Towards Formal Semantics for Associations in UML2*

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Abstract. In fact, UML2 offers two related yet different definitions of associations. One is implicit in several *Description* and *Semantics* sections of the specification and belongs to the UML folklore. It simply says that an association is a set of links. The other – official and formal – definition is explicitly fixed by the UML metamodel and shows that there is much more to associations than just being sets of links. Particularly, association ends can be owned by either participating classes or by the very association (with a striking difference between binary and multiary associations), be navigable or not, and have some constraints on combining ownership and navigability.

The paper presents a formal framework, based on sets and mappings, where all notions involved in the both definitions can be accurately explained and formally explicated. Our formal definitions allow us to reconcile the two views of associations, unify ownership for binary and multiary associations and, finally, detect a few flaws in the association part of the UML2 metamodel.

1 Introduction

Associations are amongst the most important modeling constructs. A clear and accurate formal semantics for them would provide a guidance for a convenient and precise syntax, and greatly facilitate their adequate usage. Moreover, in the context of model-driven software development, semantics must be crystal clear and syntax has to specify it in an unambiguous and suggestive way. An additional demand for clarifying the meaning of associations comes from UML2 metamodel that is based on binary associations.

Unfortunately, the UML2 specification [8], further referred to as the Spec, does not satisfy these requirements. While complaints about informality of semantics are common for many parts of UML, for associations even their (abstract) syntax seems to be complicated and obscure in some parts. For example, the meaning of the (meta)associations *ownedEnd* and *navigableOwnedEnd* of the Association (meta)class in the metamodel is not entirely clear. More accurately, it is not

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easy to comprehend their meaning in a way equally suitable for both binary and multiary (arity $n \geq 3$) associations. The infamous multiplicity problem for multiary associations is another point where the cases of binary and multiary associations are qualitatively different in UML (see, e.g., [3]). Even the very definition of association, in fact, bifurcates for the binary and multiary cases, though this fact is hidden in the excessively fragmented presentation of the UML metamodel via packages. A sign of distortion of the association part of the metamodel is that many modeling tools do not implement multiary associations (not to mention qualified associations - a rarity among the implemented modeling elements).

We will show in the paper that all these problems grow from the same root, and can be readily fixed as soon as the root problem is fixed. The point is that UML mixes up three conceptually and technically different sides of the association construct. In the most popular view, an association is just a collection of tuples or a table. For example, a ternary association between classes X_1, X_2, X_3 is a three-column table $T = (R, p_1, p_2, p_3)$ with R the set of rows or tuples of the association and p_1, p_2, p_3 the columns, that is, mappings $p_i\colon R \to X_i, i = 1, 2, 3$, called association ends. This is a purely extensional view and the roles of the classes are entirely symmetric.

A more navigation-oriented view of the same association is to consider it as a triple of binary mappings

$$f_1\colon X_2 \times X_3 \to X_1,\ f_2\colon X_1 \times X_3 \to X_2,\ \text{and}\ f_3\colon X_1 \times X_2 \to X_3 \qquad (1)$$

which we call *structural* (Table 1 on p.240 presents it in visual form). Note that each of the structural mappings is asymmetric and has a designated target, or *goal*, class. Yet the set of three mappings $M_S = (f_1, f_2, f_3)$ retains the symmetry of the tabular view. We will call such sets *structural maps* of associations.

When we think about implementation of structural maps, we need to decide, first of all, which of the possible navigation directions should be most effective and which of the classes will implement it. For example, the mapping f_1 can be implemented as either a retrieval operation in class X_2 with a formal parameter of type X_3, $f_{12}(x{:}X_3)\colon X_2 \to X_1$,[1] or as a retrieval operation in class X_3 with a formal parameter of type X_2, $f_{13}(x{:}X_2)\colon X_3 \to X_1$. We will call such mappings *operational* or *qualified*, since UML calls formal parameters *qualifiers*. Thus, the same association can be viewed as a six-tuple M_Q of qualified mappings f_{ij} (see Table 1 where only three of them shown). Note that each of the qualified mappings brings more asymmetry/navigational details to its structural counterpart yet their full set M_Q retains the symmetry of the entire association; we will call such sets *operational* or *qualified maps*.

Thus, in general an association is a triple $A = (T, M_S, M_Q)$ of mutually derivable components, with T, M_S and M_Q also consisting of multiple member mappings. Unfortunately, for specifying this rich instrumentary of extensional and navigational objects, the UML metamodel offers just one concept of the

[1] Which might be written as $f_{12}\colon X_2 \to [X_3 \to X_1]$ in the functional programming style.

association *memberEnd*. For example, a ternary association consists of the total of twelve mappings while the UML metamodel states only the existence of its three ends. Not surprisingly, that in different parts of the Spec the same notion of *memberEnd* is interpreted as either a projection mapping (column), or a structural mapping, or a qualified mapping (operation). Inevitably, it leads to ambiguities and misconceptions, only part of which was mentioned above.[2]

In the paper we build a formal framework, where the notions outlined above together with their relationships can be accurately defined and analyzed. In a sense, we disassemble the rich intuition of the association construct into elementary building blocks and then join them together in various ways to model different views of associations. Particularly, if association is a triple $A = (T, M_S, M_Q)$ as above, we can consider the pair $A_S = (T, M_S)$ as its *structural view* and the pair $A_O = (T, M_Q)$ as its *operational view*. The metamodel in Fig. 3 on p.243 presents our building blocks and their relationships in a concise way. It shows a few remarkable symmetries between the components and views of associations, which is interesting to discuss (see Section 4.3). On the other hand, it forms a useful frame of reference for analyzing the UML metamodel (Section 4.4).

