On the Effectiveness of the Linear Programming Relaxation of the 0-1 Multi-commodity Minimum Cost Network Flow Problem

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Abstract. Several studies have reported that the linear program relaxation of integer multi-commodity network flow problems often provides integer optimal solutions. We explore this phenomenon with a 0-1 multi-commodity network with mutual arc capacity constraints. Characteristics of basic solutions in the linear programming relaxation problem of the 0-1 multi-commodity problem are identified. Specifically, necessary conditions for a linear programming relaxation to have a non-integer solution are presented. Based on the observed characteristics, a simple illustrative example problem is constructed to show that its LP relaxation problem has integer optimal solutions with a relatively high probability. Furthermore, to investigate whether or not and under what conditions this tendency applies to large-sized problems, we have carried out computational experiments by using randomly generated problem instances. The results of our computational experiment indicate that there exists a narrow band of arc density in which the 0-1 multi-commodity problems possess no integer optimal solutions.

1 Introduction

The integer multi-commodity minimum cost network flow problem (IMNFP), which has been applied in various fields such as transportation, production, and communication systems, involves finding optimal integral flows that satisfy arc capacity constraints on an underlying network. The problem is known to be NP-hard even in its simplest form, viz. in a planar graph with unit arc capacities [1]. Moreover, coupled by various side constraints, many IMNFP problems in practice usually take further complication. Subsequently, several studies have developed heuristic procedures or efficient branch-and-bound based procedures for IMNFP problems with side constraints ([2],[3],[5],[7],[9],[11], [12]).

Some of these studies have reported that the linear program (LP) relaxation of instances of the IMNFP with or without side constraints often provides integer optimal solutions or excellent bounds ([5], [9], [12]). Löbel [9] considered the IMNFP with arc cover constraints as coupling constraints for vehicle scheduling in public transit

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and suggested the column generation technique for solving large-scale linear problems. In his computational experiments using instances based on real-world data, the LP relaxation gave tight bounds in several problem instances. In fact, the LP relaxation was observed to yield integer optimal solutions for a few problem instances. Faneyte, Spieksma, and Woeginger[5] studied the 0-1 MNFP with node cover constraints as coupling constraints for the crew-scheduling problem and presented a branch-and-price algorithm for arc-chain formulation. Their computational experiments also showed that the LP relaxation for most of the instances based on practical data for a crane rental company gave an integer optimal solution. In addition, they indicated the short length of feasible paths in their instances as one possible explanation for this phenomenon. For some instances with longer possible paths, however, the LP relaxation provided an integer optimal solution. For the IMNFP on a ring network, Ozdaglar and Bertsekas [12] reported that the LP relaxation gave an integer optimal solution for almost all instances. Moreover, similar findings for some instances of the IMNFP with side constraints, viz. the LP relaxation often provides an integer optimal solution, are observed in [2], [3], [7], and [11].

In some special classes of the IMNFP, it has been shown that the LP relaxation gives an integer optimal solution. Evans [4] described a sufficient condition under which the IMNFP can be transformed into an equivalent single-commodity problem, and Kleitmann, Martin-Lof, Rothschild, and Whinston [8] showed that if each node in a network were a source or sink for at least (k-1) of the k commodities, the optimal solution would be integral. Along this line of research, we investigate the 0-1 MNFP, a sub-class problem of IMNFP, to explore the effectiveness of the LP relaxation. The 0-1 MNFP has been applied to several practical problems, such as the crew-scheduling problem and telecommunications ([2], [5], [11]).

We have identified some characteristics of basic feasible solutions of the LP relaxation. Also, by using them, we have constructed an example to show that its LP relaxation problem has integer optimal solutions with a relatively high probability. As the construction of the example is pathological, we have conducted computational experiments by using randomly generated problem instances in order to investigate whether or not and under what conditions the LP relaxation provides integer optimal solutions.

This paper is organized as follows. Section 2 below describes the problem under consideration, along with some definitions of notation. It also includes the characteristics of basic feasible solutions of the LP relaxation and a simple illustrative example. Section 3 discusses the results of computational experiments and Section 4 contains concluding remarks.

