Canonical Gentzen-Type Calculi with (n,k)-ary Quantifiers

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Abstract. Propositional canonical Gentzen-type systems, introduced in [1], are systems which in addition to the standard axioms and structural rules have only logical rules in which exactly one occurrence of a connective is introduced and no other connective is mentioned. [1] provides a constructive coherence criterion for the non-triviality of such systems and shows that a system of this kind admits cut-elimination iff it is coherent. The semantics of such systems is provided using two-valued non-deterministic matrices (2Nmatrices). [14] extends these results to systems with unary quantifiers of a very restricted form. In this paper we substantially extend the characterization of canonical systems to (n, k)-ary quantifiers, which bind k distinct variables and connect n formulas. We show that the coherence criterion remains constructive for such systems, and that for the case of $k \in \{0, 1\}$: (i) a canonical system is coherent iff it has a strongly characteristic 2Nmatrix, and (ii) if a canonical system is coherent, then it admits cut-elimination.

1 Introduction

An (n, k)*-ary quantifier* (for $n > 0$, $k \geq 0$) is a generalized logical connective, which binds k variables and connects n formulas. Any n-ary propositional connective can be thought of as an $(n, 0)$-ary quantifier. For instance, the standard $\wedge$ connective binds no variables and connects two formulas: $\wedge(\psi_1, \psi_2)$. The standard first-order quantifiers $\exists$ and $\forall$ are $(1, 1)$-quantifiers, as they bind one variable and connect one formula: $\forall x\psi, \exists x\psi$. Bounded universal and existential quantification used in syllogistic reasoning ($\forall x(p(x) \to q(x))$ and $\exists x(p(x) \wedge q(x))$) can be represented as (2,1)-ary quantifiers $\overline{\forall}$ and $\overline{\exists}$, binding one variable and connecting two formulas: $\overline{\forall} x(p(x), q(x))$ and $\overline{\exists} x(p(x), q(x))$. An example of (n, k)-ary quantifiers for $k > 1$ are Henkin quantifiers[1] ([9,10]). The simplest Henkin quantifier Q_H binds 4 variables and connects one formula:

$$Q_H \begin{matrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{matrix} \psi(x_1, x_2, y_1, y_2)$$

According to a long tradition in the philosophy of logic, established by Gentzen in his classical paper *Investigations Into Logical Deduction* ([7]), an "ideal" introduction rule for a logical connective is a rule which determines the *meaning*

[1] Although the semantic interpretation of quantifiers used in this paper is not sufficient for treating such quantifiers.

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of the connective. In [1,2] the notion of a "canonical propositional Gentzen-type rule" was first defined in precise terms. A constructive *coherence* criterion for the non-triviality of such systems was then provided, and it was shown that a system of this kind admits cut-elimination iff it is coherent. It was further proved that the semantics of such systems is provided by two-valued non-deterministic matrices (2Nmatrices), which form a natural generalization of the classical matrix. In fact, a characteristic 2Nmatrix was constructed for every coherent canonical propositional system.

In [14] the results were extended to systems with unary quantifiers. A characterization of a "canonical unary quantificational rule" in such calculi was proposed (the standard Gentzen-type rules for $\forall$ and $\exists$ are canonical according to it), and a constructive extension of the coherence criterion of [1,2] for canonical systems of this type was given. 2Nmatrices were extended to languages with unary quantifiers, using a *distribution* interpretation of quantifiers ([12]). Then it was proved that again a canonical Gentzen-type system of this type admits cut-elimination iff it is coherent, and that it is coherent iff it has a characteristic 2Nmatrix. However, the canonical systems in [14] are of a very restricted form: they use unary quantifiers and only one atomic (monadic) formula is allowed in each clause.

In this paper we make the intuitive notion of a "well-behaved" introduction rule for (n, k)-ary quantifiers formally precise. We considerably extend the scope of the characterizations of [1,2,14] to "canonical (n, k)-ary quantificational rules", so that both the propositional systems of [1,2] and the restricted quantificational systems of [14] are specific instances of the proposed definition. However, in contrast to the systems in [14], there are no limitations on the size of the clauses in our formulation. It is then shown that the coherence criterion for the defined systems remains constructive. Then we turn to the class of canonical systems with (n, k)-ary quantifiers for $k \in \{0, 1\}$ and show that every coherent canonical calculus G has a characteristic 2Nmatrix and admits cut-elimination. The other direction, however, does not hold: we shall see that in contrast to the canonical systems of [1,14], the ability to eliminate cuts in a canonical calculus G does *not* necessarily imply its coherence.

2 Preliminaries

For any $n > 0$ and $k \geq 0$, if a quantifier $\mathcal{Q}$ is of arity (n, k), then $\mathcal{Q}x_1...x_k$ $(\psi_1, ..., \psi_n)$ is a formula whenever $x_1, ..., x_k$ are distinct variables and $\psi_1, ..., \psi_n$ are formulas of L.

For interpretation of quantifiers, we use a generalized notion of *distribution functions* ([12]). Given a set S, $P^+(S)$ is the set of all the nonempty subsets of S.

Definition 1. *Given a set of truth value* $\mathcal{V}$, *a* distribution *of a (1,1)-ary quantifier* $\mathcal{Q}$ *is a function* $\lambda_{\mathcal{Q}} : P^+(\mathcal{V}) \to \mathcal{V}$.

(1,1)-ary distribution quantifiers have been extensively studied and axiomatized in many-valued logic. See, for instance, [5,13,8].

In what follows, L is a language with (n,k)-ary quantifiers, that is with quantifiers $\mathcal{Q}_1, ..., \mathcal{Q}_m$ with arities (n_1,k_1), ..., (n_m,k_m) respectively. Denote by Frm_L^{cl} the set of closed L-formulas and by Trm_L^{cl} the set of closed L-terms.

$\equiv_\alpha$ is the α-equivalence relation between formulas, i.e identity up to the renaming of bound variables. We use [ ] for application of functions in the meta-language, leaving the use of ( ) to the object language. $A\{\mathbf{t}/x\}$ denotes the formula obtained from A by substituting $\mathbf{t}$ for x. Given an L-formula A, $Fv[A]$ is the set of variables occurring free in A. We denote $\mathcal{Q}x_1...x_kA$ by $\mathcal{Q}\overrightarrow{x}A$, and $A(x_1,...,x_k)$ by $A(\overrightarrow{x})$.

3 Canonical Systems with (n,k)-ary Quantifiers

In this section we formulate a precise definition of a "canonical (n,k)-ary quantificational Gentzen-type rule".