Formalities as such can be boring or interesting to play with. When they are intended to model engineering artifacts, the first and crucial requirements to them is to be an adequate and careful formalization of the intuitions behind the artifacts to be modeled. We have paid a close attention to deducing our formalization from the Spec rather than from our own perception of what the association should be. To achieve this goal, we have read the Spec as carefully as possible, and discussed possible interpretations with the experts [10, 7]. Sections 2 and 3 present the results together with an outline of some preliminary framework of main constructs. Section 4 presents an accurate formal model and sets the stage for our discussion of what is association in UML2; the culmination is in Sections 4.3 and 4.4.

Remark: What is *not* in the paper. Semantics for the concepts of association/relationship and particularly, of aggregation and role is a well-known research issue that can be traced back to the pioneering works on data semantics by Abrial, Brodie, Chen, Mylopoulos, Tsichritzis and Lochovsky in seventies-early eighties. Since then a vast body of work on the subject was done and reported in the literature, see [5] for an early survey. Certainly, UML's concept of association is built on top of this work, and it might be an interesting research issue to study the evolution of ideas and their realization in the standard (see [2] for some results). Moreover, we believe that a real understanding of such a software phenomenon as UML does need evolutionary studies, particularly, for association and related concepts, and for many other parts of UML as well. However, such a discussion would go far beyond our goals in the paper. The latter are purely technical: take the standard as the only source of information about the association construct and provide an accurate formal semantics for it.

[2] Even the much more formally precise OCL confuses operational and projection mappings when it borrows UML's notation (abstract syntax) for association classes.

2 What Is a Property? The *Structural* View of Associations

According to UML metamodel ([8, Fig.7.12], see our Fig. 1) an association A between classifiers $X_1 ... X_n$, $n \geq 2$, is an n-tuple of properties $(f_1, ..., f_n)$ called A's *memberEnds* or just *ends*.

Each of the properties has its *type* [8, Figures 7.5 and 7.10], and explanations in Sect. 7.3.3 and 7.3.44 allow us to set the correspondence $f_i.type = X_i$ for all $i = 1..n$. The main question is *what is the semantic meaning of* property *in this definition?* The Spec says [8, Sect.7.3.44, p.121]:

> when instantiated, a property represents a value or collection of values associated with an instance of one (or, in the case of a ternary or higher-order association, more than one) type. This set of classifiers is called the *context* for the property; in the case of an attribute the context is the owning classifier, and in the case of an association end the context is the set of types at the other end or ends of the association.[3]

A natural way to interpret this definition is to consider a property in general as a mapping from some source set called the *context* (and whose elements play the role of instances "owning" the property), to a target set called the *type* of the property (whose elements play the role of values that the property takes). In particular, if the properties in question are the ends of some association, then the quote above says that each f_i is a mapping

$$f_i \colon X_{j_1} \times ... \times X_{j_{n-1}} \twoheadrightarrow X_i, i \notin \{j_1 ... j_{n-1}\} \subset \{1...n\}, \tag{2}$$

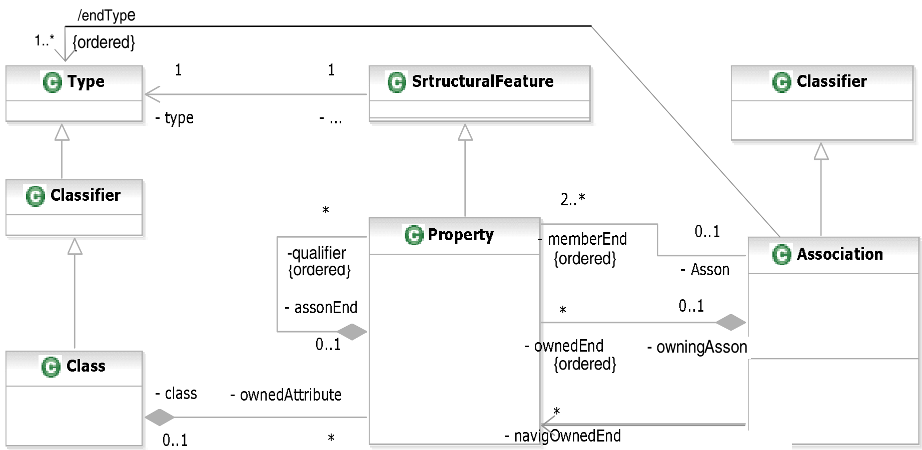
where the Cartesian product is the context, and the double-arrow head means that the actual target of the mapping is the set $\mathsf{coll}_{f_i}(X_i)$ of collections of specified (with f_i) type (sets, bags or lists) built from elements of X_i. A special case, when the value is a single element of the target class, will be denoted by the single-arrow head, and such mappings will be called *functional* or *functions*.

