The On-Line Heilbronn's Triangle Problem in d Dimensions*

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Abstract. In this paper we show a lower bound for the on-line version of Heilbronn's triangle problem in d dimensions. Specifically, we provide an incremental construction for positioning n points in the d-dimensional unit cube, for which every simplex defined by $d+1$ of these points has volume $\Omega(1/n^{(d+1)\ln(d-2)+2})$.

1 Introduction

The *off-line* version of the famous *triangle problem* was posed by Heilbronn [Ro51] more than 50 years ago. It is formulated as follows:

> Given n points in the unit square, what is $\mathcal{H}_2(n)$, the maximum possible area of the *smallest* triangle defined by some three of these points?

There is a large gap between the best currently-known lower and upper bounds on $\mathcal{H}_2(n)$, $\Omega(\log n/n^2)$ [KPS82] and $O(1/n^{8/7-\varepsilon})$ (for any $\varepsilon > 0$) [KPS81]. Jiang et al. [JLV02] showed that the expected area of the smallest triangle, when the n points are put uniformly at random in the unit square, is $\Theta(1/n^3)$. Barequet [Ba01] generalized the off-line problem to d dimensions:

> Given n points in the d-dimensional unit cube, what is $\mathcal{H}_d(n)$, the maximum possible volume of the *smallest* simplex defined by some $d+1$ of these points?

The best currently-known lower bound on $\mathcal{H}_d(n)$ is $\Omega(\log n/n^d)$ [Le03]. Other versions, in which the dimension of the optimized simplex is lower than that of the cube, were investigated in [Le04, BN05, Le05].

The *on-line* version of the triangle problem is harder than the off-line version because the value of n is not specified in advance. In other words, the points are positioned one after the other in a d-dimensional unit cube, while n is incremented by one after every point-positioning step. The procedure can be stopped at any time, and the already-positioned points must have the property that every

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subset of $d+1$ points defines a polytope whose volume is at least some quantity $\mathcal{H}_d^{\text{on-line}}(n)$, where the goal is to maximize this quantity. Schmidt [Sc71] showed that $\mathcal{H}_2^{\text{on-line}}(n) = \Omega(1/n^2)$. Barequet [Ba04] used nested packing arguments to demonstrate that $\mathcal{H}_3^{\text{on-line}}(n) = \Omega(1/n^{10/3}) = \Omega(1/n^{3.333\ldots})$ and $\mathcal{H}_4^{\text{on-line}}(n) = \Omega(1/n^{127/24}) = \Omega(1/n^{5.292\ldots})$.

In this paper we present a nontrivial generalization of the latter method to d dimensions, showing that for a fixed value of d we have $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(\frac{1}{n^{(d+1)\ln(d-2)+2}})$. Specifically, we provide an incremental procedure for positioning n points (one by one) in a d-dimensional unit cube so that no subset of up to $d+1$ points is "too dense." Specifically, the distance between any two points is at least $a_1/n^{1/d}$ (for some constant $a_1 > 0$), no three points define a triangle whose area is less than $a_2/n^{2/(d-1)}$ (for some constant $a_2 > 0$), and so on. The values of the constants are tuned at the end of the construction. It is then proven that all the d-dimensional simplices defined by $(d+1)$-tuples of the points have volume $\Omega(1/n^{(d+1)\ln(d-2)+2})$.

2 The Construction

2.1 Notation and Plan

We use the following notation. Let $p_{i_1}, p_{i_2}, ..., p_{i_q}$ be any q points in $\Re^d$. Then, $|p_{i_1}p_{i_2}|$ denotes the distance between two points p_{i_1}, p_{i_2}; $|p_{i_1}p_{i_2}p_{i_3}|$ denotes the area of the triangle $p_{i_1}p_{i_2}p_{i_3}$; $|p_{i_1}p_{i_2}p_{i_3}p_{i_4}|$ denotes the 3-dimensional volume of the tetrahedron $p_{i_1}p_{i_2}p_{i_3}p_{i_4}$; and, in general, $|p_{i_1}p_{i_2}\ldots p_{i_q}|$ denotes the volume of the $(q-1)$-dimensional simplex $p_{i_1}p_{i_2}\ldots p_{i_q}$. We denote by C^d the d-dimensional unit cube, and by B_r^d a d-dimensional ball of radius r. The line defined by the pair of points p_{i_1}, p_{i_2} is denoted by $\ell_{i_1 i_2}$.