Using an introduction rule for an (n,k)-ary quantifier $\mathcal{Q}$, we should be able to derive a sequent of the form $\Gamma \Rightarrow \mathcal{Q}x_1...x_k(\psi_1,...,\psi_n), \Delta$ or of the form $\Gamma, Qx_1...x_k(\psi_1,...,\psi_n) \Rightarrow \Delta$, based on some information about the subformulas of $\mathcal{Q}x_1...x_k(\psi_1,...,\psi_n)$ contained in the premises of the rule. For instance, consider the following standard rules for the (1,1)-ary quantifier $\forall$:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}\ (\Rightarrow \forall)$$

where $\mathbf{t}, z$ are free for w in A and z does not occur free in the conclusion. Our key observation is that the internal structure of A, as well as the exact term $\mathbf{t}$ or variable w used, are immaterial for the meaning of $\forall$. What is important here is the side of the sequent, on which A appears, as well as whether a term variable $\mathbf{t}$ or an eigenvariable z is used.

Hence, the internal structure of the formulas of L can be abstracted by using a simplified first-order language, i.e. the formulas of L in an introduction rule of a (n,k)-ary quantifier, will be represented by *atomic* formulas with predicate symbols of arity k. The case when the substituted term is any L-term, will be signified by a constant, and the case when it is a variable satisfying the above conditions - by a variable. In other words, constants serve as term variables, while variables are eigenvariables.

Hence, in addition to our original language L with (n,k)-ary quantifiers we define another, simplified language.

Definition 2. *For $k \geq 0$, $n \geq 1$ and a set of constants Con, $L_k^n(Con)$ is the language with n k-ary predicate symbols $p_1,...,p_n$ and the set of constants Con (and no quantifiers or connectives).*

Definition 3 (Canonical Rules)

1. *Let Con be some set of constants. A* canonical quantificational rule of arity *(n,k) is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$, where $m \geq 0$, C is*

either $\Rightarrow \mathcal{Q}x_1...x_k(p_1(x_1,...,x_k),...,p_n(x_1,...,x_k))$ *or* $\mathcal{Q}x_1...x_k(p_1(x_1,...,x_k),...,p_n(x_1,...,x_k)) \Rightarrow$ *for some* (n,k)*-ary quantifier* $\mathcal{Q}$ *of* L *and for every* $1 \leq i \leq m$*:* $\Pi_i \Rightarrow \Sigma_i$ *is a clause*[2] *over* $L_k^n(Con)$.

2. *Let* $R = \Theta/C$ *be an* (n,k)*-ary canonical rule, where* C *is of one of the forms* $(\mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}),...,p_n(\overrightarrow{x})) \Rightarrow)$ *or* $(\Rightarrow \mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}),...,p_n(\overrightarrow{x})))$.
Let Con_Θ *be the set of constants occurring in* Θ*. Let* Γ *be a set of* L*-formulas and* $z_1,...,z_k$ *- distinct variables.*
An $\langle R,\Gamma,z_1,...,z_k\rangle$-mapping *is any function* χ *from the predicate symbols and terms of* $L_k^n(Con_\Theta)$ *to formulas and terms of* L*, satisfying the following conditions:*
   - *For every* $1 \leq i \leq n$*,* $\chi[p_i]$ *is an* L*-formula.*
   - $\chi[y]$ *is a variable of* L*.*
   - $\chi[x] \neq \chi[y]$ *for every two variables* $x \neq y$*.*
   - $\chi[c]$ *is an* L*-term.*
   - *For every* $1 \leq i \leq n$*, every* $p_i(\boldsymbol{t}_1,...,\boldsymbol{t}_k)$ *occurring in* Θ *and every* $1 \leq j \leq k$*:* $\chi[\boldsymbol{t}_j]$ *is a term free for* z_j *in* $\chi[p_i]$*, and if* $\boldsymbol{t}_j$ *is a variable, then* $\chi[\boldsymbol{t}_j]$ *does not occur free in* $\Gamma \cup \{\mathcal{Q}z_1...z_k(\chi[p_1],...,\chi[p_n])\}$*.*

   We extend χ *to* $L_k^n(Con_\Theta)$*-formulas and sets of* $L_k^n(Con)$*-formulas as follows:*

$$\chi[p_i(\boldsymbol{t}_1,...,\boldsymbol{t}_k)] = \chi[p_i]\{\chi[\boldsymbol{t}_1]/z_1,...,\chi[\boldsymbol{t}_k]/z_k\}$$
$$\chi[\Gamma] = \{\chi[\psi] \mid \psi \in \Gamma\}$$

   An application *of a canonical quantificational rule of arity* (n,k) $R = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\mathcal{Q}x_1...x_k(p_1(x_1,...,x_k),...,p_n(x_1,...,x_k)) \Rightarrow$ *is any inference step of the form:*

$$\frac{\{\Gamma,\chi[\Pi_i] \Rightarrow \Delta,\chi[\Sigma_i]\}_{1\leq i\leq m}}{\Gamma,\mathcal{Q}z_1...z_k\ (\chi[p_1],...,\chi[p_n]) \Rightarrow \Delta}$$

   where $z_1,...,z_k$ *are variables,* Γ,Δ *are any sets of* L*-formulas and* χ *is some* $\langle R,\Gamma\cup\Delta,z_1,...,z_k\rangle$*-mapping.*
   An application of a canonical quantificational rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \Rightarrow \mathcal{Q}x_1...x_k(p_1(x_1,...,x_k),...,p_n(x_1,...,x_k))$ *is defined similarly.*

In other words, an application of an (n,k)-ary canonical rule $\Theta/ \Rightarrow \mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}),...,p_n(\overrightarrow{z}))$ is obtained by "instantiating" the rule, i.e. by replacing every $L_k^n(Con_\Theta)$-formula $p_i(c)$ in Θ by some L-formula $\psi_i\{\mathbf{t}_c/z\}$, every $p_j(x)$ - by some L-formula $\psi_j\{y_x/z\}$, and $\mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}),...,p_n(\overrightarrow{z}))$ - by $\mathcal{Q}z(\psi_1,...,\psi_n)$, with the restrictions on $\mathbf{t}_c$ and y_x which are specified above.

Below we demonstrate the above definition by a number of examples.

Example 1. The standard right introduction rule for $\wedge$, which can be thought of as an $(2,0)$-ary quantifier is $\{\Rightarrow p_1, \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$. Its application is of the form:

$$\frac{\Gamma \Rightarrow \psi_1,\Delta \quad \Gamma \Rightarrow \psi_2,\Delta}{\Gamma \Rightarrow \psi_1 \wedge \psi_2,\Delta}$$

[2] By a clause we mean a sequent containing only atomic formulas.