The left column of Table 1 on p.240 shows examples of mappings of this form for association arities $n = 2$ and $n = 3$. The term *multiary*, will be used generically to refer to the cases $n \geq 3$. Thus, an n-ary association is an n-element set of $(n-1)$-ary mappings called Properties. This definition still lacks a crucial condition. Namely, we need to require that all mappings $f_1, .., f_n$ are just different parts of the same association, or, as we will say, are *mutually inverse*, meaning that they all are mutually derivable by inverting/ permuting sources and targets (this condition is well known for the binary case).

Formally, this can be captured as follows. Given an n-ary mapping $f \colon X_1 \times .. \times X_n \twoheadrightarrow Y$, its *extension* $\mathsf{ext}(f)$ is the collection of tuples

$$\text{((extension))} \quad [(x_1, \ldots, x_n, y) : x_1 \in X_1, ..., x_n \in X_n, y \in f(x_1 ... x_n) \in \mathsf{coll}_f(Y)] \, ,$$

[3] In this piece, the terms "type" and "classifier" are used interchangeably and, hopefully, can be considered synonyms here.



Constraints for Association context in OCL
(to shorten expressions we write end for memberEnd):

(2) self.end->includesAll(self.ownedEnd) ->includesAll(navigOwnedEnd)

(3) def: self.endType = self.end->collect(type)

(4) if self.end->size() > 2 then self.ownedEnd = self.end[a]

[a] this is the Constraint 5 in [8, p.37],

Fig. 1. A piece of UML metamodel extracted from [8, Fig. 7.12] with additions from [8, Fig. 7.5, 7.10, 7.17]

which is a bag if f is bag-valued.[4] The most natural way of presenting such a collection is to store it in a table. In fact, we have a mapping $\mathsf{ext}\colon \mathsf{Mapping} \to \mathsf{Table}$ sending any n-ary mapping to a (n+1)-column table recording its extension.

Now we can formulate the condition in the following way.

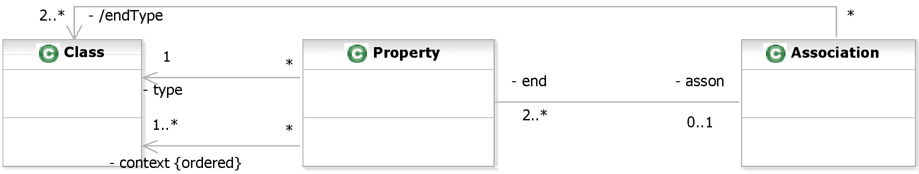
2.1 Definition: Let $\mathbf{X} = (X_1 ... X_n)$ be a family of classes.
(i) Any $(n-1)$-ary mapping of the form (2) is called a *structural mapping* over $\mathbf{X}$. Its source tuple of classes is called the *context*, and the target class the *type* of the mapping.
(ii) Two or more structural mappings $f_1 ... f_k$ over $\mathbf{X}$ are called *mutually inverse* if they have the same extension (up to renaming of the tables' columns)

$$\mathsf{ext}(f_1) = \mathsf{ext}(f_2) = ... = \mathsf{ext}(f_k). \qquad \text{((inverse))}$$

(iii) An n-element set $M_S = \{f_1 ... f_n\}$ of mutually inverse structural mappings over $\mathbf{X}$ is called a *structural map* over $\mathbf{X}$. In other words, a structural map is a maximal set of mutually-inverse structural mappings.

Thus, the Spec defines associations as nothing but structural maps.

[4] If f is list-valued, we can either disregard the ordering information by considering the underlying bag, or consider the extensional set to be partially-ordered.



Constraints for Association:

(6) def: self.endType = self.end->collect(type)

(7) self1≠ self2 implies disjoint(self1.end, self2.end)=true

(8) self.end satisfies the constraint (inverse) in Definition 2.1(ii) [a]

Constraints for Property:

(9) self.asson.endType->includesAll(self.context)

(10) self.context->size() +1 = self.asson.end->size() [b]

[a] constraints (7) and (8) are missed in the Spec
[b] constraints (9) and (10) cannot be declared in the Spec because the meta-association *context* is not there

Fig. 2. Metamodel for the *structural* view of associations

2.2 Definition: Structural view of association. An n-ary association, *structurally*, is an n-element set of mutually inverse $(n-1)$-ary mappings (called properties in UML).

Precise details and terminology associated with this definition are presented in Fig. 2. This (formal) metamodel accurately describes the corresponding part of the Spec, and it is instructive to compare it with the UML metamodel in Fig. 1 (disregarding there, for a while, the ownership aspect).

2.3 UML metamodel of associations in the light of formalization, I.

We note that the Spec misses two important constraints on associations: disjointness, (7), and being inverse, (8), in Fig. 2 (though, of course, implicitly they are assumed). Note also that our formal metamodel does not require the set of ends to be ordered. Indeed, ends are analogous to labels in labeled records: ordering is needed when there are no labels for record fields (and means, in fact, using natural numbers as labels). Thus, ordering of meta-association *memberEnd* required in the UML metamodel is redundant.

