Geometric Representation of Graphs in Low Dimension

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Abstract. An axis-parallel k–dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where R_i (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph G, its *boxicity* $\mathrm{box}(G)$ is the minimum dimension k, such that G is representable as the intersection graph of (axis–parallel) boxes in k–dimensional space. The concept of boxicity finds applications in various areas such as ecology, operation research etc.

A number of NP-hard problems are either polynomial time solvable or have much better approximation ratio on low boxicity graphs. For example, the max-clique problem is polynomial time solvable on bounded boxicity graphs and the maximum independent set problem has $\log n$ approximation ratio for boxicity 2 graphs. In most cases, the first step usually is computing a low dimensional box representation of the given graph. Deciding whether the boxicity of a graph is at most 2 itself is NP-hard.

We give an efficient randomized algorithm to construct a box representation of any graph G on n vertices in $1.5(\Delta + 2)\ln n$ dimensions, where Δ is the maximum degree of G. We also show that $\mathrm{box}(G) \leq (\Delta+2)\ln n$ for any graph G. Our bound is tight up to a factor of $\ln n$. The only previously known general upper bound for boxicity was given by Roberts, namely $\mathrm{box}(G) \leq n/2$. Our result gives an exponentially better upper bound for bounded degree graphs.

We also show that our randomized algorithm can be derandomized to get a polynomial time deterministic algorithm.

Though our general upper bound is in terms of maximum degree Δ, we show that for almost all graphs on n vertices, its boxicity is upper bound by $c\cdot(d_{av}+1)\ln n$ where d_{av} is the average degree and c is a small constant. Also, we show that for any graph G, $\mathrm{box}(G) \leq \sqrt{8nd_{av}\ln n}$, which is tight up to a factor of $b\sqrt{\ln n}$ for a constant b.

1 Introduction

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe U, where V is an index set. The intersection graph $\Lambda(\mathcal{F})$ of $\mathcal{F}$ has V as vertex set, and two distinct

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vertices x and y are adjacent if and only if $S_x \cap S_y \neq \emptyset$. Representations of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. Probably the most well studied class of intersection graphs are the *interval graphs*, where each S_x is a closed interval on the real line.

A well known concept in this area of graph theory is the *boxicity*, which was introduced by F. S. Roberts in 1969 [17]. This concept generalizes the concept of interval graphs. A k–dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where R_i (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph G, its *boxicity* is the minimum dimension k, such that G is representable as the intersection graph of (axis–parallel) boxes in k–dimensional space. We denote the boxicity of a graph G by $\text{box}(G)$. The graphs of boxicity 1 are exactly the class of interval graphs. This concept finds applications in niche overlap in ecology and to problems of fleet maintenance in operations research. (See [12].)

In many algorithmic problems related to graphs, the availability of certain convenient representations turn out to be extremely useful. Probably, the most well-known and important examples are the tree decompositions and path decompositions [5]. Many NP-hard problems are known to be polynomial time solvable given a tree(path) decomposition of the input graph that has bounded width. Similarly, the representation of graphs as intersections of "disks" or "spheres" lies at the core of solving problems related to frequency assignments in radio networks, computing molecular conformations etc. For the maximum independent set problem which is hard to approximate within a factor of $n^{(1/2)-\epsilon}$ for general graphs, a PTAS is known for disk graphs given the disk representation [13,1] and an FPTAS is known for unit disk graphs [22]. In a similar way, the availability of box representation in low dimension make some well known NP hard problems like the max-clique problem, polynomial time solvable since there are only $O((2n)^k)$ maximal cliques in boxicity k graphs. Though the complexity of finding the maximum independent set is hard to approximate within a factor $n^{(1/2)-\epsilon}$ for general graphs, it is approximable to a $\log n$ factor for boxicity 2 graphs (the problem is NP-hard even for boxicity 2 graphs) given a box representation [2,4].

It was shown by Cozzens [11] that computing the boxicity of a graph is NP–hard. This was later improved by Yannakakis [23], and finally by Kratochvil [16] who showed that deciding whether the boxicity of a graph is at most 2 itself is NP–complete. Therefore it is interesting to design efficient algorithms to represent small boxicity graphs in low dimensions. To the best of our knowledge, the only known strategy till date for computing a box representation for general graphs is by Roberts [17], but it guarantees only a box representation in $n/2$ dimensions for any graph G on n vertices and m edges. In this paper, we give a randomized algorithm that guarantees an exponentially better bound ($O(\ln n)$ instead of $n/2$) for the dimension in case of bounded degree graphs. To be precise, our approach yields a box representation for any graph G on n vertices and maximum degree Δ in $1.5(\Delta + 2) \ln n$ dimensions in $O(\Delta m \ln n)$ time with high probability. We also derandomize our algorithm to obtain a deterministic polynomial time algorithm to do the same.

