f-SWRL: A Fuzzy Extension of SWRL*

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Abstract. Although the combination of OWL and Horn rules results in the creation of a highly expressive language, i.e. SWRL, there are still many occasions where this language fails to accurately represent knowledge of our world. In particular, SWRL fails at representing vague and imprecise knowledge and information. Such type of information is apparent in many applications like multimedia processing and retrieval, information fusion, etc. In this paper, we propose f-SWRL, a fuzzy extension to SWRL to include fuzzy assertions (such as 'Mary is tall in the degree of 0.9') and fuzzy rules (such as 'being healthy is more important than being rich to determine if one is happy').

1 Introduction

According to widely known proposals for a Semantic Web architecture, Description Logics (DLs)-based ontologies will play a key role in the Semantic Web [Pan04]. This has led to considerable efforts to developing a suitable ontology language, culminating in the design of the OWL Web Ontology Language [BvHH+04b], which is now a W3C recommendation. Although OWL adds considerable expressive power with respect to languages such as RDF, it does have expressive limitations, particularly with respect to what can be said about properties. E.g., there is no composition constructor, so it is impossible to capture relationships between a composite property and another (possibly composite) property. One way to address this problem would be to extend OWL with some form of "rules language" [HPS04]. One such proposed extension is SWRL (Semantic Web Rule Language) [HPSB+04], which is a Horn clause rules extension to OWL DL[1] that overcomes many of these limitations.

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[1] OWL DL is a key sub-language of OWL.

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Even though the combination of OWL and Horn rules results in the creation of a highly expressive language, there are still many occasions where this language fails to accurately represent knowledge of our world. In particular these languages fail at representing vague and imprecise knowledge and information [Kif05]. Such type of information is very useful in many applications like multimedia processing and retrieval [SST+05, BvHH+04a], information fusion [Mat05], and many more. Experience has shown that in many cases dealing with such type of information would yield more efficient and realistic applications [AL05, ZYZ+05]. Furthermore, in many applications, like ontology alignment and modularization, the interconnection of disparate and distributed ontologies and modules is hardly ever a true or false situation, but rather a matter of a confidence or relatedness degree.

In order to capture imprecision in rules, we propose a fuzzy extension of SWRL, called f-SWRL. In f-SWRL, fuzzy individual axioms can include a specification of the "degree" (a truth value between 0 and 1) of confidence with which one can assert that an individual (resp. pair of individuals) is an instance of a given class (resp. property); and atoms in f-SWRL rules can include a "weight" (a truth value between 0 and 1) that represents the "importance" of the atom in a rule. For example, the following fuzzy rule asserts that being healthy is more important than being rich to determine if one is happy:

$$\mathsf{Rich}(?p) * 0.5 \wedge \mathsf{Healthy}(?p) * 0.9 \rightarrow \mathsf{Happy}(?p),$$

where Rich, Healthy and Happy are classes, and 0.5 and 0.9 are the weights for the atoms Rich(?p) and Healthy(?p), respectively. Additionally, observe that the classes Rich, Healthy and Happy are best represented by fuzzy classes, since the degree to which someone is Rich is both subjective and non-crisp.

In this paper, we will present the formal syntax and semantics of f-SWRL. In particular, we specify a set of key constraints of the desired semantics of f-SWRL. These constraints provides a unified framework for model theoretic semantics of f-SWRL based on fuzzy and weight operations. We will provide several examples illustrate the features of f-SWRL. To the best of our knowledge, this is the first effort on fuzzy extensions of the SWRL language.

2 Preliminaries

2.1 OWL

OWL is a standard (W3C recommendation) for expressing ontologies in the Semantic Web. The OWL language facilitates greater machine understandability of Web resources than that supported by RDFS by providing additional constructors for building class and property descriptions (vocabulary) and new axioms (constraints), along with a formal semantics. The OWL recommendation actually consists of three languages of increasing expressive power: OWL Lite, OWL DL and OWL Full. *OWL Lite* and *OWL DL* are, like DAML+OIL,

Table 1. OWL Class and Property Descriptions

Abstract Syntax	DL Syntax	Semantics
Class(A)	A	$\mathsf{A}^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
Class(owl:Thing)	$\top$	$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
Class(owl:Nothing)	$\bot$	$\bot^{\mathcal{I}} = \emptyset$
intersectionOf($C_1, C_2, \ldots$)	$C_1 \sqcap C_2$	$(C_1 \sqcap C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
unionOf($C_1, C_2, \ldots$)	$C_1 \sqcup C_2$	$(C_1 \sqcup C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
complementOf(C)	$\neg C$	$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
oneOf($o_1, o_2, \ldots$)	$\{o_1\} \sqcup \{o_2\}$	$(\{o_1\} \sqcup \{o_2\})^{\mathcal{I}} = \{o_1{}^{\mathcal{I}}, o_2{}^{\mathcal{I}}\}$
restriction(R someValuesFrom(C))	$\exists R.C$	$(\exists R.C)^{\mathcal{I}} = \{x \mid \exists y.\langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
restriction(R allValuesFrom(C))	$\forall R.C$	$(\forall R.C)^{\mathcal{I}} = \{x \mid \forall y.\langle x, y\rangle \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\}$
restriction(R hasValue(o))	$\exists R.\{o\}$	$(\exists R.\{o\})^{\mathcal{I}} = \{x \mid \langle x, \mathsf{o}^{\mathcal{I}}\rangle \in R^{\mathcal{I}}\}$
restriction(R minCardinality(m))	$\geqslant mR$	$(\geqslant mR)^{\mathcal{I}} = \{x \mid \sharp\{y.\langle x, y\rangle \in R^{\mathcal{I}}\} \geq m\}$
restriction(R maxCardinality(m))	$\leqslant mR$	$(\leqslant mR)^{\mathcal{I}} = \{x \mid \sharp\{y.\langle x, y\rangle \in R^{\mathcal{I}}\} \leq m\}$
restriction(T someValuesFrom(u))	$\exists T.u$	$(\exists T.u)^{\mathcal{I}} = \{x \mid \exists t.\langle x, t\rangle \in T^{\mathcal{I}} \wedge t \in u^{\mathbf{D}}\}$
restriction(T allValuesFrom(u))	$\forall T.u$	$(\forall T.u)^{\mathcal{I}} = \{x \mid \exists t.\langle x, t\rangle \in T^{\mathcal{I}} \rightarrow t \in u^{\mathbf{D}}\}$
restriction(T hasValue(w))	$\exists T.\{w\}$	$(\exists T.\{w\})^{\mathcal{I}} = \{x \mid \langle x, w^{\mathbf{D}}\rangle \in T^{\mathcal{I}}\}$
restriction(T minCardinality(m))	$\geqslant mT$	$(\geqslant mT)^{\mathcal{I}} = \{x \mid \sharp\{t \mid \langle x, t\rangle \in T^{\mathcal{I}}\} \geq m\}$
restriction(T maxCardinality(m))	$\leqslant mT$	$(\leqslant mT)^{\mathcal{I}} = \{x \mid \sharp\{t \mid \langle x, t\rangle \in T^{\mathcal{I}}\} \leq m\}$
ObjectProperty(S)	S	$S^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
ObjectProperty(S' inverseOf(S))	S^{-}	$(S^{-})^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
DatatypeProperty(T)	T	$T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$

basically very expressive Description Logics (DLs); they are almost[2] equivalent to the $\mathcal{SHIF}(\mathbf{D}^{+})$ and $\mathcal{SHOIN}(\mathbf{D}^{+})$ DLs. *OWL Full* provides the same set of constructors as OWL DL, but allows them to be used in an unconstrained way (in the style of RDF). It is easy to show that OWL Full is undecidable, because it does not impose restrictions on the use of transitive properties [HST99]; therefore, when we mention OWL in this paper, we usually mean OWL DL.