Finally, the most serious (and even striking) distinction is that the meta-association *context* is absent in the UML metamodel. As we have seen, the Spec does talk about this fundamental component of the association constructs, yet formally it is not entered into the metamodel. Is it hidden or lost in the long package merge chains in which the UML metamodel is separated? Note that even if the (meta)association *context* can be derived from other parts of the metamodel, its explicit presence in Figure 7.12 of the Spec, the main part of the UML association metamodel, is essential. Indeed, without this association we cannot formulate important structural constraints (9,10) in Fig. 2 and, which maybe even more important, without *context* the understandability of the metamodel is essentially lessened.

3 A Battle of Ownerships: The *Operational* View of Associations

In this section we consider that part of the UML association metamodel, which specifies ownership relations between Classes, Properties and Associations. The Spec considers two specific subsets of the set $A.memberEnd = \{f_1..f_m\}$ of association ends: the set of ends *owned* by the association, $A.ownedEnd \subseteq \{f_1..f_m\}$, and the set of *navigable owned ends*, $A.navOwnedEnd \subseteq A.ownedEnd$. Unfortunately, there is no direct explanation of the meaning of these two notions and we need to extract it from semi-formal considerations in Sect. 7.3.3 and 7.3.44.

Since for multiary association (when $n \geq 3$), the notions of *memberEnd* and *ownedEnd* coincide due to the constraint (4) in Fig. 1, we have to consider binary associations to understand the difference.

It appears that the Spec assumes (though does not state it explicitly) that if an end, say, f_1, is not owned by the association, $f_1 \notin A.ownedEnd$, then it is owned by its source classifier X_2, $f_1 \in X_2.ownedAttribute$. In this case, f_1 is considered to be an X_2's attribute [8, p.121]. What is, however, the meaning of the other end, f_2, owned by A?

We have two subcases:
(+), when f_2 is a navigable end, $f_2 \in A.navOwnedEnd$, and (–), when it is not.

In case (+), the association is navigable from X_1 to X_2 (Sect.7.3.3, p.36) and hence we have a mapping $f_2 : X_1 \twoheadrightarrow X_2$ yet f_2 is not an attribute of X_1 (otherwise it would be owned by X_1 rather than A). The only reasonable explanation that we could find for this situation is that mapping f_2 is not supposed to be stored in the instantiations of X_1 yet it can be derived from other data. Namely, we assume that mapping f_1 is actually stored (with the instantiations of classifier X_2 as its attribute) while f_2 can be derived from (the extension of) f_1 by taking the inverse. Strictly speaking, in case (+) association A consists of only one end f_1 (stored and owned by X_2!) but can be augmented with the other end, f_2, by a suitable derivation procedure (of inverting a mapping).

Case (–): the end f_2 is owned by A and is *not* navigable. The Spec says that in this case A is *not* navigable from X_1 to X_2 (Sect.7.3.3, p.36) and, hence, f_2 cannot be considered as a mapping. Then the only visible role of f_2 is to serve as a place-holder for the respective multiplicity constraint, m_2. We can consider this situation as that semantically association A consists of the only end/mapping $f_1 : X_2 \twoheadrightarrow X_1$, whose extension (graph, table) is constrained by a pair of multiplicity expressions $C = (m_1, m_2)$. In this treatment, the second end f_2 appears only in the *concrete* syntax as a way to visualize the second component of a single constraint $C = (m_1, m_2)$ rather than have any semantic meaning.

We can reformulate this situation by saying that some constraint to mapping f_1 is specified by setting a constraint m_2 to a mapping f_2 derived from f_1.In such a formulation case (−) becomes close to case (+). In both cases, association A consists, in fact, from the only end f_1 (owned by X_2) while the second end is derivable rather than storable and serves for (i) specifying the m_2-half of the multiplicity constraint to A and, (ii, optionally) for navigation from X_1 to X_2.

Thus, with help of implementation concepts, we were able to explain the mixed ownership cases (+) and (−). To be consistent, now we need to reconsider the case when both ends are owned by the association. Thinking along the lines we have just used, we conclude that in this case we deal with a situation when information about the association is stored somewhere but not in the participating classifiers (otherwise the ends were attributes owned by the classifiers). Hence, to make the ends derivable mappings we need to have a source of storable data for deriving the mappings, and the classifiers X_1, X_2 cannot be used for that.