Throughout the construction we refer to d as a fixed constant. Therefore, we omit factors that depend solely on d, except when they appear in powers of n.

We want to construct a set S of n points in C^d such that

[1] $|p_{i_1}p_{i_2}| \geq V_2 = a_1/n^{1/d}$, for any pair of distinct points $p_{i_1}, p_{i_2} \in S$ and for some constant $a_1 > 0$.

[2] $|p_{i_1}p_{i_2}p_{i_3}| \geq V_3 = a_2/n^{2/(d-1)}$, for any triple of distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in S$ and for some constant $a_2 > 0$.

[3] $|p_{i_1}p_{i_2}p_{i_3}p_{i_4}| \geq V_4 = a_3/n^{\frac{4d^2-5d-1}{d(d-1)(d-2)}}$, for any quadruple of distinct points $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4} \in S$ and for some constant $a_3 > 0$.

$\vdots$

[$q-1$] $|p_{i_1}p_{i_2}\ldots p_{i_q}| \geq V_q = a_{q-1}V_{q-1}/(a_{q-2}n^{\frac{d(q-2)+q-3}{d(d-q+2)}})$, for any q-tuple ($4 \leq q \leq d+1$) of distinct points $p_{i_1}, p_{i_2}, \ldots, p_{i_q} \in S$ and for some constant $a_{q-1} > 0$.

The goal is to construct S incrementally. That is, assume that we have already constructed a subset S_v of v points, for $v < n$, which satisfies the above conditions [1]–[$q-1$]. We want to show that there exists a new point $p \in C^d$ that satisfies

[1'] $|pp_{i_1}| \geq V_2 = a_1/n^{1/d}$, for each point $p_{i_1} \in S$.
[2'] $|pp_{i_1}p_{i_2}| \geq V_3 = a_2/n^{2/(d-1)}$, for any pair of distinct points $p_{i_1}, p_{i_2} \in S$.
[3'] $|pp_{i_1}p_{i_2}p_{i_3}| \geq V_4 = a_3/n^{\frac{4d^2-5d-1}{d(d-1)(d-2)}}$, for any triple of distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in S$.

⋮

[(q − 1)'] $|pp_{i_1}p_{i_2}\ldots p_{i_{q-1}}| \geq V_q = a_{q-1}V_{q-1}/(a_{q-2}n^{\frac{d(q-2)+q-3}{d(d-q+2)}})$, for any q-tuple $(4 \leq q \leq d+1)$ of distinct points $p_{i_1}, p_{i_2}, \ldots, p_{i_q} \in S$.

We will show this by summing up the volumes of the "forbidden" portions of C^d where one of the inequalities [1']–[(q − 1)'] is violated, and by showing that the sum of these volumes is less than 1. This implies the existence of the desired point p, which we then add to S_v to form S_{v+1}. We continue in this manner until the entire set S is constructed.

2.2 Forbidden Balls

The forbidden regions where one of the inequalities [1'] is violated are v d-dimensional balls of radius $r_1 = a_1/n^{1/d}$.[1] Their total volume is at most

$$v|B^d_{r_1}| = O\left(\frac{v}{n}\right) = O(1).$$

2.3 Forbidden Cylinders

The forbidden regions where one of the inequalities [2'] is violated are $\binom{v}{2}$ d-dimensional "cylinders" G_{ij}, for $1 \leq i < j \leq v$. The cylinder G_{ij} is centered at ℓ_{ij}, its length is at most $\sqrt{d}$, and its cross-section perpendicular to ℓ_{ij} is a $(d-1)$-dimensional sphere of radius $r_2 = \frac{2V_3}{V_2} = \frac{2a_2}{n^{2/(d-1)} \cdot |p_ip_j|} = \Theta\left(\frac{1}{n^{2/(d-1)}|p_ip_j|}\right)$ (see Figure 1).