In a recent manuscript [8] the authors showed that for any graph G, $\mathrm{box}(G) \leq \mathrm{tw}(G) + 2$, where $\mathrm{tw}(G)$ is the treewidth of G. This result implies that the class of 'low boxicity' graphs properly contains the class of 'low treewidth graphs'. It is well known that almost all graphs on n vertices and $m = c \cdot n$ edges (for a sufficiently large constant c) have $\Omega(n)$ treewidth [15]. In this paper we show that almost all graphs on n vertices and m edges have boxicity at most $c' \frac{m}{n} \ln n$ for a small constant c'. An implication of this result is that for almost all graphs, there is an exponential gap between its boxicity and treewidth. Hence it is interesting to take a relook at those NP-hard problems that are polynomial time solvable in bounded treewidth graphs and see whether they are also polynomial time solvable for bounded boxicity graphs.

Researchers have also tried to bound the boxicity of graph classes with special structure. Scheinerman [18] showed that the boxicity of outer planar graphs is at most 2. Thomassen [20] proved that the boxicity of planar graphs is bounded above by 3. Upper bounds for the boxicity of many other graph classes such as chordal graphs, AT-free graphs, permutation graphs etc. were shown in [8] by relating the boxicity of a graph with its treewidth. Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [21], the rectangle number [9], grid dimension [3], circular dimension [14,19] and the boxicity of digraphs [10] are some examples.

1.1 Our Results

We summarize below the results of this paper.

1. We show that for any graph G on n vertices, $\mathrm{box}(G) \leq (\Delta + 2) \ln n$. This bound is tight up to a factor of $\ln n$.
2. In fact, we show a randomized algorithm to construct a box representation of G in $1.5(\Delta + 2) \ln n$ dimensions, that runs in $O(\Delta m \ln n)$ time with high probability, where m is the number of edges in G.
3. Next we show a polynomial time deterministic algorithm to construct a box representation in $(\Delta + 2) \ln n$ dimensions by derandomizing the above randomized algorithm.
4. Though the general upper bound that we show is in terms of the maximum degree Δ, we also investigate the relation between boxicity and average degree. We show that for almost all graphs on n vertices and m edges, the boxicity is $O((d_{av} + 1) \ln n)$, where d_{av} is the average degree.
5. We also derive a upper bound for boxicity of general graphs in terms of average degree. We show that for any graph G, $\mathrm{box}(G) \leq \sqrt{8 n d_{av} \ln n}$, which is tight up to a factor of $b\sqrt{\ln n}$ for a constant b.

We refer the reader to the complete version [7] for the missing proofs.

1.2 Definitions and Notations

Let G be a undirected simple graph on n vertices. The vertex set of G is denoted as $V(G) = \{1, \cdots, n\}$ (or V in short). Let $E(G)$ denote the edge set of G.

We denote by $\overline{G}$, the complement of G. We say the edge e is missing in G, if $e \in E(\overline{G})$. A graph G' is said to be a super graph of G where $V(G) = V(G')$, if $E(G) \subseteq E(G')$. For a vertex $u \in V$, let $N(u)$ denote the set of neighbors of u in G and let $d(u)$ denote the degree of u in G, i.e. $d(u) = |N(u)|$. Let Δ denote the maximum degree of G.

Definition 1 (Projection). *Let π be a permutation of the set $\{1, \cdots, n\}$. Let $X \subseteq \{1, \cdots, n\}$. The projection of π onto X denoted as π_X is defined as follows. Let $X = \{u_1, \ldots, u_r\}$ such that $\pi(u_1) < \pi(u_2) < ... < \pi(u_r)$. Then $\pi_X(u_1) = 1, \pi_X(u_2) = 2, \cdots, \pi_X(u_r) = r$.*

Definition 2 (Interval Representation). *An interval graph can be represented as the intersection graph of closed intervals on real line. To define an interval representation of an interval graph G, we define the two functions $l : V \rightarrow \mathrm{R}$ and $r : V \rightarrow \mathrm{R}$. The interval corresponding to a vertex v denoted as $I(v)$ is given by $[l(v), r(v)]$, where $l(v)$ and $r(v)$ are the left and right end points of the interval corresponding to v.*