A reasonable idea is to introduce onto the stage a new set, say, R, immediately storing links between instances of X_1, X_2, that is, pairs (x_1, x_2) with $x_1 \in X_1, x_2 \in X_2$, together with two projection mappings $p_i\colon R \to X_i$. In other words, we store the links in a table $T = (R, p_1, p_2)$ with R the set of rows and p_1, p_2 the columns so that if for a row r we have $r.p_1 = o_1 \in X_1$ and $r.p_2 = o_2 \in X_2$, it means that the row stores the link (o_1, o_2) (see Table 1). We can advance this interpretation even further and identify R with A and projection mappings p_i with A's ends f_i, $i = 1, 2$. This new view of associations (though may look somewhat unusual for the UML style) possesses a few essential advantages:

1. It perfectly fits in with the UML idea that an association is a classifier whose extension consists of links.
2. It is generalized for n-ary associations in a quite straitforward way: just consider R with a family of n projections $p_i\colon R \to X_i$, $i = 1..n$, which automatically makes R a collection of n-ary tuples/links.
3. A *property* is again a mapping and, moreover,
   3.1 the classifier owning the property is again the source of the mapping,
   3.2 the type of the property is the target of the mapping as before.

This interpretation brings an essential unification to the metamodel, and possesses a clear sets-and-mappings semantics. It also shows that the "ownership-navigability" part of the UML metamodel implicitly switches the focus from the analysis/structural view of association (Definition 2.2) to more technical (closer to design) view, where the modeler begins to care about which parts of the association will be stored, and which will be derived (with an eye on how to implement that later). We will call this latter view of associations *operational*.

The UML metamodel attributes the operational view to binary associations only (see Constraint (4) in Fig. 1). It appears to be an irrelevant restriction as in the next section we show that the operational view, including all nuances of ownership relations, can be developed for the general case of n-ary associations as well.

4 Formal Model for UML Associations: Separation and Integration of Concerns

In this section we build a formal framework for an accurate definition of the concepts that appeared above. We also introduce a new, and important, actor on the stage: qualified or operational mappings, which are an analog of attributes for multiary associations. It is this actor whose improper treatment in the UML metamodel leads to a striking difference between binary and multiary associations.

4.1 Basic Definitions and Conventions

Our first concern is to set a proper framework for working with names/labels in labeling records and similar constructs.

4.1.1 Definition: Roles and contexts. Let $\mathcal{L} = \{\ell_1 ... \ell_n\}$ be a *base* set of n different labels/symbols called *role names.*

(i) A *role* is a pair $\ell{:}X$ with $\ell \in \mathcal{L}$ a role name and X a class. A(n *association*) *context* is a set of roles $\mathbf{X} = \{\ell_1{:}X_1, \dots, \ell_n{:}X_n\}$ such that all role names are distinct (while the same class may appear with different roles). We write X_ℓ for the class X in the pair $(\ell{:}X)$. Cardinality of the base set is called the *arity* of the context. For example, the set {*course:Subject, student:Person, professor:Person*} is a ternary context.
(ii) We use the term class and set interchangeably. For our goals in this section, classes are just sets of elements (called objects). We write $\bigcup \mathbf{X}$ for $\bigcup \{X_\ell \mid \ell \in \mathcal{L}\}$. We also remind the reader our convention about distinguishing general and functional mappings (presented in Section 2 immediately below formula (2)).
(iii) All our definitions will be parameterized by some context $\mathbf{X}$. We will say that the notions are defined *over the context* $\boldsymbol{X}$*.*

4.1.2 Definition: Links, products, relations.
(i) A *link* over $\mathbf{X}$ is a functional mapping $r\colon \mathcal{L} \to \bigcup \mathbf{X}$ s.t. $r(\ell) \in X_\ell$. The set of all links over $\mathbf{X}$ will be denoted by $\prod_{\ell \in \mathcal{L}} X_\ell$ or $\prod_{\mathcal{L}} \mathbf{X}$ or just $\prod \mathbf{X}$. If $\{(\ell : X) \mid \ell \in \mathcal{K}\}$ is a sub-context of $\mathbf{X}$ for some $\mathcal{K} \subset \mathcal{L}$, we will write $\prod_{\mathcal{K}} \mathbf{X}$ for the set of the corresponding sub-links.
(ii) A *(multi)relation* over $\mathbf{X}$ is a (multi)set of links over $\mathbf{X}$. If R is a multirelation, $R!$ will denote R with duplicates eliminated, thus, $R! \subset \prod \mathbf{X}$. Note that R can be written down as a table whose column names are role names from the base set and rows are links occurring in R. Since each column name must be assigned with its domain, actually column names are pairs $\ell{:}X$, that is, roles.

4.1.3 Construction: Tables vs. relations. In the relational data model a table is viewed as a collection of rows (links). However, it is possible to switch the focus from rows-links to columns-roles and consider the same table as a collection of columns. Each column $\ell{:}X$ gives rise to a functional mapping $[\![\, \ell \,]\!]\colon R \to X$, $[\![\, \ell \,]\!](r) \stackrel{\text{def}}{=} r(\ell)$. Note that R is always a set but it may happen that two different rows $r \neq r'$ store the same link if $[\![\, \ell_i \,]\!](r) = [\![\, \ell_i \,]\!](r')$ for all $\ell_i \in \mathcal{L}$.

(i) A *table* over $\mathbf{X}$ is an n-tuple $T = (p_1...p_n)$ of functions $p_i \colon R \to X_i$, $i = 1..n$ with a common source R called the *head*. Elements of R will be also called *rows*, and functions p_i *columns* or, else, *projections*, of the table. We will often make the head explicit and write a table as an $(n+1)$-tuple $T = (R, p_1...p_n)$.
(ii) We can identify projections p_i with semantics of the roles in the context, and set $p_i = [\![\ell_i]\!]$.