Let $\mathbf{C}$, $\mathbf{R_I}$, $\mathbf{R_D}$ and $\mathbf{I}$ be the sets of URIrefs that can be used to denote classes, *individual-valued* properties, *data-valued* properties and individuals respectively. An OWL DL *interpretation* is a tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathcal{I}})$ where the individual domain $\Delta^{\mathcal{I}}$ is a nonempty set of individuals, the datatype domain $\Delta_{\mathbf{D}}$ is a nonempty set of data values, $\cdot^{\mathcal{I}}$ is an individual interpretation function that maps

- each individual name $a \in \mathbf{I}$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$,
- each class name $\mathsf{CN} \in \mathbf{C}$ to a subset $\mathsf{CN}^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$,
- each *individual-valued* property name $RN \in \mathbf{R_I}$ to a binary relation $RN^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and
- each *data-valued* property name $TN \in \mathbf{R_D}$ to a binary relation $TN^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$.

Let $RN \in \mathbf{R_I}$ an *individual-valued* property URIref, R an *individual-valued* property, $TN \in \mathbf{R_D}$ a *data-valued* property URIref and T a *data-valued* property. Valid OWL DL *individual-valued* properties are defined by the DL syntax: $R ::= RN \mid R^{-}$; valid OWL DL *data-valued* properties are defined by the DL

[2] They also provide annotation properties, which Description Logics do not.

syntax: $T ::= TN$. Let $\mathsf{CN} \in \mathbf{C}$ be a class name, C, D class descriptions, $o \in \mathbf{I}$ an individual, u an OWL datatype range and $m \in \mathbb{N}$ an integer. Valid OWL DL class descriptions are defined by the DL syntax:

$$\begin{aligned} C ::= & \top \mid \bot \mid \mathsf{CN} \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \{o\} \\ & \exists R.C \mid \forall R.C \mid \geqslant mR, \mid \leqslant mR \\ & \exists T.u \mid \forall T.u \mid \geqslant mT, \mid \leqslant mT \end{aligned}$$

The individual interpretation function can be extended to give semantics to class and property descriptions shown in Tables 2, where $\mathsf{A} \in \mathbf{C}$ is a class URIref, $C, C_1, \ldots, C_n$ are class descriptions, $S \in \mathbf{R_I}$ is an *individual-valued* property URIref, R is an *individual-valued* property description and $o, o_1, o_2 \in \mathbf{I}$ are individual URIrefs, u is a data range, $T \in \mathbf{R_D}$ is a *data-valued* property and $\sharp$ denotes cardinality.

An OWL DL ontology can be seen as a DL knowledge base [HPSvH03], which consists of a set of *axioms*, including class axioms, property axioms and individual axioms.[3] A DL knowledge base consists of a TBox, an RBox and an ABox. A *TBox* is a finite set of class inclusion axioms of the form $C \sqsubseteq D$, where C, D are $\mathcal{L}$-classes. An interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. An *RBox* is a finite set of role axioms, such as role inclusion axioms ($R \sqsubseteq S$), functional role axioms ($\mathsf{Func}(R)$) and transitive role axioms ($\mathsf{Trans}(R)$); the kinds of role axioms that can appear in an RBox depend on the expressiveness of $\mathcal{L}$. An interpretation $\mathcal{I}$ satisfies $R \sqsubseteq S$ if $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$; $\mathcal{I}$ satisfies $\mathsf{Func}(R)$ if, for all $x \in \Delta^{\mathcal{I}}$, $\sharp\{y \in \Delta^{\mathcal{I}} \mid \langle x, y\rangle \in R^{\mathcal{I}}\} \leq 1$ ($\sharp$ denotes cardinality); $\mathcal{I}$ satisfies $\mathsf{Trans}(R)$ if, for all $x, y, z \in \Delta^{\mathcal{I}}$, $\{\langle x, y\rangle, \langle y, z\rangle\} \subseteq R^{\mathcal{I}} \to \langle x, z\rangle \in R^{\mathcal{I}}$. An *ABox* is a finite set of individual axioms of the form $a : C$, called *class assertions*, or $\langle a, b\rangle : R$, called *role assertions*. An interpretation $\mathcal{I}$ satisfies $a : C$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, and it satisfies $\langle a, b\rangle : R$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in R^{\mathcal{I}}$. An interpretation $\mathcal{I}$ satisfies a knowledge base Σ if it satisfies all the axioms in Σ. Σ is *satisfiable* (*unsatisfiable*) iff there exists (does not exist) such an interpretation $\mathcal{I}$ that satisfies Σ. Let C, D be $\mathcal{L}$-classes, C is *satisfiable* w.r.t. Σ iff there exist an interpretation $\mathcal{I}$ of Σ s.t. $C^{\mathcal{I}} \neq \emptyset$; C subsumes D w.r.t. Σ iff for every interpretation $\mathcal{I}$ of Σ we have $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

2.2 SWRL

SWRL is proposed by the Joint US/EU ad hoc Agent Markup Language Committee.[4] It extends *OWL DL* by introducing *rule axioms*, or simply *rules*, which have the form:

$$\mathsf{antecedent} \to \mathsf{consequent},$$

where both $\mathsf{antecedent}$ and $\mathsf{consequent}$ are conjunctions of atoms written $a_1 \wedge \ldots \wedge a_n$. Atoms in rules can be of the form C(x), P(x,y), Q(x,z), sameAs(x,y), differentFrom(x,y) or builtIn(*pred*,$\mathrm{z}_1, \ldots, \mathrm{z}_n$), where C is an OWL DL description, P is an OWL DL *individual-valued* property, Q is an OWL DL *data-valued*

[3] Individual axioms are called *facts* in OWL.

[4] See `http://www.daml.org/committee/` for the members of the Joint Committee.

property, *pred* is a datatype predicate URIref, x,y are either *individual-valued* variables or OWL individuals, and z, $z_1, \ldots, z_n$ are either *data-valued* variables or OWL data literals. An OWL data literal is either a typed literal or a plain literal; see [BvHH+04b, PH05] for details. Variables are indicated using the standard convention of prefixing them with a question mark (e.g., $?x$). For example, the following rule asserts that one's parents' brothers are one's uncles:

$$parent(?x, ?p) \wedge brother(?p, ?u) \rightarrow uncle(?x, ?u), \tag{1}$$

where $parent, brother$ and $uncle$ are all *individual-valued* properties.

In SWRL, URI references (URIrefs) are used to identify ontology elements such as classes, *individual-valued* properties and *data-valued* properties. A *URI reference* (or URIref) is a URI, together with an optional fragment identifier at the end. Uniform Resource Identifiers (URIs) are short strings that identify Web resources [Gro01]. The reader is referred to [HPSB+04] for full details of the model-theoretic semantics and abstract syntax of SWRL.

2.3 Fuzzy Sets

While in classical set theory any element belongs or not to a set, in fuzzy set theory [Zad65] this is a matter of degree. More formally, let X be a collection of elements (the universe of discourse) with cardinality m, i.e $X = \{x_1, x_2, \ldots, x_m\}$. A fuzzy subset A of X, is defined by a membership function $\mu_A(x)$, or simply $A(x)$, $x \in X$. This membership function assigns any $x \in X$ to a rational number between 0 and 1 that represents the degree in which this element belongs to X. The *support*, *Supp(A)*, of A is the crisp set $Supp(A) = \{x \in X \mid A(x) \neq 0\}$.