2 The 0-1 MNFP and Characteristics of the Problem

We consider the 0-1 MNFP on a digraph G(V, E) with a node set V and an arc set E. Given a set of commodities K, each commodity k is assumed to have a single origin and single destination. Let i and j be an arc and path index, respectively. Let Ψ^k and P_j denote the set of origin-destination paths of commodity k and the set of arcs in path j, respectively. Also, let c_j ($j \in \Psi^k$) and u_i represent the cost of shipping commodity k along path j and the bundle (mutual arc) capacity of arc i, respectively. Without loss of generality, we will assume that all arcs have capacities. Moreover, δ_{ij} denotes the Kronecker delta to indicate whether an arc i belongs to P_j ; i.e. δ_{ij} equals 1 if $i \in P_j$ and equals zero otherwise. The decision variable y_j is a binary variable to indicate whether or not commodity is shipped along path j. Then, the arc-chain formulation of the 0-1 MNFP is expressed as

$$Min \quad \sum_{k} \sum_{j \in \psi^k} c_j y_j$$

$$s.t \quad \sum_{j \in \psi^k} y_j = 1 \; for \; k \in K \tag{1}$$

$$\sum_{k} \sum_{j \in \psi^k} \delta_{ij} y_j + z_i = u_i \; for \; i \in E \tag{2}$$

$$y_j \in \{0,1\} \; for \; j \in \psi^k, k \in K \tag{3}$$

In the above formulation, z_i is a slack variable. Constraints (1) represent that each commodity must be shipped on a unique path and Constraints (2) represent that total flow on an arc can not exceed its capacity. An LP relaxation problem is obtained by relaxing integrality constraints (3) and the rank of the constraint matrix in the relaxed problem is ($|K|$+$|E|$), where $|K|$ denotes the cardinality of set K. Given a basic feasible solution of the LP relaxation problem, let B_y and B_z denote index sets of basic path variables and basic slack variables, respectively. Moreover, we use B_y^k as an index set of basic path variables for commodity k. Also, let N_z denote a set of arcs of which slack variables are nonbasic. As the arcs that belong to N_z are saturated, we will call them *saturated nonbasic arcs*. Note that there can be saturated basic arcs because of degeneracy.

In a basic feasible solution of the LP relaxation problem, at least one path variable should be basic for each commodity, i.e. $|B_y^k| \geq 1$ for all k. Thus, a basic matrix B of the LP relaxation problem can be expressed as

$$B = \begin{bmatrix} \overbrace{I}^{|K|} & \overbrace{D_1}^{|N_z|} & \overbrace{0}^{|B_z|} \\ D_2 & D_3 & 0 \\ D_4 & D_5 & I \end{bmatrix} \begin{matrix} \}|K| \\ \}|N_z| \\ \}|B_z| \end{matrix} \tag{4}$$

Each of the first $|K|$ columns corresponds to one path variable for each commodity, which we shall call a *primary path variable*. The next $|N_z|$ columns correspond to non-primary path variables in the basis and the last $|B_z|$ columns correspond to slack variables of the arcs in B_z. The first $|K|$ rows are flow constraints and the next $|N_z|$ and last $|B_z|$ rows correspond to the capacity constraints of saturated arcs in N_z and basic arcs in B_z, respectively. Now, let $D = D_3 - D_2D_1$. Then, the inverse of the basic matrix is expressed as

$$B^{-1} = \begin{bmatrix} I + D_1D^{-1}D_2 & -D_1D^{-1} & 0 \\ -D^{-1}D_2 & D^{-1} & 0 \\ -D_4 + (D_5 - D_4D_1)D^{-1}D_2 & -(D_5 - D_4D_1)D^{-1} & I \end{bmatrix}$$

The integrality of basic solutions is closely related to the characteristics of matrix D. The matrix D is square and its rank is $|N_z|$. Moreover, it can be described by using the relationship of paths and saturated arcs. We assume that, in the matrix B, the $(i+|K|)^{th}$ row corresponds to the capacity constraints of arc i and the j^{th} column corresponds to path j. Also, we assume the r^{th} column (for r=1,2,…,$|K|$) corresponds to the primary path of commodity r. Let i=1,2,…, $|N_z|$ and j=$|K|$+1, …, $|K|+|N_z|$ be the arc and path index, respectively. Also, let k_j be an commodity index of path j. In addition, $(D)_{ij}$, $(D)_{i.}$, and $(D)_{.j}$ denote an element, row, and column of matrix D, respectively. Element $(D_1)_{r,j-|K|}$ = 1 if $r= k_j$; otherwise it is zero. It notes that $(D_1)_{r,j-|K|}$ = 1 if path j is a path of commodity r. Moreover, element $(D_2)_{ir}$=1 if $i \in P_r$, otherwise $(D_2)_{ir}$=0. Therefore,

$$(D_2D_1)_{i,j-|K|} = (D_2)_{i\cdot}(D_1)_{\cdot j-|K|} = \begin{cases} 1 \text{ if } i \in P_{k_j} \\ 0 \text{ otherwise} \end{cases}$$