4.1.4 Definition: Mappings and maps over a context.
(i) A *structural* mapping over $\mathbf{X}$ is a mapping of the form $f \colon \prod_{\mathcal{K}} \mathbf{X} \twoheadrightarrow X_\ell$, where $(\mathcal{K}, \{\ell\})$ is a partition of $\mathcal{L}$ (with the second member being a singleton). The sub-context $\{\ell{:}X_\ell \mid \ell \in \mathcal{K}\}$ is called the *source context* of f.
(ii) A *qualified* or *operational* mapping over $\mathbf{X}$ is a mapping of the form $g \colon X_{\ell'} \to [\prod_{\mathcal{P}} \twoheadrightarrow X_{\ell''}]$, where $(\{\ell'\}, \mathcal{P}, \{\ell''\})$ is a partition of $\mathcal{L}$. The square brackets denote the set of all structural mappings of the form inside the brackets.

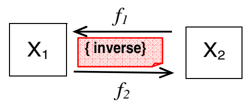
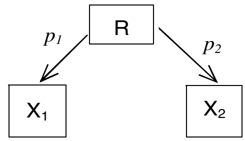
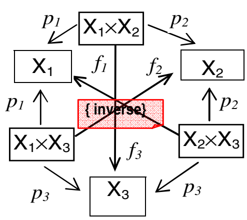
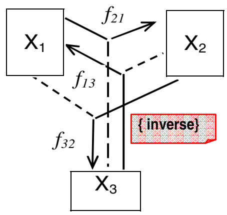
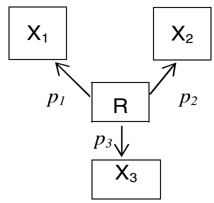
The set $X_{\ell'}$ is the source, the roles in $\mathcal{P}$ are parameters and $X_{\ell''}$ is the target (goal) set. If $\mathcal{P} = \{j_1..j_k\}$, in a standard programming notation the mapping could be written as $g(\ell_{j1}{:}X_{j1}, \ldots, \ell_{j_k}{:}X_{j_k}\ ) \colon X_{\ell'} \twoheadrightarrow X_{\ell''}$. We will call the sub-context $\mathbf{P} = \{X_\ell | \ell \in \mathcal{P}\}$ the *parameter context* or *qualifier*.
(iii) Given a structural and operational mappings f and g as above, we say that g *implements* f if $\ell'' = \ell$ (and hence $\mathcal{K} = \mathcal{P} \cup \{\ell'\}$). This is nothing but a well-known Curry construction (see, e.g., [4]), and we will also call the passages from f to g and back *Currying* of f and *unCurrying* of g. Note that they do not change the extension of mappings.

The left and middle columns of Table 1 show how it works for the cases of n=2 and n=3. For the case $n = 2$, Currying is trivial. For $n = 3$, Currying will produce six operational mappings: two for each of the structural mappings. We show only three of them. It is easy to see that any n-ary structural mapping has n operational/qualified implementations.
(iv) An *operational/qualified map* over $\mathbf{X}$ is a set M_O of $n(n-1)$ qualified mappings with the same extension. In other words, such a map is the set of all qualified mappings generated by some structural map.
(v) To ease comparison of our formal constructs with those defined in UML and avoid terminological clash, we will call the members of a qualified map *legs* while members of a structural map (Definition 2.1) will be called *arms*.

Let $f \colon \prod_{\mathcal{K}} \mathbf{X} \twoheadrightarrow Y = X_j$, $\mathcal{K} = \mathcal{L} \setminus \{j\}$, be a structural mapping as above. Its extension can be presented as a table $T = \mathsf{ext}(f)$. However, during this passage the information about which of the columns of the table corresponds to f's target is lost. Any other mapping with the same extension will result in the same table, and conversely, by looking table T up in different "directions", we will obtain n different structural mappings including f. We remind the reader that we have called such sets of structural mappings *structural maps* (Definition 2.1(iii)). Thus, a table is an exact extensional representation of maps rather than mappings.

4.1.5 Construction: Adding navigation to tables. We can enrich tables with "navigational" information about the mapping generated the table if the

Table 1. Three views of associations

	Structural: maps of (structural) mappings	Operational: maps of operations (parameterized mappings)	Extensional: tables (i.e., maps of projection mappings)
n=2			
n=3			
n=4	...	...	...

corresponding column name will be marked (say, by a star). Similarly, if a table stores the extension of a qualified mapping, we can keep this information by marking the two corresponding columns. In this way we come to the notions of (i) *star-table*, a table with one column specially designated and called the *goal*, and (ii) *double-star table*, a star-table with one more column designated/marked as the *source* or, in programming terms, *self*.

4.2 Formalization of *Ownership* in the UML Metamodel of Associations

As it was noticed in sect.3, the ownership meta-associations in the UML metamodel are related to possible implementations of structural associations. The latter can be implemented either by a table, or/and by a number of qualified mappings between the participating classes. Which implementation is most suitable depends on which navigation directions need to be implemented efficiently.