Using the above idea, the most important operations defined on crisp sets and relations (complement, union, intersection) are extended in order to cover fuzzy sets and fuzzy relations. These operations are now being performed by mathematical functions over the unit interval. More precisely, the complement $\neg A$ of a fuzzy set A is given by $(\neg A)(x) = c(A(x))$ for any $x \in X$, where the function $c : [0, 1] \rightarrow [0, 1]$ is called a fuzzy complement (or simply c-norm), which should satisfy the *boundary conditions*, $c(0) = 1$ and $c(1) = 0$, and be *monotonic decreasing*, i.e. for $a \leq b$, $c(a) \geq c(b)$. Examples of c-norms include the Lukasiewicz negation $c(a) = 1 - a$, which additionally is *continuous* and *involutive*, i.e., for each $a \in [0, 1]$, $c(c(a)) = a$ holds. The intersection of two fuzzy sets A and B is given by $(A \cap B)(x) = t[A(x), B(x)]$, where the function $t : [0, 1]^2 \rightarrow [0, 1]$ is called a triangular norm (t-norm) that should satisfy *boundary condition*, i.e. $t(a, 1) = a$, be *monotonic increasing*, *commutative*, i.e. $t(a, b) = t(b, a)$, and *associative*, i.e., $t(a, t(b, c)) = t(t(a, b), c)$. Examples of t-norms include the Gödel t-norm $t(a, b) = min(a, b)$, which additionally is *idempotent*, i.e. $min(a, a) = a$. The union of two fuzzy sets A and B is given by $(A \cup B)(x) = u[A(x), B(x)]$, where the function $u : [0, 1]^2 \rightarrow [0, 1]$ is called a triangular conorm (or simply s-norm, or u-norm), which should satisfy *boundary condition*, i.e. $u(a, 0) = a$, be *monotonic increasing*, *commutative* and *associative*. Examples of u-norms include the Gödel u-norm $u(a, b) = max(a, b)$, which additionally is *idempotent*.

A binary fuzzy relation R over two countable crisp sets X and Y is a function $R : X \times Y \to [0,1]$. The composition of two fuzzy relation $R_1 : X \times Y \to [0,1]$ and $R_2 : Y \times Z \to [0,1]$ is given by $[R_1 \circ^t R_2] = \sup_{y \in Y} t[R_1(x,y), R_2(y,z)]$. Such a type of composition is referred to as $\sup$-t composition.

Another important operation in fuzzy logics is the *fuzzy implication*, which gives a truth value to the predicate $A \Rightarrow B$. A fuzzy implication is a function ω of the form $\omega : [0,1]^2 \to [0,1]$, which is *monotonic decreasing (increasing)* on the first (second) argument. In fuzzy logics, we are usually interested in two kinds of fuzzy implications, i.e.,

- S-implication: $\omega_{u,c}(a,b) = u(c(a), b)$,
- R-implication: $\omega_t(a,b) = \sup\{x \in [0,1] \mid t(a,x) \leq b\}$,

where a, b are the truth values for A and B, respectively.

We now recall some properties of the above two fuzzy operations that we are going to use in the investigation of the properties of the f-SWRL language.

Lemma 1. *[KY95] For any $a, b, c \in [0,1]$, t a t-norm, ω_t the respective R-implication and ω an R or an S-implication, the following properties are satisfied:*

1. $t(a,b) \leq c$ *iff* $\omega_t(a,c) \geq b$,
2. $\omega_t(a,b) = 1$ *iff* $a \leq b$,
3. $\omega(0,b) = 1$, *(dominance of falsity)*
4. $\omega(1,b) = b$ *(neutrality of truth)*

The last two properties follow easily from the definitions of the fuzzy implications and the boundary conditions of t-norms and u-norms.

The reader is referred to [KY95, Haj98] for details of fuzzy logics and their applications.

3 A Motivating Use Case

In this section, we discuss a motivating use case from a casting company, which has a knowledge base that consists of person-models. Advertisement companies are using this knowledge base to look for models to be used in advertisements or other activities. Each entry in the knowledge base contains a photo of the model, personal information and some body and face characteristics. The casting company has created a user interface for inserting the information of the models as instances of a predefined ontology. It also provides a query engine to search for models with specific characteristics. A user can query the knowledge base providing high-level information about the models (such as the name, the height, the type of the hair etc.).

Now we suppose that we have only information about the following two models in the knowledge base:

- Mary has height 172cm and has weight 50kg.
- Susan has height 180cm and has weight 61kg.

If an advertisement company requires a *thin* female model. Since thinness can be regarded as a function of both the weight as well as the height of a person, one can define thinness as follows.

- One is *thin* iff one is both tall and light.
- One is *tall* iff one's height is larger than 175cm.
- One is *light* iff one's weight is less than 60kg.

Under such definitions, it is obvious that there are no thin female models in the knowledge base. Susan is over 175cm tall but is not under 60kg, while Mary is under 60kg but not over 175cm. Although Mary fails to satisfy the height requirement for only 3cm, which in fact is a rather small value, she satisfies the weight condition; in fact, she is 10kg lighter than the required weight. In fact, the advertisement company might classify her as a thin model if it regards weight a more important factor than height in terms of thinness.

The above problems can be solved if we use a fuzzy knowledge representation, instead of a crisp knowledge representation. In particular, we can define tall and light in a fuzzy way, i.e., by using degrees of confidence. For instance, based on the above data of the two models as well as the policy of the advertisement company, we can have the following fuzzy assertions.

- Mary is tall with a degree no less than 0.65.
- Mary is light with a degree no less than 0.9.
- Susan is tall with a degree no less than 0.8.
- Susan is light with a degree no less than 0.6.

Note that the above membership degrees of the individuals Mary and Susan to the fuzzy classes "tall" and "light" have resulted by providing a *fuzzy partition* [KY95] of the space of the possible values that ones height and weight can obtain. For example, the fuzzy partitions in our example can be depicted in Fig. 1.

In addition to the fuzzy assertion, we can also deduce "one is thin" in a fuzzy way. For instance, we can introduce the following fuzzy rule about thinness: One is thin if one is tall (with importance factor 0.7) and light (with importance factor 0.8).

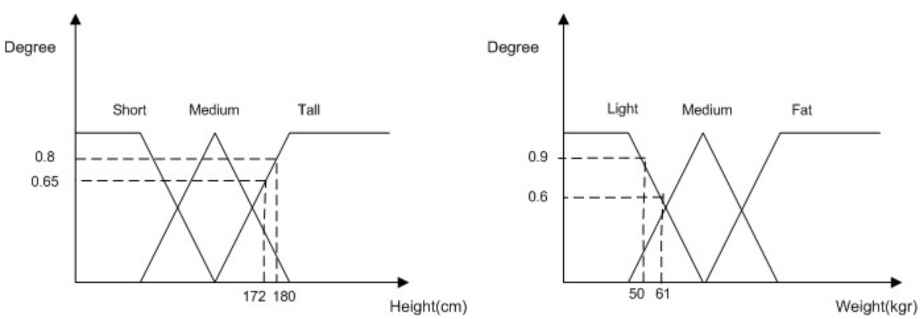

Fig. 1. The fuzzy partition of Height and Weight

After introducing the syntax and semantics of f-SWRL, we will revisit this use case in Section 4.