Example 2. The two standard introduction rules for the $(1,1)$-ary quantifier $\forall$ can be formulated as follows:

$$\{p(c) \Rightarrow\}/\forall x\, p(x) \Rightarrow \qquad \{\Rightarrow p(y)\}/ \Rightarrow \forall x\, p(x)$$

Applications of these rules have the forms:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta}\ (\forall \Rightarrow) \quad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}\ (\Rightarrow \forall)$$

where z is free for w in A, z is not free in $\Gamma \cup \Delta \cup \{\forall w A\}$, and $\mathbf{t}$ is any term free for w in A.

Example 3. Consider the bounded existential and universal $(2,1)$-ary quantifiers $\overline{\forall}$ and $\overline{\exists}$ (corresponding to $\forall x.p_1(x) \to p_2(x)$ and $\exists x.p_1(x) \wedge p_2(x)$ used in syllogistic reasoning). Their corresponding rules can be formulated as follows:

$$\{p_2(c) \Rightarrow\ ,\ \Rightarrow p_1(c)\}/\overline{\forall} x\ (p_1(x), p_2(x)) \Rightarrow \qquad \{p_1(y) \Rightarrow p_2(y)\}/ \Rightarrow \overline{\forall} x\ (p_1(x), p_2(x))$$

$$\{p_1(y), p_2(y) \Rightarrow\}/\overline{\exists}\, x(p_1(x), p_2(x)) \Rightarrow \qquad \{\Rightarrow p_1(c)\ ,\ \Rightarrow p_2(c)\}/ \Rightarrow \overline{\exists} x(p_1(x), p_2(x))$$

Applications of these rules are of the form:

$$\frac{\Gamma, \psi_2\{\mathbf{t}/z\} \Rightarrow \Delta \quad \Gamma \Rightarrow \psi_1\{\mathbf{t}/z\}, \Delta}{\Gamma, \overline{\forall} z\ (\psi_1, \psi_2) \Rightarrow \Delta} \qquad \frac{\Gamma, \psi_1\{y/z\} \Rightarrow \psi_2\{y/z\}, \Delta}{\Gamma \Rightarrow \overline{\forall} z\ (\psi_1, \psi_2), \Delta}$$

$$\frac{\Gamma, \psi_1\{y/z\}, \psi_2\{y/z\} \Rightarrow \Delta}{\Gamma, \overline{\exists} z\ (\psi_1, \psi_2) \Rightarrow \Delta} \qquad \frac{\Gamma, \psi_1\{\mathbf{t}/x\} \Rightarrow \Delta \quad \Gamma \Rightarrow \psi_2\{\mathbf{t}/x\}, \Delta}{\Gamma \Rightarrow \overline{\exists} z\ (\psi_1, \psi_2), \Delta}$$

where $\mathbf{t}$ and y are free for z in ψ_1 and ψ_2, y does not occur free in $\Gamma \cup \Delta \cup \{\overline{\exists} z(\psi_1, \psi_2)\}$.

Example 4. Consider the (2,2)-ary rule

$$\{p_1(x, z) \Rightarrow\ ,\ p_1(y, d) \Rightarrow p_2(c, d)\}/ \Rightarrow \mathcal{Q} z_1 z_2(p_1(z_1, z_2), p_2(z_1, z_2))$$

Its application is of the form:

$$\frac{\Gamma, \psi_1\{w_1/z_1, w_2/z_2\} \Rightarrow \Delta \quad \Gamma, \psi_1\{w_3/z_1, \mathbf{t}_1/z_2\} \Rightarrow \Delta, \psi_2\{\mathbf{t}_2/z_1, \mathbf{t}_1/z_2\}}{\Gamma \Rightarrow \Delta, \mathcal{Q} z_1 z_2(\psi_1, \psi_2)}$$

where $w_1, w_2, w_3, \mathbf{t}_1, \mathbf{t}_2$ satisfy the appropriate conditions.

Henceforth, in cases where the set of constants Con_Θ is clear from the context (it is the set of all constants occurring in a canonical rule), we will write L_k^n instead of $L_k^n(Con_\Theta)$.

Definition 4. *A Gentzen-type calculus G is* canonical *if in addition to the α-axiom $A \Rightarrow A'$ for $A \equiv_\alpha A'$ and the standard structural rules, G has only canonical quantificational rules, such that the sets of constants and variables of every two rules are disjoint.*

Although we can define arbitrary canonical systems using our simplified language L_k^n, our quest is for systems, the syntactic rules of which define the semantic meaning of logical connectives. Thus we are interested in calculi with a "reasonable" or "non-contradictory" set of rules, which allows for defining a sound and complete semantics for the system. This can be captured syntactically by the *coherence* criterion of [1,14]:

Definition 5 (Coherence). [3] *A canonical calculus G is* coherent *if for every two canonical rules of G of the form $\Theta_1/ \Rightarrow A$ and $\Theta_2/A \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent.*

Proposition 1 (Decidability of coherence). *The coherence of a canonical calculus G is decidable.*

Proof. The question of classical consistency of a finite set of clauses without quantifiers can be easily shown to be equivalent to satisfiability of a finite set of universal formulas in a language with no function symbols, which is decidable.

Notation. (Following [1], notations 3-5.) Let $-t = f, -f = t$ and $ite(t, A, B) = A$, $ite(f, A, B) = B$. Let Φ, A^s (where Φ may be empty) denote $ite(s, \Phi \cup \{A\}, \Phi)$. For instance, the sequents $A \Rightarrow$ and $\Rightarrow A$ are denoted by $A^a \Rightarrow A^{-a}$ for $a = t$ and $a = f$ respectively.

According to this notation, a (n, k)-ary canonical rule is of the form

$$\{\Sigma_j \Rightarrow \Pi_j\}_{1 \leq j \leq m}/$$

$$\mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}), ..., p_n(\overrightarrow{z}))^s \Rightarrow \mathcal{Q}\overrightarrow{z}(p_1(\overrightarrow{z}), ..., p_n(\overrightarrow{z}))^{-s}$$

for $s \in \{t, f\}$. For further abbreviation, we denote such rule by $\{\Sigma_j \Rightarrow \Pi_j\}_{1 \leq j \leq m}/\mathcal{Q}(s)$.