Definition 3. *We define a map $\mathcal{M}(G, \pi)$ which associates a permutation π of the vertices $\{1, 2, \cdots, n\}$ to an interval super graph G' of G, as follows: Consider any vertex $u \in V(G)$. Let $n_u \in N(u) \cup \{u\}$ be the vertex such that $\pi(n_u) = \min_{w \in N(u) \cup \{u\}} \pi(w)$. Then associate the interval $[\pi(n_u), \pi(u)]$ to the vertex u, and let G' be the resulting interval graph. It is easy to verify that G' is a super graph of G. We define $\mathcal{M}(G, \pi) = G'$.*

1.3 Box Representation and Interval Graph Representation

Let $G = (V, E(G))$ be a graph and let $I_1, \ldots, I_k$ be k interval graphs such that each $I_j = (V, E(I_j))$ is defined on the same set of vertices V. If

$$E(G) = E(I_1) \cap \cdots \cap E(I_k),$$

then we say that $I_1, \ldots, I_k$ is an *interval graph representation* of G. The following equivalence is well-known.

Theorem 1 (Roberts [17]). *The minimum k such that there exists an interval graph representation of G using k interval graphs $I_1, \ldots, I_k$ is the same as* $\mathrm{box}(G)$.

Recall that a k–dimensional box representation of G is a mapping of each vertex $u \in V$ to $R_1(u) \times \cdots \times R_k(u)$, where each $R_i(u)$ is a closed interval of the form $[\ell_i(u), r_i(u)]$ on the real line. It is straightforward to see that an interval graph representation of G using k interval graphs $I_1, \ldots, I_k$, is equivalent to a k–dimensional box representation in the following sense. Let $R_i(u) = [\ell_i(u), r_i(u)]$ denote the closed interval corresponding to vertex u in an interval realization of I_i. Then the k–dimensional box corresponding to u is simply $R_1(u) \times \cdots \times R_k(u)$. Conversely, given a k–dimensional box representation of G, the set of intervals $\{R_i(u) : u \in V\}$ forms the ith interval graph I_i in the corresponding interval graph representation.

When we say that a box representation in t dimensions is output by an algorithm, the algorithm actually outputs the interval graph representation: that is, the interval representation of the constituent interval graphs.

2 The Randomized Construction

Consider the following randomized procedure **RAND** which outputs an interval super graph of G. Let Δ be the maximum degree of G.

RAND
  Input: G.
  Output: G' which is an interval super graph of G.
begin
  step1. Generate a permutation π of $\{1, \dots, n\}$ uniformly at random.
  step2. Return $G' = \mathcal{M}(G, \pi)$.
end.

Lemma 1. *Let $e = (u, v) \in E(\overline{G})$. Let G' be the output of **RAND**(G). Then,*

$$\mathbf{Pr}\left[e \notin E(\overline{G'})\right] = \frac{1}{2}\left(\frac{d(u)}{d(u)+2} + \frac{d(v)}{d(v)+2}\right) \leq \frac{\Delta}{\Delta+2}.$$

Proof. We have to estimate the probability that u and v are adjacent in G'. That is, $I(u) \cap I(v) \neq \emptyset$.

Let $n_u \in N(u)$ be a vertex such that it minimizes $\min_{w \in N(u)} \pi(w)$. Similarly, let $n_v \in N(v)$ be a vertex such that it minimizes $\min_{w \in N(v)} \pi(w)$.

It is easy to see that $I(u) \cap I(v) \neq \emptyset$ if (a) $\pi(n_u) < \pi(v) < \pi(u)$. This is because, if the above condition holds, then, recalling the definition of $\mathcal{M}(G, \pi)$, it follows that $l(u) < r(v) < r(u)$, which implies that $r(v) \in I(u) \cap I(v)$. Similarly, if (b) $\pi(n_v) < \pi(u) < \pi(v)$ then also $I(u) \cap I(v) \neq \emptyset$. On the other hand, it is easy to see that $I(u) \cap I(v) \neq \emptyset$ *only if* either (a) or (b) hold. Again, the above two events ((a) and (b)) are mutually exclusive. Hence

$$\mathbf{Pr}\left[e \notin E(\overline{G'})\right] = \mathbf{Pr}[\pi(n_u) < \pi(v) < \pi(u)] + \mathbf{Pr}[\pi(n_v) < \pi(u) < \pi(v)].$$