4 f-SWRL

Fuzzy rules are of the form antecedent → consequent, where atoms in both the antecedent and consequent can have weights (i.e., importance factors), i.e., numbers between 0 and 1. More specifically, atoms can be of the forms C(x)$*w$, P(x,y)$*w$, Q(x,z)$*w$, sameAs(x,y)$*w$, differentFrom(x,y)$*w$ or builtIn(pred,$z_1, \ldots, z_n$), where $w \in [0,1]$ is the weight of an atom, and omitting a weight is equivalent to specifying a value of 1. For instance, the following fuzzy rule axiom, inspired from the field of emotional analysis, asserts that if a man has his eyebrows raised enough and his mouth open then he is happy, and that the condition that he has his eyebrows raised is a bit more important than the condition that he has his mouth open.

$$\mathsf{EyebrowsRaised}(?a) * 0.9 \wedge \mathsf{MouthOpen}(?a) * 0.8 \rightarrow \mathsf{Happy}(?a), \tag{2}$$

In this example, EyebrowsRaised, MouthOpen and Happy are class URIrefs, ?a is a *individual-valued* variable, and 0.9 and 0.8 are the weights of the atoms Eyebrows- Raised(?a) and MouthOpen(?a), respectively.

In this paper, we only consider *atomic* fuzzy rules, i.e., rules with only one atom in the consequent. The weight of an atom in a consequent, therefore, can be seen as indicating the weight that is given to the rule axiom in determining the degree with which the consequent holds. Consider, for example, the following two fuzzy rules:

$$\mathsf{parent}(?x, ?p) \wedge \mathsf{Happy}(?p) \rightarrow \mathsf{Happy}(?x) * 0.8 \tag{3}$$

$$\mathsf{brother}(?x, ?b) \wedge \mathsf{Happy}(?b) \rightarrow \mathsf{Happy}(?x) * 0.4, \tag{4}$$

which share Happy($?x$) in the consequent. Since $0.8 > 0.4$, more weight is given to rule (3) than to rule (4) when determining the degree to which an individual is Happy.

In what follows, we formally introduce the syntax and model-theoretic semantics of fuzzy SWRL.

4.1 Syntax

In this section, we present the syntax of fuzzy SWRL. To make the presentation simple and clear, we use DL syntax (see the following definition) instead of the XML, RDF or abstract syntax of SWRL.

Definition 1. *Let* a, b *be individual URIrefs, l a OWL data literal, C, D OWL class descriptions, r, s OWL* individual-valued *property descriptions, r_1, r_2 individual-valued property URIrefs, s, s_1* data-valued *property URIrefs, pred a datatype predicate, $w, w_1, \ldots, w_n \in [0,1]$, $\vec{v}, \vec{v}_1, \ldots, \vec{v}_n$ are (unary or binary)* *tuples of variables and/or individual URIrefs,* $a_1(\overrightarrow{v_1}), \ldots, a_n(\overrightarrow{v_n})$ *and* $c(\overrightarrow{v})$ *are of the forms* $C(x)$, $r(x,y)$, $s(x,z)$, $sameAs(x,y)$, $differentFrom(x,y)$, $\overline{m}$ *or builtIn(pred,* $z_1, \ldots, z_n$*), where* x, y *are* individual-valued *variables or individual URIrefs,* $\overline{m}$ *is a truth constant, which is a rational number between 0 and 1, and* z, $z_1, \ldots, z_n$ *are* data-valued *variables or OWL data literals.*

An f-SWRL ontology can have the following kinds of axioms:

- *class axioms:* $C \sqsubseteq D$ *(class inclusion axioms);*
- *property axioms:* $r \sqsubseteq r_1$ *(*individual-valued *property inclusion axioms),* $\mathsf{Func}(r_1)$ *(functional* individual-valued *property axioms),* $\mathsf{Trans}(r_2)$ *(transitive property axioms),* $s \sqsubseteq s_1$ *(*data-valued *property inclusion axioms),* $\mathsf{Func}(s_1)$ *(functional* data-valued *property axioms);*
- *individual axioms (facts):* $(\mathtt{a} : C) \geq m$, $(\mathtt{a} : C) \leq m$ *(fuzzy class assertions),* $(\langle \mathtt{a,b} \rangle : r) \geq m$, $(\langle \mathtt{a,b} \rangle : r) \leq m$ *(fuzzy* individual-valued *property assertions),* $(\langle \mathtt{a,l} \rangle : r) \geq m$, $(\langle \mathtt{a,l} \rangle : r) \leq m$ *(fuzzy* data-valued *property assertions),* $\mathtt{a} = \mathtt{b}$ *(individual equality axioms) and* $\mathtt{a} \neq \mathtt{b}$ *(individual inequality axioms);*
- *rule axioms:* $a_1(\overrightarrow{v_1}) * w_1 \wedge \cdots \wedge a_n(\overrightarrow{v_n}) * w_n \rightarrow c(\overrightarrow{v}) * w$ *(fuzzy rule axioms).*

Omitting a degree or a weight is equivalent to specifying the value of 1. ◇

According to the above definition, f-SWRL extends SWRL with fuzzy class assertions, fuzzy property assertions and fuzzy rule axioms. We have some remarks here. Firstly, in f-SWRL, there are two (i.e. $\geq$ and $\leq$) kinds of fuzzy assertions; as first pointed out in [HKS02], we can simulate the form of $(a : C) = m$ by considering two assertions of the form $(a : C) \geq m$ and $(a : C) \leq m$. Secondly, although f-SWRL supports degrees in fuzzy assertions, it does not support degrees in fuzzy class axioms and fuzzy property axioms because it is not very clear how to obtain degrees for them. Nevertheless, it is worth noting that fuzzy class axioms and fuzzy property axioms *have* fuzzy interpretations instead of crisp interpretations (see Section 4.3). Furthermore, we allow the use of truth constants $\overline{m}$ [Pav79, Haj98] in the consequence of a fuzzy rule axiom. This could enable us to simulate fuzzy assertions of the form $(a : C) \leq m$ with fuzzy rule axioms (see Section 4.3).

4.2 Constraints on Semantics

In order to make the semantics of f-SWRL more intuitive, in this section we briefly clarify the constraints of our desired semantics for f-SWRL. The proposed constraints provide a unified framework for giving model theoretic semantics for f-SWRL based on fuzzy intersections (t-norms), fuzzy union (u-norms), fuzzy negations (c-norms), fuzzy implications (ω-norms) and weight operations $g(w,d) : [0,1]^2 \rightarrow [0,1]$, i.e. how to handle the degree d of an atom (in antecedents) and its weight w.

Firstly, one of the most useful relationships which is used to manipulate expressions in propositional logic is the *modus ponen*, which states that $A \cap (A \Rightarrow B) \Rightarrow B$ (if A is true and A implies B, then B is also true). This suggests the following constraint on fuzzy implications.

Constraint 1. *The fuzzy implications used in the semantics of f-SWRL should satisfy the modus ponen:*

$$\omega(t(a, \omega(a, b)), b) = 1.$$

It is easy to verify that, e.g., the following two sets of fuzzy operations satisfy the above constraint:

- $\{t(a, b) = min(a, b), \omega_t(a, b) = \sup\{x \in [0, 1] \mid t(a, x) \leq b\}\}$,
- $\{t(a, b) = a \cdot b, \omega_t(a, b) = \sup\{x \in [0, 1] \mid t(a, x) \leq b\}\}$,

while the set of fuzzy operations $\{t(a, b) = min(a, b), u(a, b) = max(a, b), c(a) = 1 - a, \omega_{u,c}(a, b) = u(c(a), b)\}$ does not (e.g., when $a = 0.4, b = 0.5$). In short, R-implication satisfies Constraint 1, while S-implication does not.