4 The Semantic Framework

4.1 Non-deterministic Matrices

Our main semantic tool are non-deterministic matrices (Nmatrices), first introduced in [1] and used in [2,14]. These structures are a generalization of the standard concept of a many-valued matrix, in which the truth-value of a formula is chosen non-deterministically from a given non-empty set of truth-values. Thus, given a set of truth-values $\mathcal{V}$, we can generalize the notion of a distribution function of an (n, k)-ary quantifier $\mathcal{Q}$ (from Definition. 1) to a function $\lambda_{\mathcal{Q}} : P^+(\mathcal{V}^n) \rightarrow P^+(\mathcal{V})$. In other words, given some distribution Y of n-ary vectors of truth values, the interpretation function non-deterministically chooses the truth value assigned to $\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)$ out from $\lambda_{\mathcal{Q}}[Y]$.

[3] Strongly related coherence criterions are defined in [11], where linear logic is used to reason about various sequent systems, and in [6], where a characterization of cut-elimination is given for a general class of propositional single-conclusion sequent calculi.

Definition 6 (Non-deterministic matrix). *A non-deterministic matrix (henceforth Nmatrix) for L is a tuple $\mathcal{M} =< \mathcal{V}, \mathcal{G}, \mathcal{O} >$, where:*

- *$\mathcal{V}$ is a non-empty set of truth values.*
- *$\mathcal{G}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.*
- *$\mathcal{O}$ is a set of interpretation functions: for every (n, k)-ary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$ includes the corresponding distribution function $\tilde{\mathcal{Q}}_{\mathcal{M}} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$.*

At this point a remark on our treatment of propositional connectives is in order. In [1,14], an Nmatrix includes an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective of the language; given a valuation v, the truth value $v[\diamond(\psi_1, ..., \psi_n)]$ is chosen non-deterministically from $\tilde{\diamond}[\langle v[\psi_1], ..., v[\psi_n]\rangle]$. In the definition above, the interpretation of a propositional connective $\diamond$ is a function of another type: $\tilde{\diamond} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$. This can be thought as a generalization of the previous definition, identifying the tuple $\langle v[\psi_1], ..., v[\psi_n]\rangle$ with the singleton $\{\langle v[\psi_1], ..., v[\psi_n]\rangle\}$. The advantage of this generalization is that it allows for a uniform treatment of both quantifiers and propositional connectives.

Definition 7 (L-structure). *Let $\mathcal{M}$ be an Nmatrix for L. An L-structure for $\mathcal{M}$ is a pair $S = \langle D, I\rangle$ where D is a (non-empty) domain and I is a function interpreting constants, predicate symbols and function symbols of L, satisfying the following conditions: $I[c] \in D$, $I[p^n] : D^n \to \mathcal{V}$ is an n-ary predicate, and $I[f^n] : D^n \to D$ is an n-ary function.*
I is extended to interpret closed terms of L as follows:

$$I[f(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[f][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_n]]$$

Definition 8 (L(D)). *Let S=$\langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. $L(D)$ is the language obtained from L by adding to it the set of* individual constants $\{\overline{a} \mid a \in D\}$. *$S' = \langle D, I'\rangle$ is the $L(D)$-structure, such that I' is an extension of I satisfying: $I'[\overline{a}] = a$.*

Given an L-structure $S = \langle D, I\rangle$, we shall refer to the extended $L(D)$-structure $\langle D, I'\rangle$ as S and to I' as I when the meaning is clear from the context.

Definition 9 (Congruence of terms and formulas). *Let S be an L-structure for an Nmatrix $\mathcal{M}$. The relation $\sim^S$ between terms of $L(D)$ is defined inductively as follows:*

- *$x \sim^S x$*
- *For closed terms $\boldsymbol{t}, \boldsymbol{t}'$ of $L(D)$: $\boldsymbol{t} \sim^S \boldsymbol{t}'$ when $I[\boldsymbol{t}] = I[\boldsymbol{t}']$.*
- *If $\boldsymbol{t}_1 \sim^S \boldsymbol{t}'_1, ..., \boldsymbol{t}_n \sim^S \boldsymbol{t}'_n$, then $f(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \sim^S f(\boldsymbol{t}'_1, ..., \boldsymbol{t}'_n)$.*

The relation $\sim^S$ between formulas of $L(D)$ is defined as follows:

- *If $\boldsymbol{t}_1 \sim^S \boldsymbol{t}'_1, \boldsymbol{t}_2 \sim^S \boldsymbol{t}'_2, ..., \boldsymbol{t}_n \sim^S \boldsymbol{t}'_n$, then $p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \sim^S p(\boldsymbol{t}'_1, ..., \boldsymbol{t}'_n)$.*
- *If $\psi_1\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_1\{\overrightarrow{z}/\overrightarrow{y}\}, ..., \psi_n\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_n\{\overrightarrow{z}/\overrightarrow{y}\}$, where $\overrightarrow{x} = x_1...x_k$ and $\overrightarrow{y} = y_1...y_k$ are distinct variables and $\overrightarrow{z} = z_1...z_k$ are new distinct variables, then $\mathcal{Q}\overrightarrow{x}(\psi_1, ..., \psi_n) \sim^S \mathcal{Q}\overrightarrow{y}(\varphi_1, ..., \varphi_n)$ for any (n, k)-ary quantifier $\mathcal{Q}$ of L.*

Lemma 1. *Let S be an L-structure for an Nmatrix $\mathcal{M}$. Let ψ, ψ' be formulas of $L(D)$. Let $\boldsymbol{t}, \boldsymbol{t}'$ be closed terms of $L(D)$, such that $\boldsymbol{t} \sim^S \boldsymbol{t}'$.*

1. *If $\psi \equiv_\alpha \psi'$, then $\psi \sim^S \psi'$.*
2. *If $\psi \sim^S \psi'$, then $\psi\{\boldsymbol{t}/x\} \sim^S \psi'\{\boldsymbol{t}'/x\}$.*

Definition 10 (Legal valuation). *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An S-valuation $v : Frm_L^{\mathsf{cl}} \rightarrow \mathcal{V}$ is* legal in $\mathcal{M}$ *if it satisfies the following conditions: $v[\psi] = v[\psi']$ for every two sentences ψ, ψ' of $L(D)$, such that $\psi \sim^S \psi'$, $v[p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[p][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_n]]$, and:*

$$v[\mathcal{Q}x_1, ..., x_k(\psi_1, ..., \psi_n)] \in$$

$$\tilde{\mathcal{Q}}_{\mathcal{M}}[\{\langle v[\psi_1\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}], ..., v[\psi_n\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}]\rangle \mid a_1, ..., a_k \in D\}]$$

for every (n, k)-ary quantifier $\mathcal{Q}$ of L.