We bound $\mathbf{Pr}[\pi(n_u) < \pi(v) < \pi(u)]$ as follows. Let $X = \{u\} \cup N(u) \cup \{v\}$. Let π_X be the projection of π onto X. Clearly, the event $\pi(n_u) < \pi(v) < \pi(u)$ translates to saying that $\pi_X(v) < \pi_X(u)$ and $\pi_X(v) \neq 1$. Note that π_X can be any permutation of $|X|$ elements with equal probability, which is $\frac{1}{(d(u)+2)!}$. The number of permutations where $\pi_X(v) < \pi_X(u)$ equals $(d(u)+2)!/2$. Now the number of permutations where $\pi_X(v) = 1$ equals $(d(u)+1)!$. Note that the set of permutations with $\pi_X(v) = 1$ is a subset of the set of permutations with $\pi_X(v) < \pi_X(u)$. It follows that

$$\mathbf{Pr}[\pi_X(v) < \pi_X(u) \text{ and } \pi_X(v) \neq 1] = \frac{(d(u)+2)!/2 - (d(u)+1)!}{(d(u)+2)!}$$

which is $\frac{d(u)}{2(d(u)+2)}$. Using similar arguments, it follows that $\mathbf{Pr}[\pi(n_v) < \pi(u) < \pi(v)] = \frac{d(v)}{2(d(v)+2)}$. Combing the two bounds, the result follows. ■

Lemma 2. *Let $I_1, I_2, \cdots, I_t$ be the output generated by t invocations of **RAND**(G). If $t \geq \frac{3}{2}(\Delta + 2)\ln n$ then $E(G) = E(I_1) \cap E(I_2) \cap \cdots \cap E(I_t)$ with high probability .*

As mentioned in the proof of Lemma 2, if we fix $t = (\Delta + 2)\ln n$, the resulting intersection graph is G with probability at least $1/2$. Hence we have the following Corollary.

Corollary 1. *Let G be a graph on n vertices and with maximum degree Δ. Then* $\mathrm{box}(G) \leq (\Delta + 2)\ln n$.

The following Lemma is straightforward.

Lemma 3. *The **RAND** procedure can be implemented in $O(m + n)$ time assuming that a permutation of $\{1, \ldots, n\}$ can be generated uniformly at random in $O(n)$ time.*

The following theorem is a direct consequence of Lemma 2 and Lemma 3.

Theorem 2. *Given a graph G on n vertices and m edges, with high probability, a box representation of G in $(\Delta + 2)\ln n$ dimensions can be constructed in $O(\Delta m \ln n)$ time, where Δ is the maximum degree of G.*

Tight example: We remark that for any given Δ and $n > \Delta + 1$, we can construct a graph G on n vertices and with maximum degree Δ such that $\mathrm{box}(G) \geq \lfloor (\Delta + 2)/2 \rfloor$. We assume that Δ is even for the ease of explanation. Roberts [17] has shown that for any even number k, there exists a graph on k vertices with degree $k - 2$ and boxicity $k/2$. We call such graphs as *Roberts graph*. The Roberts graph on n vertices is obtained by removing the edges of a perfect matching from a complete graph on n vertices. We take such a graph by fixing $k = \Delta + 2$ and we let the remaining $n - (\Delta + 2)$ vertices to be isolated vertices. Clearly, the boxicity of such a graph is also $k/2 = (\Delta + 2)/2$, where as the maximum degree is Δ. Thus our upper bound is tight up to a factor of $2\ln n$.

3 Derandomization

In this section we derandomize the above randomized algorithm to obtain a deterministic polynomial time algorithm to output the box representation in $(\Delta + 2)\ln n$ dimensional space for a given graph G on n vertices with maximum degree Δ.

Lemma 4. *Let $G = (V, E)$ be the graph. Let $E(\overline{G})$ be the edge set of the complement of G. Let $H \subseteq E(\overline{G})$. Then we can construct an interval super graph G' of G in polynomial time such that $|E(\overline{G'}) \cap H| \geq \frac{2}{\Delta+2}|H|$.*

Theorem 3. *Let G be a graph on n vertices with maximum degree Δ. The box representation of G in $(\Delta+2)\ln n$ dimensions can be constructed in polynomial time,*