Secondly, we require the weight operations $g(w, d)$ in antecedents satisfy the following properties.

Constraint 2. *The weight operations $g(w, d)$ used in the semantics of f-SWRL should satisfy the following properties:*

1. *monotone in d: if $d_1 < d_2$ then $g(w, d_1) < g(w, d_2)$,*
2. *$g(0, d) = 1, g(1, d) = d$.*

The intuition of Property 1 is immediate. Property 2 ensures that the weight 0 would not affect the result of fuzzy intersections in the antecedent, and that the full membership degree would participate in fuzzy intersections when the weight is 1.

It is easy to verify that, e.g., the following two weight operations satisfy the above constraint:

- $g(a, b) = \begin{cases} a \cdot b & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$,
- $g(a, b) = \omega_t(a, b)$,

while the weight operation $g(a, b) = min(a, b)$ does not (e.g. when $a = 0$).

Thirdly, in order to enable the use of weights in the head atoms as the weights of the rule axiom, we have the following constraint.

Constraint 3. *Given a fuzzy rule $A \rightarrow c * w$, where A is the antecedent of the rule and c is the consequent atom with weight w, the semantics of f-SWRL should satisfy the following property:*

$$\omega(A(\mathcal{I}), c(\mathcal{I})) \geq w,$$

where $A(\mathcal{I})$ and $c(\mathcal{I})$ are interpretations of A and c, respectively.

Intuitively speaking, the above constraint requires that the degree of fuzzy implication should be no less than the weight. This constraint is inspired by Theorem 5 from [DP01], which shows an important property of the weighted rules of the form $A \xrightarrow{\theta} C$, where θ is a weight of the rule.

Furthermore, individual axioms (or facts) are special forms of rule axioms in SWRL. This suggests yet another constraint on the semantics of f-SWRL.

Constraint 4. *The semantics of f-SWRL should ensure that fuzzy individual axioms (fuzzy facts) are special forms of fuzzy rule axioms.*

It is worth noting that we do not require fuzzy class (or property) axioms be special forms of fuzzy rule axioms. In some decidable sub-languages of SWRL, such as the DL-safe SWRL [MSS04], class (or property) axioms are not special forms of rule axioms.

4.3 Model-Theoretic Semantics

In this section, we give a model-theoretic semantics for fuzzy SWRL, based on the constraints specified in the precious section. Although many f-SWRL axioms share the same syntax as their counterparts in SWRL, such as class inclusion axioms, they have different semantics because we use fuzzy interpretations in the model-theoretic semantics of f-SWRL.

Before we provide a model-theoretic semantics for f-SWRL, we introduce the notions of datatype predicates and datatype predicate maps.

Definition 2. (Datatype Predicate) *A* datatype predicate *(or simply* predicate*) p is characterised by an arity $a(p)$, or a minimum arity $a_{min}(p)$ if p can have multiple arities, and a predicate extension (or simply* extension*) $E(p)$.* $\diamondsuit$

For example, $=^{int}$ is a datatype predicate with arity $a(=^{int}) = 2$ and extension $E(=^{int}) = \{\langle i_1, i_2\rangle \in V(integer)^2 \mid i_1 = i_2\}$, where $V(integer)$ is the set of all integers. Datatypes can be regarded as *special* predicates with arity 1 and predicate extensions equal to their value spaces; e.g., the datatype *integer* can be seen as a predicate with arity $a(integer) = 1$ and predicate extension $E(integer) = V(integer)$.[5]

Definition 3. (Predicate Map) *We consider a predicate map $\mathbf{M}_p$ that is a partial mapping from predicate URI references to predicates.* $\diamondsuit$

Intuitively, datatype predicates (resp. datatype predicate URIrefs) in $\mathbf{M}_p$ are called built-in datatype predicates (resp. datatype predicate URIrefs) w.r.t. $\mathbf{M}_p$. Note that allowing the datatype predicate map to vary allows different implementations of f-SWRL to implement different datatype predicates.

Based on the constraints we specified in the previous section, we define the semantics of f-SWRL as follows.

Definition 4. *Let c, t, u be fuzzy negations, fuzzy intersections and fuzzy unions, g weight operations that satisfy Constraints 2. Due to Constraint 1, we choose the R-implication as the fuzzy implication. Given a datatype predicate map $\mathbf{M}_p$, a fuzzy interpretation is a triple $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathcal{I}}\rangle$, where the abstract domain $\Delta^{\mathcal{I}}$ is a non-empty set, the datatype domain contains at least all the data values in the extensions of built-in datatype predicates in $\mathbf{M}_p$, and $\cdot^{\mathcal{I}}$ is a fuzzy interpretation function, which maps*

[5] See [Pan04] for detailed discussions on the relationship between datatypes and datatype predicates.

1. *individual URIref and* individual-valued *variables to elements of* $\Delta^{\mathcal{I}}$,
2. *a class description* C *to a membership function* $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$,
3. *an* individual-valued *property URIref* r *to a membership function* $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$,
4. *an* data-valued *property URIref* q *to a membership function* $q^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}} \to [0, 1]$,
5. *a truth constant* $\overline{m}$ *to itself:* $\overline{m}^{\mathcal{I}} = m$,
6. *a built-in datatype predicate URIref pred to its extension* $pred^{\mathcal{I}} = E(\mathbf{M}_p(pred)) \in (\Delta_{\mathbf{D}})^n$*, where* $n = a(\mathbf{M}_p(pred))$*, so that*
$$builtIn^{\mathcal{I}}(pred, z_1, \ldots, z_n) = \begin{cases} 1 \ \textit{if} \ \langle z_1^{\mathcal{I}}, \ldots, z_n^{\mathcal{I}} \rangle \in pred^{\mathcal{I}} \\ 0 \ \textit{otherwise}, \end{cases}$$
7. *the built-in property sameAs to a membership function*
$$sameAs^{\mathcal{I}}(x, y) = \begin{cases} 1 \ \textit{if} \ x^{\mathcal{I}} = y^{\mathcal{I}} \\ 0 \ \textit{otherwise}, \end{cases}$$
8. *the built-in property differentFrom to a membership function*
$$differentFrom^{\mathcal{I}}(x, y) = \begin{cases} 1 \ \textit{if} \ x^{\mathcal{I}} \neq y^{\mathcal{I}} \\ 0 \ \textit{otherwise}. \end{cases}$$

The fuzzy interpretation function can be extended to give semantics for fuzzy class descriptions listed in Table 2.

A fuzzy interpretation $\mathcal{I}$ *satisfies a class inclusion axiom* $C \sqsubseteq D$*, written* $\mathcal{I} \models C \sqsubseteq D$*, if* $\forall o \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(o) \leq D^{\mathcal{I}}(o)$.