Note that in case $\mathcal{Q}$ is a propositional connective (for $k = 0$), the function $\tilde{\mathcal{Q}}$ is applied to a singleton, as was explained above.

Definition 11 (Model, $\mathcal{M}$-validity, $\vdash_{\mathcal{M}}$). *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$.*

1. *An $\mathcal{M}$-legal S-valuation v is* a model of a sentence ψ in $\mathcal{M}$*, denoted by $S, v \models_{\mathcal{M}} \psi$, if $v[\psi] \in \mathcal{G}$.*
2. *A formula ψ is* $\mathcal{M}$-valid in S *if for every S-substitution σ and every $\mathcal{M}$-legal S-valuation v, $S, v \models_{\mathcal{M}} \sigma[\psi]$. A formula ψ (a set of formulas Γ) is $\mathcal{M}$-valid if ψ (every $\psi \in \Gamma$) is $\mathcal{M}$-valid in every L-structure for $\mathcal{M}$.*
3. *A sequent $\Gamma \Rightarrow \Delta$ is* $\mathcal{M}$-valid in S *if for every $\mathcal{M}$-legal S-valuation v and every S-substitution σ: $S, v \models_{\mathcal{M}} \sigma[\Gamma]$ implies that there exists some $\psi \in \Delta$, such that $S, v \models_{\mathcal{M}} \sigma[\psi]$. A sequent is $\mathcal{M}$-valid if it is $\mathcal{M}$-valid in every structure.*
4. *The* consequence relation $\vdash_{\mathcal{M}}$ induced by $\mathcal{M}$ *is defined as follows: $\Gamma \vdash_{\mathcal{M}} \Delta$ if $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid.*
5. *An Nmatrix $\mathcal{M}$ is* sound *for a system G if $\vdash_G \subseteq \vdash_{\mathcal{M}}$. An Nmatrix $\mathcal{M}$ is* complete *for a system G if $\vdash_{\mathcal{M}} \subseteq \vdash_G$.*

Definition 12 (Strong soundness). *An Nmatrix $\mathcal{M}$ is* strongly sound *for a system G if: (i) $\mathcal{M}$ is sound for G, and (ii) for every inference rule R of G and every L-structure S: if the premises of R are $\mathcal{M}$-valid in S, the conclusion of R is $\mathcal{M}$-valid in S.*

Definition 13. *An Nmatrix $\mathcal{M}$ is a* characteristic Nmatrix *for a canonical system G if $\vdash_{\mathcal{M}} = \vdash_G$.*

A characteristic Nmatrix $\mathcal{M}$ for G is strongly characteristic *if it is strongly sound for G.*

4.2 Semantics for Simplified Languages L_k^n

In addition to L-structures for languages with (n, k)-ary quantifiers, we also use L_k^n-structures for the simplified languages L_k^n, using which the canonical rules are formulated. To make the distinction clearer, we shall use the metavariable S for the former and $\mathcal{N}$ for the latter. Since the formulas of L_k^n are always atomic, the specific 2Nmatrix for which $\mathcal{N}$ is defined is immaterial, and can be omitted. We may even speak of classical validity of sequents over L_k^n. Furthermore, instead of speaking of $\mathcal{M}$-validity of a set of clauses Θ over L_k^n, we may speak simply of validity.

Next we define the notion of a *distribution* of L_k^n-structures.

Definition 14. *Let $\mathcal{N}$ be a structure for L_k^n. $Dist_{\mathcal{N}}$, the distribution of $\mathcal{N}$ is defined as follows:*

$$Dist_{\mathcal{N}} = \{\langle I[p_1][a_1, ..., a_k], ..., I[p_n][a_1, ..., a_k]\rangle \mid a_1, ..., a_k \in D\}$$

We say that an L_k^n-structure is $\mathcal{E}$-characteristic if $Dist_{\mathcal{N}} = \mathcal{E}$.

Note that the distribution of an $\mathcal{L}_0^n$-structure $\mathcal{N}$ is $Dist_{\mathcal{N}} = \{\langle I[p_1], ..., I[p_n]\rangle\}$ and so it is always singleton. Furthermore, the validity of a set of clauses over $\mathcal{L}_0^n$ can be reduced to propositional satisfiability as stated in the following lemma.

Lemma 2. *For every $\mathcal{L}_0^n$-structure $\mathcal{N}$, such that $Dist_{\mathcal{N}} = \{\langle a_1, ..., a_n\rangle\}$, let $v_{Dist_{\mathcal{N}}}$ be any propositional valuation satisfying $v[p_i] = a_i$. A set of clauses Θ is valid in a $Dist_{\mathcal{N}}$-characteristic $\mathcal{L}_0^n$-structure $\mathcal{N}$ iff $v_{Dist_{\mathcal{N}}}$ propositionally satisfies Θ.*

Now we turn to the case $k = 1$. In this case it is convenient to define a special kind of $\mathcal{L}_1^n$-structures which we call *canonical* structures, which will be sufficient to reflect the behavior of all possible $\mathcal{L}_1^n$-structures.

Definition 15. *Let $\mathcal{E} \in P^+(\{t, f\}^n)$. A $\mathcal{L}_1^n$-structure $\mathcal{N} = \langle D, I\rangle$ is $\mathcal{E}$-canonical if $D = \mathcal{E}$ and for every $b = \langle a_1, ..., a_n\rangle \in D$ and every $1 \leq i \leq n$: $I[p_i][b] = a_i$.*

Clearly, every $\mathcal{E}$-canonical $\mathcal{L}_1^n$-structure is $\mathcal{E}$-characteristic.

Lemma 3. *Let Θ be a set of clauses over $\mathcal{L}_1^n$, which is valid in a structure $\mathcal{N} = \langle D, I\rangle$. Then there exists a $Dist_{\mathcal{N}}$-canonical structure $\mathcal{N}'$ in which Θ is valid.*

Proposition 2. *Let $\mathcal{E} \in P^+(\{t, f\}^n)$. For a finite set of clauses Θ over L_k^n, the question whether Θ is valid in a $\mathcal{E}$-characteristic structure is decidable.*

5 Canonical Systems with (n,k)-ary Quantifiers for $k \in \{0, 1\}$

Now we turn to the class of systems with (n, k)-ary quantifiers for $k \in \{0, 1\}$ and $n \geq 1$. Henceforth, unless stated otherwise, assume that $k \in \{0, 1\}$. For a uniform treatment of both $k = 0$ and $k = 1$, we use the following notation. For any variable x and any constant c, let x^0 and c^0 denote the empty string, and x^1, c^1 denote the strings 'x' and 'c' respectively. When we write $\mathcal{Q}x^k(\psi_1, ..., \psi_n)$, we mean $\mathcal{Q}x(\psi_1, ..., \psi_n)$ if $k = 1$ and $\mathcal{Q}(\psi_1, ..., \psi_n)$ if $k = 0$; when we write $\psi\{\mathbf{t}/x^k\}$, we mean $\psi\{\mathbf{t}/x\}$ for $k = 1$, and ψ for $k = 0$.