The overall volume of the "cylinders" (within C^d) is at most

$$\sum_{1 \leq i < j \leq v} (|B^{d-1}_{r_2}|\sqrt{d}) = \sum_{1 \leq i < j \leq v} O\left(\frac{1}{n^2|p_ip_j|^{d-1}}\right). \tag{1}$$

To bound this sum, we fix p_i and sum over p_j. We use a d-dimensional spherical packing argument that exploits the fact that S_v satisfies [1]. Specifically, we have

$$\sum_{j \neq i} \frac{1}{|p_ip_j|^{d-1}} \leq \sum_{t=1}^{O(n^{1/d})} \frac{M_t n^{\frac{d-1}{d}}}{a_1^{d-1} t^{d-1}}, \tag{2}$$

where M_t is the number of points of S_v that lie in the d-dimensional spherical shell centered at p_i with inner radius $a_1 t/n^{1/d}$ and outer radius $a_1(t+1)/n^{1/d}$;

[1] Recall that $|B^d_r| = \pi^{d/2}r^d/\Gamma(d/2+1) = \Theta(r^d)$, where $\Gamma(\cdot)$ is the continuous generalization of the factorial function.

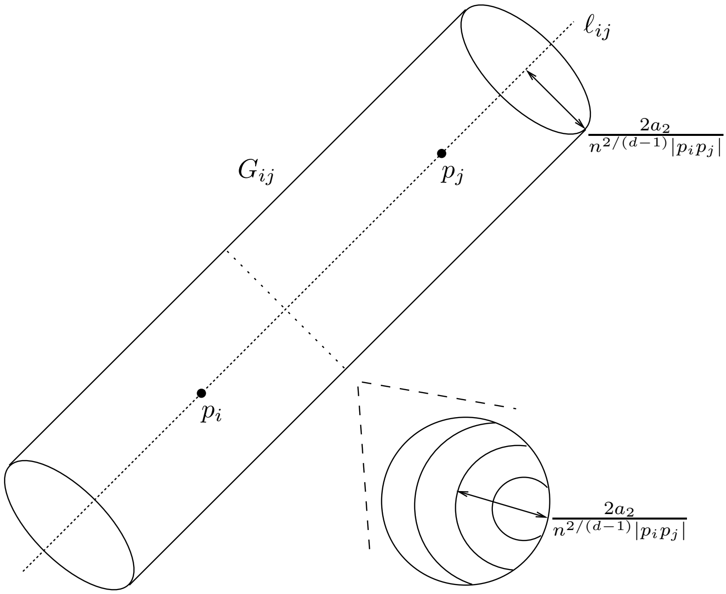

Fig. 1. A cylinder in $\Re^d$

see Figure 2. There are $O(n^{1/d})$ such spherical shells (within C^d). Because of [1], the number of such points is $M_t = O(t^{d-1})$. This follows by an argument of packing spheres of volume $\Theta(1/n)$ within a shell whose volume is $\Theta\left(\frac{t^{d-1}}{n}\right)$.

Hence, the sum in Equation (2) is $O(n)$. Summing this over all p_i, we obtain a final bound of $O(vn)$. Substituting this in Equation (1), we see that the total volume of the forbidden cylinders is $O(v/n) = O(1)$.

2.4 Forbidden Prisms

The forbidden regions where one of the inequalities [3'] is violated are $\binom{v}{3}$ d-dimensional "prisms" ϕ_{ijk}, for $1 \le i < j < k \le v$. The base area (a portion of a 2-dimensional flat) of ϕ_{ijk} is at most d, and its "height" is a $(d-2)$-dimensional sphere of radius $r_3 = \frac{3V_4}{V_3} = O\left(\frac{1}{n^{\frac{4d^2-5d-1}{d(d-1)(d-2)}}|p_i p_j p_k|}\right)$. The overall volume of the prisms (within C^d) is at most

$$\sum_{1\le i<j<k\le v} (|B_{r_3}^{d-2}| \cdot d) = \sum_{1\le i<j<k\le v} O\left(\frac{1}{n^{\frac{4d^2-5d-1}{d(d-1)}}|p_i p_j p_k|^{d-2}}\right). \quad (3)$$

To bound this sum, we fix p_i, p_j and sum over p_k. We use a d-dimensional cylindrical packing argument that exploits the fact that S_v satisfies [1] and [2].