4.2.1 Definition: Operational view of associations. *Operationally*, an association over $\mathbf{X}$ is an triple $A = (M_O, T, B)$ with M_O an operational map of qualified mappings over $\mathbf{X}$, T their common extension table, and B a non-empty subset of the set $M_O \cup \{T\}$, whose elements are called *basic* while other elements of $M_O \cup \{T\}$ are called *derived*. The intuition is that the elements of the set $M_O \cap B$ are to be implemented as retrieval operations of the corresponding classes (their attributes in the binary case); the classes then own these elements. If also $T \in B$, then the extension is to be really stored in some table T. The elements formally called "derived" can be indeed derived from the basic elements (by say looking up the extension table in the required direction, and the extension table can be derived by recording the input-output pairs).

The rightmost part of Fig. 3 present the metamodel of this definition.

4.2.2 Remark: uniqueness constraints. It was proposed in [6], that even if the extension table contains duplicates and hence all qualified mappings from M_O are bag-valued, it may be useful for navigational purposes to choose for some of them their versions with eliminated duplicates. Then, operationally, an association over $\mathbf{X}$ is defined to be a quadruple $A = (M_O, T, B, U)$ with the triple (M_O, T, B) as above and $U \subset M_O$ is the set (perhaps, empty) of those members that we have chosen to consider with eliminated duplicates. Details and a thorough discussion can be found in [6].

4.3 The Metamodel: Playing LEGO Blocks with Associations

Figure 3 on p.243 presents the metamodel of the notions and transformations we have defined above. All meta-classes in the model are parameterized by the association's arity n. It allows us to capture numerous important size constraints (like constraint (10) in Fig. 2) by stating the corresponding multiplicities. We believe that this presentation would be also useful for the UML metamodel.

In the vertical direction, the metamodel consists of two parts: the upper half presents the extensional, or tabular, view of associations, the lower half shows the procedural, or map-based, view. Each of the parts is based on the corresponding structural foundation: the role context for the maps, and the column context for the tables. These two context are in one-one correspondence via the *semantics-name* meta-association, see Construction 4.1.3(ii), and it is our conjecture that in a deeper formal setting they could be unified into a single notion.

There is also a nice parallelism between the two parts in their treatment of navigability as the consecutive augmentation of the respective constructs with additional "navigational" information (what is declared to be the source and the target of the corresponding mapping). To underline this parallelism, we have denoted the (meta) associations "source context" for structural mappings, and "parameter context" for operational mappings, by *context** and *context*** respectively. This "addition of navigability" is governed by one-to-many associations in both parts. One n-column Table generates n starTables, and each starTable generates $(n-1)$ doubleStarTables, and similarly for Maps, structuralMappings and operationalMapppings. The two parts are tightly connected by vertical meta-associations *ext-lookUp* and diagonal meta-associations (shown in dashed line) derived by the respective compositions of horizontal and vertical meta-association ends.

In the horizontal direction, the metamodel also consists of two parts: the structural view of associations (the left half) and the operational view of associations (the right half). These two views are also tightly connected by horizontal meta-associations of *Currying*-*unCurrying* and (*set* self-*column*) – (*forget* self-*column*).

In fact, our metamodel presents a toolbox of blocks for building different views/notions of associations. For example, structurally an association is a pair $A_S = (M_S, T)$ with M_S a map of mutually inverse structural mappings and T the table representing their common extension (A_S's collections of links). Operationally, an association is a triple $A_O = (M_O, T, B)$ with M_O a map of mutually inverse operational/qualified mappings, T the extension table and B sorting the elements of $M_O \cup \{T\}$ into basic-derived. We say that A_O *implements* A_S if they have the same extension T (and hence, mappings in M_O are Currying versions of mappings in M_S). Extensionally, an association is a table T, and procedurally, it is a pair of maps (M_S, M_O). We can consider an integrated notion of association by defining it as a quadruple $A = (M_S, T, M_O, B)$. Then all the views mentioned above are indeed views, that is, different projections/parts of the whole construct.

4.4 UML Metamodel in the Light of Formalization, II

It is instructive to compare our formal model of associations specified in Fig. 3 with the UML model (Fig. 1). Our formalization clearly shows three components of the association concept: extensional, structural and operational (Table 1). They all have the same underlying structure: a host object (a table/ structural map/ qualified map) holds a number of member mappings (columns/ arms/ legs respectively). Though these components are closely related and, in fact, mutually derivable, they consist of *different* elements: a simple calculation shows that an n-ary association $A = (T, M_S, M_O)$ consists of the total of $m(A) = n + n + n(n-1) = n(n+1)$ mappings (columns, arms and legs) plus one set of links (the head). Note that all these association's elements appear in one or another way in different *Semantics* and *Description* sections of the Spec, and are used for defining associations' (meta)properties like ownerships, navigability, multiplicity. However, as formally defined by the UML metamodel, an n-ary association A consists of only $m_{UML}(A) = n$ elements, its *memberEnd* Properties (UML's analog of mappings). Thus, UML metamodel offers only n-elements to name and manipulate $n(n+1)$ constructs. In Fig. 3, we have pointed out UML counterparts of our formal constructs by their names in square brackets, which makes the shortage of constructs in the UML metamodel explicit. Not surprisingly, this shortage leads to ambiguities in practical usage of associations reported by experts [7].