Proof. Let $h = |E(\overline{G})|$. It follows from Lemma 4 that we can construct t interval graphs such that the number of edges of $E(\overline{G})$ which is not missing in any of these t interval graphs is at most $\left(\frac{\Delta}{\Delta+2}\right)^t h$. If $\left(\frac{\Delta}{\Delta+2}\right)^t h < 1$, then we are done. That is, we are done if $t \ln\left(\frac{\Delta}{\Delta+2}\right) + \ln h < 0$ is true. Clearly this is true, if $t > \frac{\ln h}{\ln\left(\frac{\Delta+2}{\Delta}\right)}$. Using the fact that $\ln\frac{\Delta+2}{\Delta} \geq \frac{2}{\Delta} - \frac{1}{2}(\frac{2}{\Delta})^2$, we obtain box$(G) \leq \frac{\Delta^2}{2(\Delta-1)} \ln h \leq (\Delta+2)\ln n$. By Lemma 4, each interval graph is constructed in polynomial time. Hence the total running time is still polynomial. Thus the theorem follows. ∎

Proof (Lemma 4). We derandomize the **RAND** algorithm to devise a deterministic algorithm to construct G'.

Our deterministic strategy defines a permutation π on the vertices $\{1, \cdots, n\}$ of G. The desired G' is then obtained as $\mathcal{M}(G, \pi)$. Let the ordered set $V_n =< v_1, \cdots, v_n >$ denote the final permutation given by π. We construct V_n in a step by step fashion. At the end of step i, we have already defined the first i elements of the permutation, namely the ordered set $V_i =< v_1, \cdots, v_i >$, where each v_j is distinct. Let V_0 denote the empty set. Having obtained V_i for $i \geq 0$, we compute V_{i+1} in the next step as follows.

Given an ordered set V_i of i vertices $< v_1, v_2, \cdots, v_i >$, let $V_i \diamond u$ denote the ordered set of the $i+1$ vertices $< v_1, v_2, \cdots, v_i, u >$. (We will abuse notation and use V_i to denote the underlying unordered set also, when there is no chance of confusion.) Let $V_0 \diamond u$ denote $< u >$.

Consider the **RAND** algorithm whose output is denoted as G''. For each $e \in H$, let x_e denote the indicator random variable which is 1 if $e \in E(\overline{G''})$, and 0 otherwise. Let $X_H = \sum_{e \in H} x_e$.

Let $\mathcal{Z}(V_i)$ for $i \geq 0$ denote the event that the first i elements of the random permutation generated by **RAND** is given by the ordered set $V_i =< v_1, \cdots, v_i >$. Note that $\mathbf{Pr}[\mathcal{Z}(V_0)] = 1$ since the first 0 elements of any permutation is the empty set V_0.

Let $x_e|\mathcal{Z}(V_i)$ denote the indicator random variable corresponding to x_e conditioned on the event $\mathcal{Z}(V_i)$.

Similarly, let the random variable $X_H|\mathcal{Z}(V_i)$ denote the number of missing edges in G'' conditioned on the event $\mathcal{Z}(V_i)$.

For $i \geq 0$, Let $f_e(V_i)$ denote $\mathbf{Pr}[x_e = 1 \mid \mathcal{Z}(V_i)]$ and let $F(V_i)$ denote $\mathbf{E}[X_H \mid \mathcal{Z}(V_i)]$

Note that $f_e(V_0)$ denote $\mathbf{Pr}[x_e = 1]$ and $F(V_0)$ denote $\mathbf{E}[X_H]$.

Clearly

$$F(V_i) = \sum_{e \in H} f_e(V_i).$$

By Lemma 1, we know that for any $e \in H$, $f_e(V_0) \geq \frac{2}{\Delta+2}$. Thus $F(V_0) \geq \frac{2|H|}{\Delta+2}$. Clearly,

$$\mathbf{E}[X_H|\mathcal{Z}(V_i)] = \frac{1}{|V - V_i|} \sum_{u \in V - V_i} \mathbf{E}[X_H|\mathcal{Z}(V_i \diamond u)] \,.$$

Let $u \in V - V_i$ be such that

$$\mathbf{E}[X_H|\mathcal{Z}(V_i \diamond u)] = \max_{w \in V - V_i} \mathbf{E}[X_H|\mathcal{Z}(V_i \diamond w] \,.$$

Define $V_{i+1} = V_i \diamond u$. It follows that

$$F(V_{i+1}) = \mathbf{E}[X_H|\mathcal{Z}(V_{i+1})] \geq \mathbf{E}[X_H|\mathcal{Z}(V_i)] = F(V_i).$$

In particular, it is also true that $F(V_1) \geq F(V_0)$.

After n steps, we obtain the final permutation V_n. Applying the above inequality n times, it follows that

$$F(V_n) = \mathbf{E}[X_H|\mathcal{Z}(V_n)] \geq \mathbf{E}[Z_H] = F(V_0).$$

Recalling that $F(V_0) \geq \frac{2|H|}{\Delta+2}$, we have $F(V_n) \geq \frac{2|H|}{\Delta+2}$.