In this section we show that any coherent canonical calculus G has a characteristic 2Nmatrix and admits cut-elimination. We start by defining the notion of *suitability* for G.

Definition 16 (Suitability for G). *Let G be a canonical calculus over L. A 2Nmatrix $\mathcal{M}$ is* suitable *for G if for every (n, k)-ary canonical rule $\Theta/A^{-s} \Rightarrow A^s$ of G, where $s \in \{t, f\}$ and $A = \mathcal{Q}x^k(p_1(x^k), ..., p_n(x^k))$ it holds that for every L^n_k-structure $\mathcal{N}$ in which Θ is valid: $\tilde{\mathcal{Q}}_{\mathcal{M}}[Dist_{\mathcal{N}}] = \{s\}$.*

Next we prove that any 2Nmatrix $\mathcal{M}$ suitable for G is strongly sound for G. But first we transform G into a canonical calculus G', satisfying a certain property defined below.

Lemma 4 (Elimination of constants). *Let G be a canonical calculus with a canonical (1,n)-ary rule $R = \Theta/\mathcal{Q}(s)$ for some $s \in \{t, f\}$, where there are two clauses of the form $\Sigma_1, p_i(c) \Rightarrow \Pi_1$ and $\Sigma_2 \Rightarrow p_i(c), \Pi_2$ in Θ. Let $R' = \Theta'/\mathcal{Q}(s)$, where Θ' is obtained from Θ by replacing these two clauses for the clause $\Sigma_1, \Sigma_2 \Rightarrow \Pi_1, \Pi_2$. Let G' be the calculus obtained from G by replacing R for R'. Then any 2Nmatrix strongly sound for G', is also strongly sound for G.*

Corollary 1. *Let G be a canonical calculus. Then there exists a calculus G', such that (i) any 2Nmatrix strongly sound for G', is also strongly sound for G, and (ii) for every (n,1)-ary rule $\Theta/\mathcal{Q}(s)$ of G' and every clause $\Sigma_1, p_i(c)^r \Rightarrow \Pi_1, p_i(c)^{-r} \in \Theta$: there is no clause of the form $\Sigma_2, p_i(c)^{-r} \Rightarrow \Pi_2, p_i(c)^r$ in Θ.*

Proof. Easily follows by repeatedly applying lemma 4.

Theorem 1. *Let G be a canonical calculus over L and $\mathcal{M}$ - a 2Nmatrix suitable for G. Then $\mathcal{M}$ is strongly sound for G.*

Proof. Clearly, we may assume that G satisfies condition (ii) from corollary 1. Let S be an L-structure. Let R be an (n, k)-ary rule $R = \{\Sigma_j \Rightarrow \Pi_j\}_{1 \leq j \leq m}/ \Rightarrow \mathcal{Q}x^k(p_1(x^k), ..., p_n(x^k))$ of G'. Consider an application of R:

$$\frac{\{\Gamma, \chi[\Sigma_j] \Rightarrow \chi[\Pi_j], \Delta\}_{1 \leq j \leq m}}{\Gamma \Rightarrow \Delta, \mathcal{Q}z^k(\chi[p_1], ..., \chi[p_n])}$$

where χ is some $\langle R, \Gamma \cup \Delta, z^k\rangle$-mapping. It suffices to show that if the premises are $\mathcal{M}$-valid in S, then the conclusion is $\mathcal{M}$-valid in S. Let σ be an S-substitution and v an $\mathcal{M}$-legal valuation, such that $S, v \models_{\mathcal{M}} \sigma[\Gamma]$ and for every $\psi \in \Delta$: $S, v \not\models_{\mathcal{M}} \sigma[\psi]$. Denote by $\overrightarrow{\psi}$ the L-formula obtained from a formula ψ by substituting every free occurrence of $w \in Fv[\psi] - \{z^k\}$ for $\sigma[w]$. Let

$$\mathcal{E} = \{\langle v[\overrightarrow{\chi[p_1]}\{\overline{a}/z^k\}], ..., v[\overrightarrow{\chi[p_n]}\{\overline{a}/z^k\}]\rangle \mid a \in D\}$$

Define the L_k^n-structure $\mathcal{N} = \langle D', I' \rangle$: $D = D'$ and I' is defined as follows:

- For every $p_i(c) \in ite(s, \Sigma_j, \Pi_j)$ for some $1 \leq j \leq m$ and $s \in \{t, f\}$: if there is some $a \in D$, such that $v[\overrightarrow{\chi[p_i]}\{\overline{a}/z\}] = -s$, choose $I'[c]$ to be any such a (note that in this case $\Pi_j \Rightarrow \Sigma_j$ becomes valid); otherwise, choose $I'[c]$ to be any $a \in D$. It is important to stress that this is well-defined due to the special property of G', namely that $p_i(c)$ cannot occur on two different sides of a clause.
- For every $a \in D$: $I'[p_i][a^k] = v[\overrightarrow{\chi[p_i]}\{\overline{a}/z^k\}]$

It is easy to show that $\{\Sigma_j \Rightarrow \Pi_j\}_{1 \leq j \leq m}$ is valid in $\mathcal{N}$. Obviously, $Dist_{\mathcal{N}} = \mathcal{E}$ and since $\mathcal{M}$ is suitable for G: $\tilde{\mathcal{Q}}_{\mathcal{M}}[\mathcal{E}] = \{t\}$ and so $v[\sigma[\mathcal{Q}z(\chi[p_1], ..., \chi[p_n])]] = t$ and the conclusion of the application is $\mathcal{M}$-valid in S. □

Now we come to the construction of a characteristic 2Nmatrix for a coherent canonical calculus.