Element $(D_3)_{ij}$ is given as

$$(D_3)_{i,j-|K|} == \begin{cases} 1 \text{ if } i \in P_j \\ 0 \text{ otherwise} \end{cases}$$

Then, the element of matrix D is given as, for i=1,2,…, $|N_z|$ and j=$|K|$+1, …, $|K|+|N_z|$

$$(D)_{i,j-|K|} = (D_3)_{i,j-|K|} - (D_2)_{i\cdot}(D_1)_{\cdot j-|K|} = \begin{cases} 1 \ \ \textit{if } i \in P_j, i \notin P_{k_j} \\ -1 \ \textit{if } i \notin P_j, i \in P_{k_j} \\ 0 \quad \text{otherwise} \end{cases} \tag{5}$$

The above observation clearly indicates that that matrix D can be obtained from the relationship of the basic path variables and saturated nonbasic arcs and that the elements in matrix D are 0, 1, or -1. Similar observation applies to the term D_5 - D_4D_1, which is a term in the inverse of a basic matrix. As a parenthetical note, if D is unimodular, then the corresponding basic feasible solution is integral.

Now, we state some properties of basic solutions.

Proposition 1. Every integer feasible solution of the 0-1 MNFP is a basic feasible solution of its LP relaxation problem.

Proof) For a given integer feasible solution, each commodity has exactly one path with its flow equal to 1. Let the index set of these variables be B_y and the arc set E be B_z. Then, from a constraint matrix, a matrix B consisting of columns corresponding to the path variables in B_y and slack variables of arcs in B_z is given as

$$B = \begin{bmatrix} I_{|K|\times|K|} & 0 \\ D_4 & I_{|E|\times|E|} \end{bmatrix}$$

Matrix B is nonsingular and its rank is ($|K|+|E|$). Therefore, the matrix B is a basis of an LP relaxation. ■

As one of the properties for non-integer basic solutions, a relationship between $|K|$, $|N_z|$, and $|B_y|$ is established in [6] and [10].

Proposition 2. (By Maurras and Vaxès [10] and Farvoleden *et al* [6]) Every non-integer basic feasible solution of the LP relaxation problem satisfies $|B_y| = |N_z|+|K|$.

Proof. From equation (5), the result immediately follows. ■

In the above proposition, if $|N_z|=0$, then $|B_y|=|K|$ and the corresponding solution is integral.

Proposition 3. There should be at least two saturated nonbasic arcs in a non-integer basic feasible solution of the LP relaxation problem; viz. $|N_z| \geq 2$.

Proof. Suppose that $|N_z| < 2$. If $|N_z|=0$, the solution is integral. Moreover, if $|N_z|=1$, matrix D is [1] and the solution is integral. ■

Proposition 3 above can be strengthened by the main proposition of the paper below. It states that there should be at least two saturated nonbasic arcs on the same path for a non-integer basic solution.

Proposition 4. For a non-integer basic feasible solution of the LP relaxation problem with a basic matrix B of the form given in (4), $| N_z \cap P_j | \geq 2$ for some $j \in B_y$.

Proof. Suppose that $| N_z \cap P_j | \leq 1$ for all $j \in B_y$. By (5), each column of matrix D has at most two nonzero elements because $| N_z \cap P_j | \leq 1$ for all $j \in B_y$. Since in this case the matrix D takes the form of node-arc incidence matrix obtained by removing a node in a general network, it is unimodular. This is a contradiction because the solution is assumed to be nonintegral. ■

The above two propositions describe necessary conditions for the LP relaxation problem to have non-integer vertices and they are closely related to the form of basic matrices.