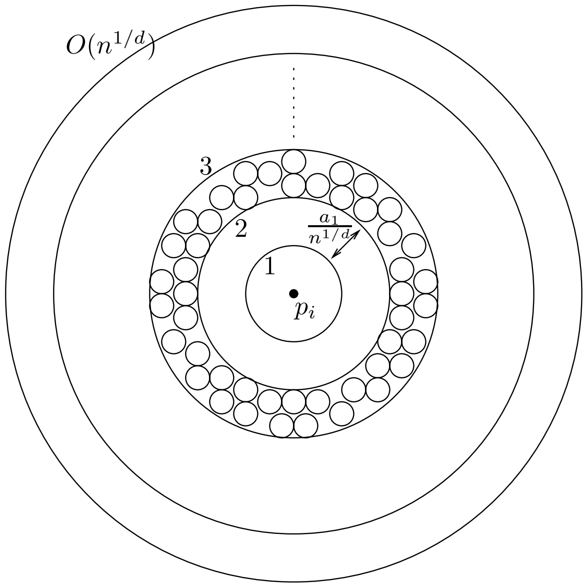

Fig. 2. A spherical packing of balls in $\Re^d$

The cylinders are centered at ℓ_{ij}; see Figure 3, where the line ℓ_{ij} emanates from p_i toward p_j through the dth dimension. Specifically, we have

$$\sum_{k \neq i,j} \frac{1}{|p_i p_j p_k|^{d-2}} \leq \frac{N_0 n^{\frac{2(d-2)}{d-1}}}{a_2^{d-2}} + \sum_{t=1}^{O(n^{1/d})} \frac{2^{d-2} N_t n^{\frac{d-2}{d}}}{a_1^{d-2} t^{d-2} |p_i p_j|^{d-2}}, \qquad (4)$$

where N_0 is the number of points of S_v that lie in the innermost d-dimensional cylinder of the packing (centered at ℓ_{ij} and of radius $a_1/n^{1/d}$), and N_t is the number of points of S_v that lie in the cylindrical shell centered at ℓ_{ij} with inner radius $a_1 t/n^{1/d}$ and outer radius $a_1(t+1)/n^{1/d}$.

Obviously, $N_0 = O(n^{1/d})$, since the volume of the $(d-1)$-dimensional cross-sectional sphere of the innermost cylinder is $O(1/n^{\frac{d-1}{d}})$ and because of [1]. Also, we have $N_t = O(t^{d-2} n^{1/d})$. This follows by an argument of packing spheres of volume $\Theta(1/n)$ within a shell whose volume is $\Theta\left(\frac{t^{d-2}}{n^{\frac{d-1}{d}}}\right)$.

Hence, the quantity in Equation (4) is

$$O\left(n^{\frac{2d^2-3d-1}{d(d-1)}} + \frac{n}{|p_i p_j|^{d-2}}\right).$$

Substituting this in Equation (3), we obtain the upper bound on the total volume of the forbidden prisms