The comparison also reveals two more flaws in the UML metamodel. First is the absence of meta-association *context* for meta-class Property. In fact, it means that the fundamental notion of *property* is not completely defined in UML. We consider this as one of the most serious problem of the entire UML metamodel (see [2] on the value of the property construct in semantics of OO visual modeling).

The second problem is less fundamental yet is important for practical modeling: the meta-association *qualifier* is improperly defined in the metamodel. Our formalization clearly shows that the target of this meta-association is the meta-class of Roles rather than that of Properties. This mistake in the metamodel can lead to mistakes in practical modeling with qualified associations. Space limitations do not allow us to demonstrate the issue with a few remarkable examples we have in our archive (see [1] for one of them).

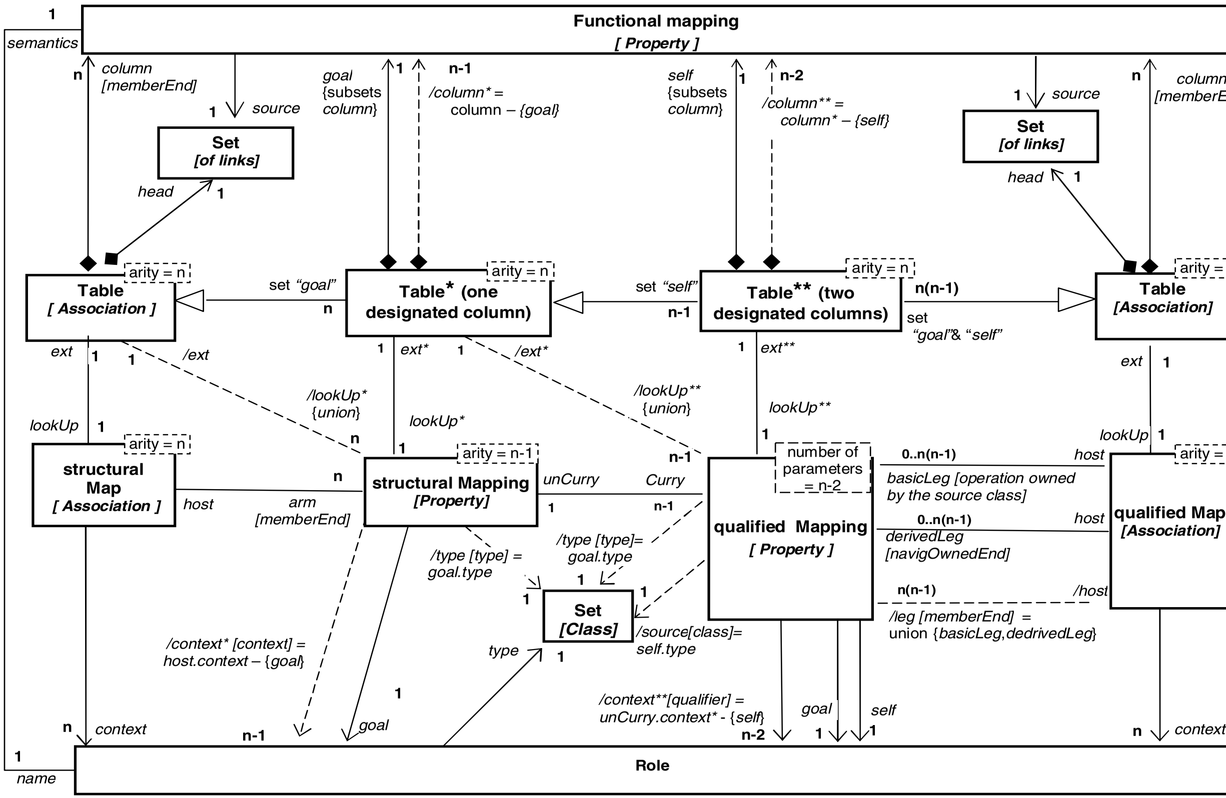


Fig. 3. Metamodel of our formal model for associations. Italic terms in square brackets refer to UML counterparts of our formal constructs. *Warning:* The node Table with its meta-associations is repeated twice to avoid clutter!

5 Conclusion

We have developed a formal framework where the complex notion of association can be disassembled into a few basic blocks. We then built from these blocks a few constructs that formally model different aspects of associations as described and used in UML2. We have found that semantics of the association construct can be uncovered in a few *Semantics* and *Description* sections of the specification, and is presented there in a sufficiently consistent way. However, the part of this semantics formally captured in the UML2 metamodel is much poorer, which makes the latter incomplete and ambiguous.

Our formal model allowed us to explain a few known problems with associations and to detect several omissions in the metamodel, which have been unnoticed so far (see sections 2.3 and 4.4). We have also proposed a few general suggestions on augmenting and restructuring the metamodel for associations to capture their semantics in a precise and unambiguous way.

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