It is possible to approximate the probability that LP relaxations have integer optimal solutions for simple problems, in which we relax the assumption that all arcs have capacities. As an example, we consider an instance with $|K|=2$, $|E|=2$, and four feasible paths for each commodity, in which E is a set of arcs with capacities only. Fig. 1(a) shows the constraint and basic matrices corresponding to a non-integer basic solution and one of their extended forms. The four paths include (i) a path including none of the two capacity-constrained arcs, (ii) a path including both of the two capacity-constrained arcs, and (iii) a couple of paths including exactly one of the two capacity-constrained arcs. Each of the paths may include some uncapacitated arcs that are not included in E, but they will not appear in the constraint matrix. There are nine feasible integer solutions and, by Proposition 1, all of them are basic solutions of the LP relaxation. By Propositions 2 and 3, $|N_z| = 2$, $|B_y| = 4$, $|B_y^k| = 2$ (for k=1,2). Then, two non-integer basic feasible solutions can be drawn to satisfy the necessary condition in Proposition 4. The extended form of a basic matrix corresponds to the case in which one path of commodity 1 includes all saturated nonbasic arcs and each path of commodity 2 includes exclusively one saturated nonbasic arc.

Suppose that the arc capacity is set equal to 1 and the cost for each path is selected randomly among integer values between 1 and C_{max} in the above example. Consider the pseudo-probability P_u that all optimal solutions are integral and the pseudo-probability P_o that at least one of the optimal solutions is integral. To get P_u and P_o for a given C_{max}, we obtained optimal solutions for all possible cases (C_{max}^8 cases) of path

(a) Constraint and basic matrices

A constraint matrix

$$\mathbf{A}=\begin{bmatrix}1&1&1&1&0&0&0&0&0&0\\0&0&0&0&1&1&1&1&0&0\\1&1&0&0&1&1&0&0&1&0\\1&0&1&0&1&0&1&0&0&1\end{bmatrix}\begin{matrix}\}k=1\\\}k=2\\\}i_1\\\}i_2\end{matrix}$$

Basic matrices of non-integer basic solutions

$$\mathbf{B}=\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&1&0&0\\1&0&0&1\end{bmatrix}\qquad \mathbf{B}=\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&1&0&0\\0&1&1&0\end{bmatrix}$$

An extended form of a basic matrix

$$\mathbf{B}=\begin{bmatrix}1&1&0&\cdots&0\\0&0&1&\cdots&1\\1&0&1&0&0\\\vdots&\vdots&0&\ddots&0\\1&0&0&0&1\end{bmatrix}\begin{matrix}\left.\begin{matrix}\\ \\\end{matrix}\right\}|K|=2\\\left.\begin{matrix}\\ \\ \\\end{matrix}\right\}|N_z|\end{matrix}$$

(b) Pseduo-probabilities P_o and P_u according to C_{max}

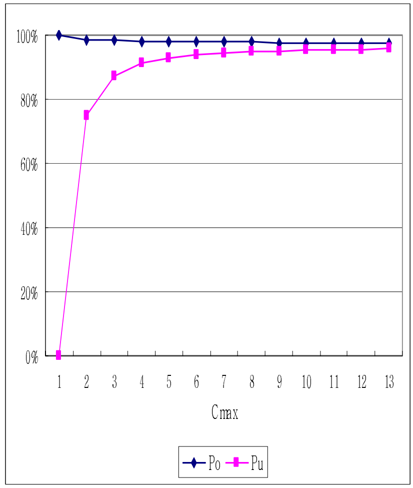

Fig. 1. The constraint and basic matrices of a non-integer basic solution and the pseudo-probabilities P_o and P_u according to C_{max} for an instance with |K|=2, |E|=2

costs and counted the number of corresponding cases. The term "pseudo" is used to indicate that the sample space generated as such provides "pseudo" elementary events, which may not be equally likely. Fig. 1(b) shows the results. When all path costs are identically one, i.e. C_{max}=1, P_u =0 and P_o =1. Moreover, P_u =95.7% and P_o=97.5% at C_{max}=13. Also, pseudo-probability P_u increases while pseudo-probability P_o decreases as C_{max} increases and that they converges as C_{max} increases. Because P_o is always greater than or equals to P_u for a specific value of C_{max} the pseudo-probability P_o is greater than 95.7%. Considering the pseudo-probability and the simple ratio of the number of integer solutions to that of basic feasible solutions (9/11), we conclude that the probability that the LP relaxation has an integer optimal solution must be high in our example problem. For simple instances like the example above, it may be possible to identify all basic feasible solutions and to approximate the probabilities P_o and P_u. In general, however, it will be difficult to calculate the probability for a large-sized problem instance.

3 Computational Results

Conceivably, as the number of uncapacitated arcs increases, so does the chance of obtaining integer solution of the LP relaxation problem. Moreover, there might be a specific band of arc density within which the chance of obtaining non-integer optimal solution of the LP relaxation problem is high. If so, then many observations made by earlier studies on the integrality of the LP relaxation solution could be partially explained.