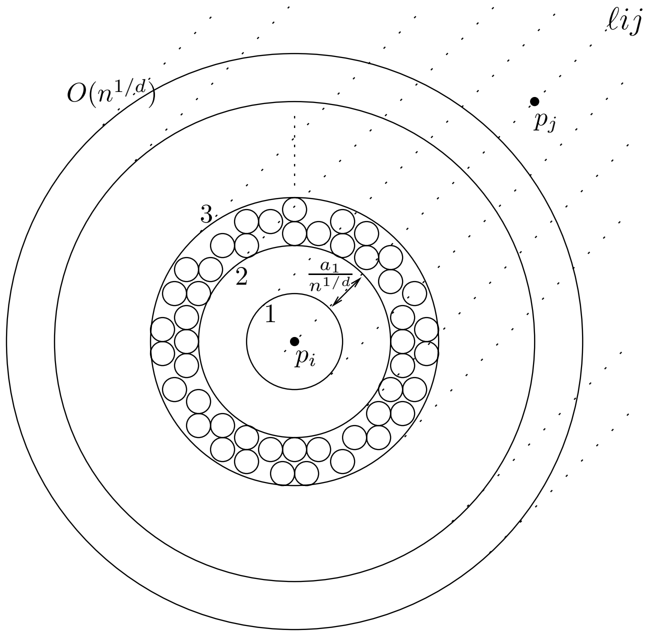

Fig. 3. A d-dimensional cylindrical packing (an extruded $(d-1)$-dimensional spherical packing) of balls in $\Re^d$

$$O\left(\sum_{1\le i<j\le v}\left(\frac{1}{n^2}+\frac{1}{n^{\frac{3d^2-4d-1}{d(d-1)}}|p_ip_j|^{d-2}}\right)\right)$$
$$=O\left(\frac{v^2}{n^2}+\frac{1}{n^{\frac{3d^2-4d-1}{d(d-1)}}}\sum_{1\le i<j\le v}\frac{1}{|p_ip_j|^{d-2}}\right). \quad (5)$$

We bound the sum in the second summand similarly to our bounding of the term in Equation (2) (in Section 2.3). We fix p_i and use a d-dimensional spherical packing argument within spherical shells centered at p_i. Arguing as above, we obtain

$$\sum_{j\neq i}\frac{1}{|p_ip_j|^{d-2}} \le \sum_{t=1}^{O(n^{1/d})}\frac{M_t n^{\frac{d-2}{d}}}{a_1^{d-2}t^{d-2}} = \sum_{t=1}^{O(n^{1/d})}\frac{O(t^{d-1})n^{\frac{d-2}{d}}}{a_1^{d-2}t^{d-2}} = O(n).$$

Summing this over all p_i, we obtain a final bound of $O(vn)$. Substituting this in Equation (5), we see that the total volume of the forbidden prisms is

$$O\left(\frac{v^2}{n^2}+\frac{v}{n^{\frac{2d^2-3d-1}{d(d-1)}}}\right)=O(1).$$

2.5 General Forbidden Zones

In Sections 2.2, 2.3, and 2.4 we computed the total volume of the forbidden zones in which the respective inequalities [1']–[3'] are violated. These zones correspond to $q = 2, 3, 4$, respectively. In this section we analyze the general case $4 < q \leq d + 1$.

The forbidden regions where one of the inequalities $[(q-1)']$ is violated are $\binom{v}{q-1}$ d-dimensional zones $\psi_{i_1 i_2 \ldots i_{q-1}}$ (for $1 \leq i_1 < i_2 < \ldots < i_{q-1} \leq v$ and $4 < q \leq d+1$), whose "bases" are portions of $(q-2)$-dimensional flats with volume at most $d^{(q-2)/2}$. The "height" of the zone $\psi_{i_1 i_2 \ldots i_{q-1}}$ is a $(d-q+2)$-dimensional sphere of radius $r_{q-1} = O(V_q/V_{q-1}) = O\left(\frac{V_q}{|p_{i_1} p_{i_2} \cdots p_{i_{q-1}}|}\right)$. The total volume of the zones (within C^d) is at most

$$\sum_{1 \leq i_1 < \ldots < i_{q-1} \leq v} O(|B^{d-q+2}_{r_{q-1}}| d^{\frac{q-2}{2}}) = \sum_{1 \leq i_1 < \ldots < i_{q-1} \leq v} O\left(\frac{V_q^{d-q+2}}{|p_{i_1} p_{i_2} \cdots p_{i_{q-1}}|^{d-q+2}}\right). \tag{6}$$