Definition 17. *Let G be a coherent canonical calculus. The Nmatrix $\mathcal{M}_G$ for L is defined as follows. For every (n,k)-ary quantifier $\mathcal{Q}$ of L, every $s \in \{t, f\}$ and every $\mathcal{E} \in P^+(\{t, f\}^n)$:*

$$\tilde{\mathcal{Q}}_{\mathcal{M}_G}[\mathcal{E}] = \begin{cases} \{s\} & \textit{if } \Theta/\mathcal{Q}(s) \in G \textit{ and} \\ & \Theta \textit{ is valid in some } \mathcal{E} - \textit{characteristic structure} \\ \{t, f\} & \textit{otherwise} \end{cases}$$

First let us show that $\mathcal{M}_G$ is well-defined. Assume by contradiction that there are two rules $\Theta_1/ \Rightarrow A$ and $\Theta_2/A \Rightarrow$, such that both Θ_1 and Θ_2 are valid in some $\mathcal{E}$-characteristic structures $\mathcal{N}_1 = \langle D_1, I_1 \rangle, \mathcal{N}_2 = \langle D_2, I_2 \rangle$ respectively. If $k = 0$, by lemma 2, the set of clauses $\Theta_1 \cup \Theta_2$ is propositionally satisfiable by $v_{\mathcal{E}}$ and is thus classically consistent, in contradiction to the coherence of G.

If $k = 1$, by lemma 3 there are $\mathcal{E}$-canonical structures $\mathcal{N}_1', \mathcal{N}_2'$ in which Θ_1, Θ_2 are valid. Recall that the only difference between different $\mathcal{E}$-canonical structures is in the interpretation of constants, and since the sets of constants occurring in Θ_1 and Θ_2 are disjoint, an $\mathcal{E}$-canonical structure $\mathcal{N}' = \langle D', I' \rangle$ (for the extended language containing the constants of both Θ_1 and Θ_2) can be constructed, in which $\Theta_1 \cup \Theta_2$ are valid. Thus the set $\Theta_1 \cup \Theta_2$ is classically consistent, in contradiction to the coherence of G.

Let us demonstrate the construction of a characteristic 2Nmatrix for some simple coherent canonical calculi.

Example 5. It is easy to see that for any canonical coherent calculus G including the standard (1,1)-ary rules for $\forall$ and $\exists$ from Example 2:

$$\tilde{\forall}_{\mathcal{M}_G}[\{t, f\}] = \tilde{\forall}_{\mathcal{M}_G}[\{f\}] = \tilde{\exists}_{\mathcal{M}_G}[\{f\}] = \{f\}$$

$$\tilde{\forall}_{\mathcal{M}_G}[\{t\}] = \tilde{\exists}_{\mathcal{M}_G}[\{t, f\}] = \tilde{\exists}_{\mathcal{M}_G}[\{t\}] = \{t\}$$

Example 6. Consider the canonical calculus G' consisting of the following two $(1,2)$-ary rules from Example 3:

$$\{p_1(y) \Rightarrow p_2(y)\}/ \Rightarrow \overline{\forall}x\ (p_1(x), p_2(x))$$

and

$$\{\Rightarrow p_1(c)\ ,\ \Rightarrow p_2(c)\}/ \Rightarrow \overline{\exists}x(p_1(x), p_2(x))$$

It is easy to see that G' is coherent. The 2Nmatrix $\mathcal{M}_{G'}$ is defined as follows for every $H \in P^+(\{t,f\}^2)$:

$$\tilde{\overline{\forall}}[H] = \begin{cases} \{t\} & \textit{if } \langle t,f\rangle \notin H \\ \{t,f\} & \textit{otherwise} \end{cases} \qquad \tilde{\overline{\exists}}[H] = \begin{cases} \{t\} & \textit{if } \langle t,t\rangle \in H \\ \{t,f\} & \textit{otherwise} \end{cases}$$

Remark. The construction of $\mathcal{M}_G$ above is much simpler than the constructions carried out in [1,14]: a canonical calculus there is first transformed into an equivalent normal form calculus, which is then used to construct the characteristic Nmatrix. The idea is to transform the calculus so that each rule dictates the interpretation for only one $\mathcal{E}$. However, the above definitions show that the transformation into normal form is actually not necessary and we can construct $\mathcal{M}_G$ directly from G.

Now we come to the main theorem, establishing that $\mathcal{M}_G$ is sound and complete for any coherent calculus G.

Theorem 2 (Soundness and cut-free completeness). *Let G be a coherent canonical calculus. Then a sequent $\Gamma \Rightarrow \Delta$ satisfying the free-variable condition*[4] *has a cut-free proof in G iff $\Gamma \vdash_{\mathcal{M}_G} \Delta$.*

Proof. *Soundness*: It is easy to see that $\mathcal{M}_G$ is suitable for G. By theorem 1, $\mathcal{M}_G$ is strongly sound for G, and thus $\vdash_G \subseteq \vdash_{\mathcal{M}_G}$.

Cut-free completeness: Let $\Gamma \Rightarrow \Delta$ be a sequent satisfying the free-variable condition. Suppose that $\Gamma \Rightarrow \Delta$ has no cut-free proof in G. We will show that it is not $\mathcal{M}_G$-valid.

It is easy to see that we can limit ourselves to the language L^*, which is a subset of L, consisting of all the constants and predicate and function symbols, occurring in $\Gamma \Rightarrow \Delta$. Let $\mathbf{T}$ be the set of all the terms in L^* which do not contain variables occurring bound in $\Gamma \Rightarrow \Delta$. It is a standard matter to show that Γ, Δ can be extended to two (possibly infinite) sets Γ', Δ' (where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$), satisfying the following properties:

1. For every $\Gamma_1 \subseteq \Gamma'$ and $\Delta_1 \subseteq \Delta'$, $\Gamma_1 \Rightarrow \Delta_1$ does not have a cut-free proof in G.
2. There are no $A \in \Gamma'$ and $B \in \Delta'$, such that $A \equiv_\alpha B$.
3. If $\{\Pi_j \Rightarrow \Sigma_j\}_{1 \le j \le m}/\mathcal{Q}(r)$ is an (n,k)-ary rule of G and $\mathcal{Q}z^k\ (A_1, ..., A_n) \in ite(r, \Delta', \Gamma')$, then there is some $1 \le j \le m$, such that:

[4] By the free-variable condition we mean that the set of bound variables of $\Gamma \cup \Delta$ is disjoint from its set of free variables.

- If $p_i(c^k) \in ite(r, \Pi_j, \Sigma_j)$ for some $1 \leq i \leq n$, then $A_i\{\mathbf{t}/z^k\} \in ite(r, \Gamma', \Delta')$ for every $\mathbf{t} \in \mathbf{T}$.
- If $p_i(y^k) \in ite(r, \Pi_j, \Sigma_j)$ for some $1 \leq i \leq n$, then there exists some $\mathbf{t} \in \mathbf{T}$, such that $A_i\{\mathbf{t}/z^k\} \in ite(r, \Gamma', \Delta')$.