A fuzzy interpretation $\mathcal{I}$ *satisfies an* individual-valued *property inclusion axiom* $r \sqsubseteq r_1$*, written* $\mathcal{I} \models r \sqsubseteq r_1$*, if* $\forall o, q \in \Delta^{\mathcal{I}}, r^{\mathcal{I}}(o, q) \leq r_1^{\mathcal{I}}(o, q)$. $\mathcal{I}$ *satisfies a functional* individual-valued *property axiom* $\mathsf{Func}(r_1)$*, written* $\mathcal{I} \models \mathsf{Func}(r_1)$*, if* $\forall o \in \Delta^{\mathcal{I}}, \inf_{q_1, q_2 \in \Delta^{\mathcal{I}}} u(c(r_1^{\mathcal{I}}(o, q_1)), c(r_1^{\mathcal{I}}(o, q_2))) \geq 1$. $\mathcal{I}$ *satisfies a transitive property axiom* $\mathsf{Trans}(r_2)$*, written* $\mathcal{I} \models \mathsf{Trans}(r_2)$*, if* $\forall o, q \in \Delta^{\mathcal{I}}, r_2^{\mathcal{I}}(o, q) = \sup_{p \in \Delta^{\mathcal{I}}} t[r_2^{\mathcal{I}}(o, p), r_2^{\mathcal{I}}(p, q)]$*, where* t *is a triangular norm. A fuzzy interpretation* $\mathcal{I}$ *satisfies a* data-valued *property inclusion axiom* $s \sqsubseteq s_1$*, written* $\mathcal{I} \models s \sqsubseteq s_1$*, if* $\forall \langle o, l \rangle \in \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}, s^{\mathcal{I}}(o, l) \leq s_1^{\mathcal{I}}(o, l)$. $\mathcal{I}$ *satisfies a functional* data-valued *property axiom* $\mathsf{Func}(s_1)$*, written* $\mathcal{I} \models \mathsf{Func}(s_1)$*, if* $\forall o \in \Delta^{\mathcal{I}}, \inf_{l_1, l_2 \in \Delta_{\mathbf{D}}} u(c(s_1^{\mathcal{I}}(o, l_1)), c(s_1^{\mathcal{I}}(o, l_2))) \geq 1$.

A fuzzy interpretation $\mathcal{I}$ *satisfies a fuzzy class assertion* $(\mathsf{a} : C) \geq m$*, written* $\mathcal{I} \models (\mathsf{a} : C) \geq m$*, if* $C^{\mathcal{I}}(\mathsf{a}) \geq m$. $\mathcal{I}$ *satisfies a fuzzy* individual-valued *property assertion* $(\langle \mathtt{a}, \mathtt{b} \rangle : r) \geq m_2$*, written* $\mathcal{I} \models (\langle \mathtt{a}, \mathtt{b} \rangle : r) \geq m_2$*, if* $r^{\mathcal{I}}(\mathtt{a}, \mathtt{b}) \geq m_2$. $\mathcal{I}$ *satisfies a fuzzy* data-valued *property assertion* $(\langle \mathsf{a}, l \rangle : s) \geq m_3$*, written* $\mathcal{I} \models (\langle \mathsf{a}, l \rangle : s) \geq m_3$*, if* $s^{\mathcal{I}}(\mathsf{a}, l) \geq m_3$*. The semantics of fuzzy assertions using* $\leq$ *are defined analogously.* $\mathcal{I}$ *satisfies an individual equality axiom* $\mathsf{a} = \mathsf{b}$*, written* $\mathcal{I} \models \mathsf{a} = \mathsf{b}$*, if* $\mathsf{a}^{\mathcal{I}} = \mathsf{b}^{\mathcal{I}}$. $\mathcal{I}$ *satisfies an individual inequality axiom* $\mathsf{a} \neq \mathsf{b}$*, written* $\mathcal{I} \models \mathsf{a} \neq \mathsf{b}$*, if* $\mathsf{a}^{\mathcal{I}} \neq \mathsf{b}^{\mathcal{I}}$.

A fuzzy interpretation $\mathcal{I}$ *satisfies a fuzzy rule axiom* $a_1(\overrightarrow{v_1}) * w_1 \wedge \cdots \wedge a_n(\overrightarrow{v_n}) * w_n \to c(\overrightarrow{v}) * w$*, written* $\mathcal{I} \models a_1(\overrightarrow{v_1}) * w_1 \wedge \cdots \wedge a_n(\overrightarrow{v_n}) * w_n \to c(\overrightarrow{v}) * w$*, if* $t(g(w_1, a_1^{\mathcal{I}}(\overrightarrow{v_1}^{\mathcal{I}})), \ldots, g(w_n, a_n^{\mathcal{I}}(\overrightarrow{v_n}^{\mathcal{I}}))) \leq \omega_t(w, c^{\mathcal{I}}(\overrightarrow{v}^{\mathcal{I}}))$. ◇

Table 2. Syntax and Semantics of Fuzzy Class and Property Descriptions

DL Syntax	Semantics
A	$A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$
$\top$	$\top^{\mathcal{I}}(a) = 1$
$\bot$	$\bot^{\mathcal{I}}(a) = 0$
$C_1 \sqcap C_2$	$(C \sqcap D)^{\mathcal{I}}(a) = t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
$C_1 \sqcup C_2$	$(C \sqcup D)^{\mathcal{I}}(a) = u(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
$\neg C$	$(\neg C)^{\mathcal{I}}(a) = c(C^{\mathcal{I}}(a))$
$\{o_1\} \sqcup \{o_2\}$	$(\{o_1\} \sqcup \{o_2\})^{I}(a) = 1$ if $a \in \{ o_1^I, o_2^I\}$
	$(\{o_1\} \sqcup \{o_2\})^{I}(a) = 0$ otherwise
$\exists r.C$	$(\exists r.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(r^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))$
$\forall r.C$	$(\forall r.C)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}} \omega_t(r^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))$
$\exists r.\{o\}$	$(\exists (r.\{o\})^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(r^{\mathcal{I}}(a,b), \{o\}^{\mathcal{I}}(b))$
$\geqslant mr$	$(\geqslant mr)^{\mathcal{I}}(a) = \sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} t_{i=1}^{m} r^{\mathcal{I}}(a, b_i)$
$\leqslant mr$	$(\leqslant mr)^{\mathcal{I}}(a) = \inf_{b_1,\ldots,b_{m+1} \in \Delta^{\mathcal{I}}} u_{i=1}^{m+1} c(r^{\mathcal{I}}(a, b_i))$
$\exists s.d$	$(\exists s.d)^{\mathcal{I}}(a) = \sup_{y \in \Delta_{\mathbf{D}}} t(s^{\mathcal{I}}(a,y), y \in d^{\mathcal{I}})$
$\forall s.d$	$(\forall s.d)^{\mathcal{I}}(a) = \inf_{y \in \Delta_{\mathbf{D}}} \omega_t(s^{\mathcal{I}}(a,y), y \in d^{\mathcal{I}})$
$\geqslant ms$	$(\geqslant ms)^{\mathcal{I}}(a) = \sup_{y_1,\ldots,y_m \in \Delta_{\mathbf{D}}} t_{i=1}^{m} s^{\mathcal{I}}(a, y_i)$
$\leqslant ms$	$(\leqslant ms)^{\mathcal{I}}(a) = \inf_{y_1,\ldots,y_{m+1} \in \Delta_{\mathbf{D}}} u_{i=1}^{m+1} c(s^{\mathcal{I}}(a, y_i))$
R^-	$(R^-)^{\mathcal{I}}(a,b) = R^{\mathcal{I}}(b,a)$

There are some remarks on the above definition. Firstly, as we have seen in the previous section, only R-implication satisfies Constraint 1. Therefore, we implicitly use R-implication for fuzzy rule axioms (see below). In fact, given a fuzzy rule axiom $A \to C$, Definition 4 asserts that an fuzzy interpretation $\mathcal{I}$ satisfies $A \to C$ if $A(\mathcal{I}) \leq C(\mathcal{I})$, where $A(\mathcal{I})$ and $C(\mathcal{I})$ are interpretations of the antecedent A and conclusion C, respectively. By applying Property 2 of Lemma 1, it follows that $\omega_t(A(\mathcal{I}), C(\mathcal{I})) = 1$. One of the consequences of such semantics is the support of *chaining* of rules. Suppose that we have two fuzzy rule axioms $A_1 \to C_1, C_1 \to C_2$, if an fuzzy interpretation $\mathcal{I}$ satisfies both of them, i.e. $A_1(\mathcal{I}) \leq C_1(\mathcal{I})$ and $C_1(\mathcal{I}) \leq C_2(\mathcal{I})$, it follows $A_1(\mathcal{I}) \leq C_2(\mathcal{I})$. In other words, $\mathcal{I}$ also satisfies the fuzzy rule axiom $A_1 \to C_2$.