To bound this sum, we fix $p_{i_1}, p_{i_2}, \ldots, p_{i_{q-2}}$ and sum over $p_{i_{q-1}}$. We use a packing argument that exploits the fact that S_v satisfies $[1]$–$[q-2]$. The packing consists of the Cartesian product of the $(q-3)$-dimensional flat $\pi = \pi_{i_1 i_2 \ldots i_{q-2}}$ that passes through $p_{i_1}, p_{i_2}, \ldots, p_{i_{q-2}}$, and spheres whose centers belong to π and extend to the $(d-q+3)$-dimensional space orthogonal to π. Specifically, we have

$$\sum_{i_{q-1} \neq i_1, \ldots, i_{q-2}} \frac{1}{|p_{i_1} p_{i_2} \cdots p_{i_{q-1}}|^{d-q+2}} \leq \frac{Z_0}{V_{q-1}^{d-q+2}} + \sum_{t=1}^{O(n^{1/d})} \left(Z_t \cdot O\left(\frac{n^{\frac{1}{d}}}{a_1 t |p_{i_1} p_{i_2} \cdots p_{i_{q-2}}|}\right)^{d-q+2} \right), \tag{7}$$

where Z_0 is the number of points of S_v that lie in the innermost shape of the packing (centered at the flat π and of radius $a_1/n^{1/d}$), and Z_t is the number of points of S_v that lie in the shell centered at π with inner radius $a_1 t/n^{1/d}$ and outer radius $a_1(t+1)/n^{1/d}$.

Obviously, $Z_0 = O(n^{\frac{q-3}{d}})$, since the volume of the innermost shape is $O(1/n^{\frac{d-q+3}{d}})$ and because of [1]. Also, we have $Z_t = O(t^{d-q+2} n^{\frac{q-3}{d}})$. This follows by an argument of packing spheres of volume $\Theta(1/n)$ within a shell whose volume is $\Theta(t^{d-q+2}/n^{\frac{d-q+3}{d}})$.

Hence, the sum in Equation (7) is

$$O\left(\frac{n^{\frac{q-3}{d}}}{V_{q-1}^{d-q+2}} + \frac{n}{|p_{i_1} p_{i_2} \cdots p_{i_{q-2}}|^{d-q+2}}\right). \tag{8}$$

Substituting this in Equation (6), we obtain the upper bound on the total volume of the forbidden zones

$$O\left(V_q^{d-q+2}\sum_{1\le i_1<\cdots<i_{q-2}\le v}\left(\frac{n^{\frac{q-3}{d}}}{V_{q-1}^{d-q+2}}+\frac{n}{|p_{i_1}p_{i_2}\ldots p_{i_{q-2}}|^{d-q+2}}\right)\right)$$
$$=O\left(n^{\frac{q-3}{d}}v^{q-2}\left(\frac{V_q}{V_{q-1}}\right)^{d-q+2}+\sum_{1\le i_1<\ldots<i_{q-2}\le v}n\left(\frac{V_q}{V_{q-2}}\right)^{d-q+2}\right).$$

Combining this with the equality $V_q=\frac{a_{q-1}V_{q-1}}{a_{q-2}n^{\frac{d(q-2)+q-3}{d(d-q+2)}}}$, we see that the total forbidden volume is

$$O\left(1+\sum_{1\le i_1<\ldots<i_{q-2}\le v}n\left(\frac{V_q}{V_{q-2}}\right)^{d-q+2}\right). \tag{9}$$

In order to show that the bound in Equation (9) is $O(1)$, it remains to prove that the second summand in it is smaller than 1. This amounts to proving that the second summand in Equation (8) is smaller than the first summand in it. From $[q-2]$ we know that $V_{q-1}=\frac{a_{q-2}V_{q-2}}{a_{q-3}n^{\frac{d(q-3)+q-4}{d(d-q+3)}}}$ for $4<q\le d+1$. By substituting this in Equation (8), we obtain the equal quantity

$$O\left(n^{\frac{q-3}{d}}\left(\frac{a_{q-3}n^{\frac{d(q-3)+q-4}{d(d-q+3)}}}{a_{q-2}V_{q-2}}\right)^{d-q+2}+\frac{n}{V_{q-2}^{d-q+2}}\right)$$
$$=O\left(\frac{n^{\frac{(q-3)d^2+(-q^2+7q-13)d-2q^2+12q-17}{d(d-q+3)}}}{V_{q-2}^{d-q+2}}+\frac{n}{V_{q-2}^{d-q+2}}\right). \tag{10}$$