Note that for the case of $k = 1$, $\mathbf{t}$ is free for z in A_i by the free-variable condition.

Let $S = \langle \mathbf{T}, I \rangle$ be the L^*-structure defined as follows:

- $I[c] = c$ for every constant c of L^*.
- $I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$ for every n-ary predicate symbol p.
- $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$ for every n-ary function symbol f.

Let σ^* be any S-substitution satisfying $\sigma^*[x] = \overline{x}$ for every $x \in \mathbf{T}$. (Note that every $x \in \mathbf{T}$ is also a member of the domain and thus has an individual name referring to it in $L^*(D)$.)

It is easy to show that (i) $I^*[\sigma^*[\mathbf{t}]] = \mathbf{t}$ for every $\mathbf{t} \in \mathbf{T}$, and (ii) for every $A, B \in \Gamma' \cup \Delta'$: if $\sigma^*[A] \sim^S \sigma^*[B]$, then $A \equiv_\alpha B$.

Define the S-valuation v as follows:

- $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
- For every (n, k)-ary quantifier $\mathcal{Q}$, if there is some $C \in \Gamma' \cup \Delta'$, such that $\sigma^*[C] \equiv_\alpha \mathcal{Q}x^k(\psi_1, ..., \psi_n)$, then $v[\mathcal{Q}x^k(\psi_1, ..., \psi_n)] = t$ iff $C \in \Gamma'$. Otherwise $v[\mathcal{Q}x^k(\psi_1, ..., \psi_n)] = t$ iff $\tilde{\mathcal{Q}}[\{\langle v[\psi_1\{\overline{a}/x^k\}], ..., v[\psi_n\{\overline{a}/x^k\}]\rangle \mid a \in D\}] = \{t\}$.

Lemma 5. *1. v is legal in $\mathcal{M}_G$.*
2. For every $\psi \in \Gamma' \cup \Delta'$: $v(\sigma^*[\psi]) = t$ iff $\psi \in \Gamma'$.

Since v is legal in $\mathcal{M}_G$, $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, by the above lemma v refutes $\Gamma \Rightarrow \Delta$. □

Corollary 2. *If G is coherent, then $\mathcal{M}_G$ is strongly characteristic for G.*

Corollary 3. *For any canonical calculus G, the following two statements are equivalent:*

1. *G has a strongly characteristic 2Nmatrix.*
2. *G is coherent.*

Proof. The proof of (1⇒2) is easy and is left to the reader. (2⇒1) follows from corollary 2.

Corollary 4. *The existence of a strongly characteristic 2Nmatrix for a canonical calculus G is decidable.*

Remark. It is important to note that the coherence of G is *not* a necessary condition for the existence of a characteristic 2Nmatrix for G and, consequently, for cut-elimination. Consider, for instance the canonical calculus G_1 consisting of the following two inference rules:

$$(1)\ \{p_1(y) \Rightarrow p_2(y)\ ,\ p_1(c_1) \Rightarrow\ ,\ p_2(c_1) \Rightarrow\ ,$$
$$p_1(c_2) \Rightarrow\ ,\ \Rightarrow p_2(c_2)\ ,\ \Rightarrow p_1(c_3)\ ,\ \Rightarrow p_2(c_3)\}/ \Rightarrow \mathcal{Q}z(p_1(z), p_2(z))$$
$$(2)\ \{p_1(y) \Rightarrow p_2(y)\ ,\ p_1(c_4) \Rightarrow\ ,\ p_2(c_4) \Rightarrow\ ,$$
$$p_1(c_5) \Rightarrow\ ,\ \Rightarrow p_2(c_5)\ ,\ \Rightarrow p_1(c_6)\ ,\ \Rightarrow p_2(c_6)\}/\mathcal{Q}z(p_1(z), p_2(z)) \Rightarrow$$

It is easy to see that G_1 is not coherent, but the only sequents provable in it are logical axioms, and so G_1 has a characteristic 2Nmatrix and admits cut-elimination. This is in contrast to the systems in [1,14], where the fact that a canonical calculus admits cut-elimination implies that G is coherent.

6 Summary and Further Research

In this paper we have considerably extended the characterization of canonical calculi of [1,14] to (n, k)-ary quantifiers. For the case of $k \in \{0, 1\}$, we have shown that any coherent calculus admits cut-elimination and has a characteristic 2Nmatrix, but the converse does not necessary hold (unlike in [1,14]). In fact, a calculus is coherent iff it has a strongly characteristic 2Nmatrix. In addition to some proof-theoretical results for a natural type of multiple conclusion Gentzen-type systems with $(n, 1)$-ary quantifiers, our work also provides further evidence for the thesis that the meaning of a logical constant is given by its introduction (and "elimination") rules . We have shown that at least in the framework of multiple-conclusion consequence relations, any "reasonable" set of canonical quantificational rules completely determines the semantics of the quantifier.

Some of the most immediate research directions are:

1. Defining an exact criterion for the ability to eliminate cuts in canonical systems and developing a syntactic method for cut-elimination for the case of $k \in \{0, 1\}$, i.e. a stepwise transformation of any derivation of a canonical calculus into a cut-free derivation, possibly along the lines of [3].
2. Developing a general theory, extending the results of the previous section to the case of $k > 1$. This might lead to new insights on Henkin quantifiers and other important extensions, such as Transitive Closure operations. However, already for the simplest quantifiers this is not straightforward. First of all, the coherence of a canonical calculus G with general quantifiers *does not* imply that a 2Nmatrix suitable for G exists. For instance, consider the calculus G, consisting of the following two (1,2)-ary rules:

$$\{p(c, x) \Rightarrow\}/ \Rightarrow \mathcal{Q}z_1z_2p(z_1, z_2) \quad \{\Rightarrow p(y, d)\}/\mathcal{Q}z_1z_2p(z_1, z_2) \Rightarrow$$

   G is coherent, but it is easy to see that $\mathcal{M}_G$ is not well-defined in this case. Secondly, even if a 2Nmatrix $\mathcal{M}$ suitable for G does exist, it is not necessarily sound for G. Therefore, a more complex interpretation of quantifiers is needed, which in its turn will lead to various extensions of the simplified language L^n_k (e.g. adding function symbols), and the cost of losing the decidability of the coherence criterion in this case seems inevitable.

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