We performed computational experiments using randomly generated instances of the 0-1 MNFP to search for trends that the LP relaxation had an integer optimal solution in large-size instances. For this purpose, we considered four factors: the number of commodities (|*K*|=10, 30, 50, 70), the arc density (*d*=0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15%), the number of nodes (|*V*|=100, 200, 300), and the maximum arc capacity (U_{max}=1, 0.1*|*K*|). The arc density *d*, defined as the number of arcs over the number of possible arcs, specified the number of arcs in the random instance, viz. *d**(|*V*|-1)*|*V*| arcs, where |*V*| denotes the number of nodes.

To generate instances, we used a path-based generation scheme. First, for each commodity, source and destination nodes were randomly selected and a path from the source to the destination node was constructed. The length of a path was randomly determined between 1 and the minimum of |*V*|-1 and *d**(|*V*|-1)*|*V*|/|*K*|. If needed, additional arcs were randomly generated. In addition, the arc capacities were determined randomly between 1 and the maximum arc capacity U_{max}, and each arc was assigned a random cost between 1 and 100. For each combination of the four factors, 100 instances were generated and Cplex 9.0 was used to solve the generated instances. We obtained LP optimal solutions of the generated problem instances and counted the number of instances of which the optimal solution was integral.

Tables 1 and 2 show the results for U_{max} with 1 and 0.1*|*K*|, respectively. In all but three cases, there were at least 50 instances with an integer optimal solution, particularly more than 90 instances for all cases when U_{max}=0.1*|*K*|. Moreover, the minimum number of instances with an integer optimal solution was 29, when U_{max} =1, |*V*|=300, |*K*|=70, and *d*=1%.

In Table 1, for the given number of nodes and commodities, the number of instances with an integer optimal solution is minimal at a specific arc density and increases when the arc density is far apart from the specific level. There are two explanations for this: in our procedure to generate problem instances, candidate paths likely share few arcs at low arc densities; at high arc densities, there are many arcs and the paths with low cost likely share few arcs.

Table 1. The number of instances in which the LP relaxation yields optimal solution when U_{max}=1

	|*V*|=100				|*V*|=200				|*V*|=300			
d \ |*K*|	10	30	50	70	10	30	50	70	10	30	50	70
0.5%	-	-	-	-	91	100	100	100	92	59	97	99
1%	100	100	100	100	97	64	83	100	100	91	65	28
2%	97	97	100	100	100	91	68	45	100	97	93	81
3%	97	74	71	95	99	97	90	70	100	99	97	93
4%	98	80	58	51	100	99	98	87	100	99	99	100
5%	100	92	70	49	100	98	95	85	100	100	99	98
6%	100	90	78	42	100	98	96	98	100	100	100	99
7%	99	96	88	67	100	100	99	97	100	100	99	100
8%	99	92	83	67	100	100	100	100	100	99	100	99
9%	100	98	85	68	100	100	98	98	100	100	100	99
10%	99	98	90	78	100	99	100	99	100	100	100	100
15%	100	100	97	90	100	100	98	100	100	100	100	100

Table 2. The number of instances in which the LP relaxation yields optimal solution when U_{max}=0.1*|*K*|

	|*V*|=100				|*V*|=200				|*V*|=300			
d \ |*K*|	10	30	50	70	10	30	50	70	10	30	50	70
0.5%		-	-	-	91	100	100	100	92	86	99	100
1%	100	100	100	100	97	89	98	100	100	98	97	98
2%	97	98	100	100	100	100	99	100	100	100	100	100
3%	97	95	98	100	99	100	99	100	100	100	100	99
4%	98	97	98	100	100	100	100	100	100	100	100	100
5%	100	99	100	99	100	100	100	100	100	100	100	100
6%	100	98	99	100	100	100	100	100	100	100	100	100
7%	99	99	100	100	100	100	100	98	100	100	100	100
8%	99	99	99	100	100	100	100	100	100	100	100	100
9%	100	99	100	100	100	100	100	100	100	100	100	100
10%	99	100	99	100	100	99	100	100	100	100	100	100
15%	100	99	100	100	100	100	100	100	100	100	100	100

Although the results in Tables 1 and 2 indicate that instances give integer optimal solutions with a high probability for the given number of nodes and commodities, there may be arc densities at which most instances give non-integer optimal solutions. In our cases indeed, the results reveal that the range of arc densities at which most instances have non-integer optimal solutions is very narrow, if it exists. As an example, consider the case with U_{max}=1, |*V*|=300, and |*K*|=70 in Table 1; most instances may have non-integer optimal solutions at some arc densities between 0.5 and 2%; however, the range of arc densities at which most instances have non-integer optimal solutions is very narrow for this case, if exists, as shown in Fig. 2. In Fig. 2, the minimum number of instances with an integer optimal solution was 21, when *d*=0.8%.