Secondly, there is more than one choice of semantics of fuzzy class descriptions. The one we presented in Table 2 is simply a relatively straightforward one out of many possible choices. For example, we decide to use R-implication in value restriction ($\forall r.C$) and datatype value restriction ($\forall s.d$) because we use R-implication in fuzzy rule axioms. The semantics of fuzzy number restrictions were first presented in [Str05]. They are derived by the fuzzy version of the First-Order formulae of classical number restrictions [Str05]. It is easy to see that the fuzzy interpretation of ($\geqslant 1r$) is equivalent to that of ($\exists r.\top$).

Furthermore, the semantics of fuzzy functional role axioms is equivalent to that of the fuzzy class inclusion axiom $\top \sqsubseteq \leq 1r$. Note that there are two ways to encode fuzzy disjointness axioms. For example, to assert that C is disjoint with D, one can encode it as the fuzzy class axiom $C \sqcap D \sqsubseteq \bot$ or $C \sqsubseteq \neg D$, which have different semantics. In this paper, we do not prejudge which approach is better and leave it to the users to choose, based on the modelling requirements in their applications.

Let us conclude this section by showing that f-SWRL satisfies all the constraints presented in Section 4.2.

Lemma 2. *Given a f-SWRL rule axiom $A \to c * w$, where A is the antecedent of the rule and c is the consequent atom with weight w, we have $\omega_t(A(\mathcal{I}), c(\mathcal{I})) \geq w$, where $A(\mathcal{I})$ and $c(\mathcal{I})$ are interpretations of A and c, respectively.*

Proof: According to the Definition 4, we have $A(\mathcal{I}) \leq \omega_t(w, c(\mathcal{I}))$. Due to Property 1 of Lemma 1, we have $t(w, A(\mathcal{I})) \leq c(\mathcal{I})$; i.e., $t(A(\mathcal{I}), w) \leq c(\mathcal{I})$. Due to Property 1 of Lemma 1 again, we have $\omega_t(A(\mathcal{I}), c(\mathcal{I})) \geq w$. □

Lemma 3. *In f-SWRL, fuzzy assertions are special forms of fuzzy rule axioms.*

Proof: $(a : C) \geq m$ can be simulated by $\top(a) \to C(a) * m$. According to Definition 4, we have $1 \leq \omega_t(m, C^{\mathcal{I}}(a))$. Due to Property 2 of Lemma 1, we have $C^{\mathcal{I}}(a) \geq m$, which is the interpretation of $(a : C) \geq m$.

$(a : C) \leq m$ can be simulated by $C(a) \to \overline{m}$, where $\overline{m}$ is a truth constant. According to Definition 4, we have $C^{\mathcal{I}}(a) \leq \omega_t(1, m)$. Due to Property 4 of Lemma 1, we have $C^{\mathcal{I}}(a) \leq m$, which is the interpretation of $(a : C) \leq m$.

Similarly, $(\langle a, b\rangle : r) \geq m$ can be simulated by $\top(a) \wedge \top(b) \to r(a, b) * m$, and $(\langle a, b\rangle : r) \leq m$ can be simulated by $r(a, b) \to \overline{m}$. □

Based on Definition 4, Lemma 2 and Lemma 3, we have the following theorem.

Theorem 1. *f-SWRL satisfies Constraints 1-4.*

5 Examples

In this section, we use some examples to further illustrate the semantics of f-SWRL. Firstly, let us revisit our motivating example presented in Section 3, so as to show that the use of different fuzzy and weight operations could lead to very different results.

Example 1. The corresponding f-SWRL knowledge base about models consists of the following fuzzy axioms:

- Mary is Tall with a degree no less than 0.65: $(\mathtt{Mary} : \mathsf{Tall}) \geq 0.65$.
- Mary is Light with a degree no less than 0.9: $(\mathtt{Mary} : \mathsf{Light}) \geq 0.9$.
- Susan is Tall with a degree no less than 0.8: $(\mathtt{Susan} : \mathsf{Tall}) \geq 0.8$.
- Susan is Light with a degree no less than 0.6: $(\mathtt{Susan} : \mathsf{Light}) \geq 0.6$.
- One is Thin if one is Tall (with importance factor 0.7) and Light (with importance factor 0.8):

$$\mathsf{Tall}(?p) * 0.7 \wedge \mathsf{Light}(?p) * 0.8 \to \mathsf{Thin}(?p).$$

The interpretation of the above rule axiom is as follows.

$$t(g(0.7, \mathsf{Tall}^{\mathcal{I}}(?p^{\mathcal{I}})), g(0.8, \mathsf{Light}^{\mathcal{I}}(?p^{\mathcal{I}}))) \le \omega_t(1, \mathsf{Thin}^{\mathcal{I}}(?p^{\mathcal{I}})).$$

In this example, we first use the following operations: $t(a,b) = min(a,b), \omega_t(a,b) = \begin{cases} 1 & \text{if } a \le b \\ b & \text{if } a > b \end{cases}$, $g(a,b) = \begin{cases} a \cdot b & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$. According to Definition 4, we have $\mathsf{Thin}^{\mathcal{I}}(\texttt{Mary}^{\mathcal{I}}) \ge min(0.7 \cdot 0.65, 0.8 \cdot 0.9) = min(0.455, 0.72) = 0.455$, while $\mathsf{Thin}^{\mathcal{I}}(\texttt{Susan}^{\mathcal{I}}) \ge min(0.7 \cdot 0.8, 0.8 \cdot 0.6) = min(0.56, 0.48) = 0.48$. As a result, Susan seems to be thinner than Mary in this setting.

If we choose another set of operations, the conclusion, however, can be completely different.

For example, now we use the following operations: $t(a,b) = a \cdot b$, $\omega_t(a,b) = \begin{cases} 1 & \text{if } a \le b \\ b/a & \text{if } a > b \end{cases}$, $g(a,b) = \omega_t(a,b)$. According to Definition 4, we have $\mathsf{Thin}^{\mathcal{I}}(\texttt{Mary}^{\mathcal{I}}) \ge \omega_t(0.7, 0.65) \cdot \omega_t(0.8, 0.9) = 0.929 \cdot 1 = 0.929$, while $\mathsf{Thin}^{\mathcal{I}}(\texttt{Susan}^{\mathcal{I}}) \ge \omega_t(0.7, 0.8) \cdot \omega_t(0.8, 0.6) = 1 \cdot 0.75 = 0.75$. As a result, Mary seems to be quite thinner than Susan in this setting. ♢

The above example indicates that t-torm based weights give quite different meaning than ω_t based weights.