The second summand in Equation (8) is smaller than the first summand in it if and only if the second summand in Equation (10) is smaller than the first summand in it. That is, we have to prove that

$$n^{\frac{d^2(q-3)+d(-q^2+7q-13)-2q^2+12q-17}{d(d-q+3)}}>n,$$

i.e., the inequality

$$\frac{(q-3)d^2+(-q^2+7q-13)d-2q^2+12q-17}{d(d-q+3)}>1,$$

which, after simple manipulations, is

$$(q-4)d^2-(q-4)^2d-2q^2+12q-17>0.$$

However, it is easily verified that

$$(q-4)d^2 - (q-4)^2 d - 2q^2 + 12q - 17 = (q-4)(d-q+2)(d+2) - 1 \geq 5 > 0,$$

using the facts that $q \geq 5$, $d - q \geq -1$, and $d \geq 4$.

2.6 Epilogue

We are now ready to bound $\mathcal{H}_d^{\text{on-line}}(n)$, the maximum possible volume of the smallest simplex defined by some $d+1$ of n points in the d-dimensional unit cube. In other words, we want to lower bound V_{d+1}. For this purpose we use its recursive definition and write

$$V_{d+1} = \prod_{q=4}^{d+1} \left(\frac{a_{q-1}}{a_{q-2} n^{\frac{d(q-2)+q-3}{d(d-q+2)}}} \right) \cdot V_3 = \frac{a_d}{n^{\left(\sum_{q=4}^{d+1} \frac{d(q-2)+q-3}{d(d-q+2)}\right) + \frac{2}{d-1}}}. \tag{11}$$

Let us upper bound the power of n in Equation 11:

$$\begin{aligned}
&\sum_{q=4}^{d+1} \frac{d(q-2)+q-3}{d(d-q+2)} + \frac{2}{d-1} \\
&= \sum_{q=4}^{d+1} \left(\frac{q-1-1/d}{d-q+2} \right) - \frac{1}{d} \sum_{q=4}^{d+1} \left(\frac{d-q+2}{d-q+2} \right) + \frac{2}{d-1} \\
&= \sum_{t=1}^{d-2} \frac{d+1-1/d-t}{t} - (1-2/d) + \frac{2}{d-1} \\
&< (d+1-1/d)(\ln(d-2)+1) - (d-2) - (1-2/d) + 2/(d-1) \\
&= (d+1)\ln(d-2) + 2 - (\ln(d-2)-1)/d + 2/(d-1) \\
&< (d+1)\ln(d-2) + 2,
\end{aligned}$$

where we use the facts that $\sum_{t=1}^{k} 1/t < \ln k + 1$ and $2/(d-1) - (\ln(d-2)-1)/d < 0$ for d sufficiently large ($d \geq 24$). We see that $V_{d+1} > \frac{a_d}{n^{(d+1)\ln(d-2)+2}}$.

It remains to show that the constants $a_1, a_2, \ldots, a_d$ can be fixed so that the total volume of the forbidden zones is strictly less than 1. To this aim note that among these constants, the total volume of the forbidden balls depends only on a_1, the total volume of the forbidden prisms depends only on a_1, a_2, and so on. This allows us to fix the values of the constants sequentially so that the total volume of any type of forbidden shapes is strictly less than $1/d$. (See [Ba04] for the implementation of this technique for $d = 3, 4$.)

This completes the proof of the main theorem:

Theorem 1. $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(1/n^{(d+1)\ln(d-2)+2})$. □

3 Conclusion

In this paper we show by using nested packing arguments that $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(1/n^{(d+1)\ln(d-2)+2})$. This compares favorably with the best-known lower bound [Le03] in the off-line case $\mathcal{H}_d^{\text{off-line}}(n) = \Omega(\log n / n^d)$.

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