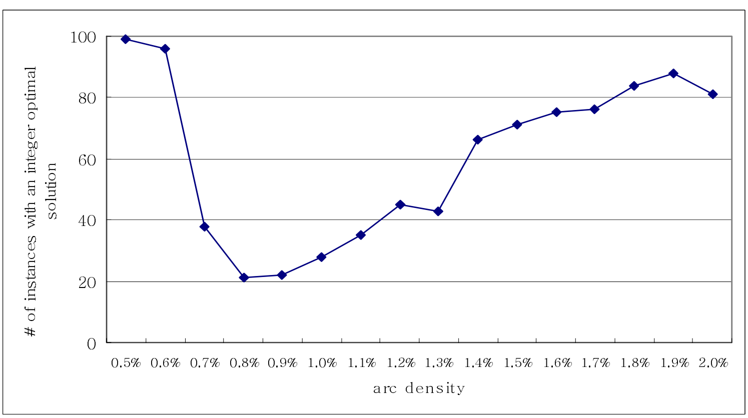

Fig. 2. The result according to the arc density when |*V*|=300 and |*K*|=70

In Table 1, the range of arc densities, at which the number of instances with integer optimal solutions is large, widens as the number of commodities decreases. In the case with U_{max}=1, |V|=100, and |K|=70, there are less than 90 problem instances with integer optimal solutions at arc densities from 4 to 10%. In contrast, for |K|=30, the number is less than 90 at arc densities of 3 and 4%. As the number of nodes increases, the range of arc densities at which there are more than 90 instances with an integer optimal widens.

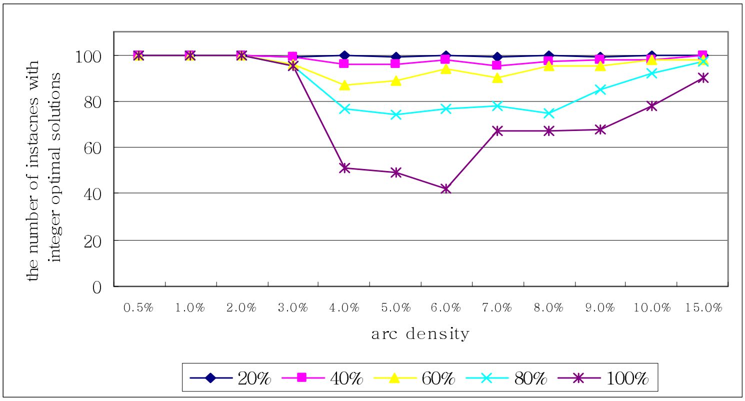

Fig. 3. The results for d_c levels of 20, 40, 60, 80, and 100% when |V|=100 and |K|=70

When the 0-1 MNFP is applied to crew or vehicle scheduling problems, it would have many arcs without capacity. To see the effects of varying the number of arcs with capacity, we performed additional experiments. For |V|=100 and |K|=70, five different ratios d_c of arcs with capacity were used to generate instances, *i.e.*, 20, 40, 60, 80, and 100%, where d_c is defined as the ratio of the number of arcs with capacity over the number of arcs. For each case, 100 instances were generated and tested. Fig. 3 shows that the number of instances with integer optimal solutions increases as d_c decreases.

4 Conclusion

Motivated by the observations made by several studies, in which the LP relaxations of instances of the IMNFP often gave integer optimal solutions or excellent bounds, we have examined the 0-1 MNFP with mutual arc capacity constraints to explore this phenomenon. The characteristics of basic feasible solutions in the LP relaxation were examined and the necessary conditions for a basis in the LP relaxation problem to be non-integral were identified. Our computational experiments showed that the LP relaxation frequently provided an integer optimal solution, except when the arc density was within a specific range. Moreover, when the capacities of the arcs were large or the proportion of arcs with capacities was small, the LP relaxation yielded an integer optimal solution.

Our results are applicable to the 0-1 MNFP with side constraints related to a single path, such as hop constraints, because the constraints are considered implicitly in subproblems used to generate a feasible path. However, further study is needed to address other side constraints such as resource constraints.

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