Let π be the permutation which maps $< 1, \cdots, n >$ to V_n. The final interval super graph G' output by our deterministic strategy is $\mathcal{M}(G, \pi)$. By definition, $F(V_n)$ is the total number of edges from H that are missing in G'. We have shown that $F(V_n) \geq \frac{2|H|}{\Delta+2}$ as claimed.

It remains to show that the above deterministic strategy takes only polynomial time. For that we need the following lemma.

Lemma 5. *For any ordered set $U_j =< u_1, \cdots, u_j >$ and any $e \in H$, $f_e(U_j)$ can be computed exactly in polynomial time.*

Given a vertex $w \in V - V_i$, $F(V_i \diamond w)$ is simply $\sum_{e \in H} f_e(V_i \diamond w)$. It follows from Lemma 5 that $F(V_i \diamond w)$ can be computed in polynomial time. Recall that given V_i, V_{i+1} is $V_i \diamond u$ where u maximizes $F(V_i \diamond w)$ among the vertices from $w \in V - V_i$. Clearly such a u can also be found in polynomial time. Since there are only n steps before computing V_n, the overall running time is still polynomial. ■

4 In Terms of Average Degree

It is natural to ask whether our upper bound of $(\Delta + 2) \ln n$ still holds even if we replace Δ by the average degree d_{av}. Unfortunately this is not the case as illustrated by the following example. Consider the following graph $G = (V, E)$ on n vertices. We take a Roberts graph on n_1 vertices such that $n_1(n_1 - 2)/n = d_{av}$ and we let the remaining $n - n_1$ vertices to be isolated vertices. The average degree of this graph is clearly d_{av} (recall the definition of Roberts graph) and its boxicity is at least $n_1/2 \geq \frac{1}{2}\sqrt{nd_{av}}$. If we substitute Δ by d_{av} in our upper bound, we obtain that the boxicity of this graph is at most $(d_{av} + 2) \ln n$, which is far below the actual boxicity. Still, we can prove the following general upper bound in terms of the average degree.

Theorem 4. *For a graph $G = (V, E)$ on n vertices and average degree d_{av},* $\text{box}(G) \leq \sqrt{8nd_{av}\ln(n)}$*. Moreover, there exists a graph G with n vertices and average degree d_{av} such that* $\text{box}(G) \geq \frac{1}{2}\sqrt{nd_{av}}$.

Proof. We show the upper bound as follows. Let $x = \sqrt{\frac{nd_{av}}{2\ln(n)}}$. Let V' denote the set of vertices in G whose degree is greater than or equal to x. It is straightforward to verify that $|V'| \leq \frac{nd_{av}}{x}$. Let G'' be the induced sub graph of G induced on $V - V'$. That is, each vertex in G'' has degree at most x. By Theorem 2, we obtain that $\text{box}(G'') \leq 2x\ln(n)$. Since $\text{box}(G'') + |V'|$ is a trivial upper bound for $\text{box}(G)$, it follows that $\text{box}(G) \leq 2x\ln(n) + \frac{nd_{av}}{x} = 2\sqrt{2nd_{av}\ln(n)}$. The example graph discussed in the beginning of this section serves as the example that illustrate the lower bound. ■

4.1 Boxicity of Random Graphs

Though in general boxicity of a graph is not upper bound by $(d_{av} + 2)\ln n$, where d_{av} is the average degree, we now show that for almost all graphs, the boxicity is at most $c(d_{av} + 1)\ln n$, for a small positive constant c. We show the following. Let G be a random graph drawn according to the $\mathcal{G}(n, m)$ model [6], where n is the number of vertices and m is the number of edges. Then $\mathbf{Pr}\big[\text{box}(G) \leq 8(\frac{2m}{n} + 1)\ln n)\big] \geq 1 - \frac{2}{n^2}$. (Note that $d_{av} = 2m/n$). It follows immediately that for almost all graphs on n vertices and m edges, the boxicity is upper bound by $8(d_{av} + 1)\ln n$.

Theorem 5. *For a random graph G on n vertices and m edges drawn according to $\mathcal{G}(n, m)$ model,*

$$\mathbf{Pr}\left[\text{box}(G) \;\leq\; 8\left(\frac{2m}{n} + 1\right)\ln n\right] \geq 1 - \frac{2}{n^2}.$$

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