Secondly, we revisit rules (3) and (4) discussed at the beginning of Section 4. Interestingly, this time the above two sets of operations lead to the agreeing result.

Example 2. Suppose we have an f-SWRL knowledge base as follows:

- Tom is Happy with a degree no less than 0.7: $(\texttt{Tom} : \mathsf{Happy}) \ge 0.7$,
- Tom is a *parent* of Jane: $\langle \texttt{Jane}, \texttt{Tom} \rangle : parent$,
- Tom is a *brother* of Kate: $\langle \texttt{Kate}, \texttt{Tom} \rangle : brother$,
- if one's *parent* is Happy, then one is Happy (with importance factor 0.8):

$$\mathsf{parent}(?x, ?p) \land \mathsf{Happy}(?p) \to \mathsf{Happy}(?x) * 0.8,$$

- if one's *brother* is Happy, then one is Happy (with importance factor 0.4):

$$\mathsf{brother}(?x, ?b) \land \mathsf{Happy}(?b) \to \mathsf{Happy}(?x) * 0.4.$$

Let us use the two sets of operations in Example 1 with this knowledge base.

Firstly, we use the following operations: $t(a,b) = min(a,b), \omega_t(a,b) = \begin{cases} 1 & \text{if } a \le b \\ b & \text{if } a > b \end{cases}$, $g(a,b) = \begin{cases} a \cdot b & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$. According to Definition 4, we have $\omega_t(0.8, \mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}})) \ge min(1 \cdot 1, 1 \cdot 0.7) = 0.7$. Due to Property 1 of Lemma 1, we have $\mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}}) \ge 0.7$. As for Kate, we have $\omega_t(0.4, \mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}})) \ge min(1 \cdot 1, 1 \cdot 0.7) = 0.7$; hence, $\mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}}) \ge 0.4$. Hence, Jane seems to be happier than Kate.

Now we use the following operations: $t(a,b) = a \cdot b$, $\omega_t(a,b) = \begin{cases} 1 & \text{if } a \le b \\ b/a & \text{if } a > b \end{cases}$, $g(a,b) = \omega_t(a,b)$. According to Definition 4, we have $\omega_t(0.8, \mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}})) \ge$

$\omega_t(1,1)\cdot\omega_t(1,0.7) = 0.7$; hence, $\mathsf{Happy}^{\mathcal{I}}(\mathtt{Jane}^{\mathcal{I}}) \geq t(0.8, 0.7) = 0.56$. As for Kate, we have $\omega_t(0.4, \mathsf{Happy}^{\mathcal{I}}(\mathtt{Kate}^{\mathcal{I}})) \geq \omega_t(1,1) \cdot \omega_t(1, 0.7) = 0.7$; hence, $\mathsf{Happy}^{\mathcal{I}}(\mathtt{Kate}^{\mathcal{I}}) \geq t(0.4, 0.7) = 0.28$. Again, Jane seems to be happier than Kate. ♢

So far we have only seen fuzzy assertions of the form $(a : C) \geq m$; in the next example, we will use fuzzy assertions of the form $(a : C) \leq m$.

Example 3. Suppose we have a slightly different f-SWRL knowledge base from that in the previous example.

- Jane is Happy with a degree no larger than 0.75: $(\mathtt{Jane} : \mathsf{Happy}) \leq 0.75$,
- Kate is Happy with a degree no larger than 0.85: $(\mathtt{Kate} : \mathsf{Happy}) \leq 0.85$,
- Tom is a *parent* of Jane: $\langle \mathtt{Jane}, \mathtt{Tom} \rangle : parent$,
- Tom is a *brother* of Kate: $\langle \mathtt{Kate}, \mathtt{Tom} \rangle : brother$,
- if one's *parent* is Happy, then one is Happy (with importance factor 0.8):

$$\mathsf{parent}(?x, ?p) \wedge \mathsf{Happy}(?p) \rightarrow \mathsf{Happy}(?x) * 0.8 \tag{5}$$

- if one's *brother* is Happy, then one is Happy (with importance factor 0.4):

$$\mathsf{brother}(?x, ?b) \wedge \mathsf{Happy}(?b) \rightarrow \mathsf{Happy}(?x) * 0.4, \tag{6}$$

Here we use the following operations: $t(a,b) = min(a,b), \omega_t(a,b) = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$, $g(a,b) = \begin{cases} a \cdot b & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$. From (5), we have $\mathsf{Happy}^{\mathcal{I}}(\mathtt{Tom}^{\mathcal{I}}) \leq \omega_t(0.8, \mathsf{Happy}^{\mathcal{I}}$ $(\mathtt{Jane}^{\mathcal{I}})) \leq \omega_t(0.8, 0.75) = 0.75$. Hence, we have $\mathsf{Happy}^{\mathcal{I}}(\mathtt{Tom}^{\mathcal{I}}) \leq 0.75$. From (6), we have $\omega_t(0.4, \mathsf{Happy}^{\mathcal{I}}(\mathtt{Kate}^{\mathcal{I}})) \geq \mathsf{Happy}^{\mathcal{I}}$ $(\mathtt{Tom}^{\mathcal{I}})$. Due to Property 1 of Lemma 1, we have $\mathsf{Happy}^{\mathcal{I}}(\mathtt{Kate}^{\mathcal{I}}) \geq min(0.4, \mathsf{Happy}^{\mathcal{I}}(\mathtt{Tom}^{\mathcal{I}}))$.

It is easy to verify that we have the same results if we use the other set of operations. ♢

6 Discussion

In this paper, we have proposed f-SWRL, a fuzzy extension to SWRL to include fuzzy assertions and fuzzy rules. We have provided formal syntax and semantics for f-SWRL, shown how weights of atoms in consequences of fuzzy rule can be used as important factors of fuzzy rules, illustrated the features of f-SWRL with several examples.

The main strength of the proposal is the openness of the use of fuzzy and weight operations. As many theoretical and practical studies [Voj01] have pointed out, the choice of these operations is usually context dependent. Therefore, it is appropriate to simply specify some key constraints of the desired semantics of f-SWRL and to allow the use of any of these operations as long as they conform to the key constraints. Like in SWRL, in f-SWRL assertions are special forms of rules. Although class and property axioms are not associated with any degrees or important factors, they have fuzzy interpretations instead of crisp interpretations. We show that f-SWRL may be applied in many applications, such as multimedia processing and retrieval. To the best of our knowledge, this is the first effort on fuzzy extensions of SWRL.

Several ways of extending Description Logics using the theory of fuzzy logic have been proposed in the literature [Yen91, TM98, Str01, Str05, SST+05]. Furthermore, in [Str04] an approach to extend *Description Logic Programs* (DLPs) with uncertainty was provided, where DLP is extended with *negation as failure*. DLPs are different from SWRL in that rules in DLPs are programs instead of axioms; therefore, the semantics of rules in DLPs are based on Herbrand models instead of model theoretic semantics. [Voj01] presents an approach to fuzzy logic programs which is similar to ours. In that approach, interpretations of rules are based on Herbrand models, instead of model theoretic semantics. The main difference from our work is that weights are only for the whole rules, not for rule atoms. The semantics of weights, accordingly, are based on *fuzzy aggregation* functions, such as linear aggregation or weighted sum.

Our future work includes further investigation of logical properties and computational aspect of f-SWRL. Another interesting direction is to extend f-SWRL to support datatype groups [Pan04], which allows the use of customised datatypes and datatype predicates in ontologies.

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