Axiomatic Systems of Generalized Rough Sets*

William Zhu[1,2,3] and Feiyue Wang[1,4]

[1] The Key Laboratory of Complex Systems and Intelligent Science,
Institute of Automation, The Chinese Academy of Sciences, Beijing 100080, China
[2] Department of Computer Science, University of Auckland, Auckland, New Zealand
[3] Computer Information Engineering College, Jiangxi Normal University, China
[4] Systems and Industrial Engineering Department,
The University of Arizona, Tucson, AZ 85721, USA
fzhu009@ec.auckland.ac.nz, feiyue@sie.arizona.edu

Abstract. Rough set theory was proposed by Pawlak to deal with the vagueness and granularity in information systems that are characterized by insufficient, inconsistent, and incomplete data. Its successful applications draw attentions from researchers in areas such as artificial intelligence, computational intelligence, data mining and machine learning. The classical rough set model is based on an equivalence relation on a set, but it is extended to generalized model based on binary relations and coverings. This paper reviews and summarizes the axiomatic systems for classical rough sets, generalized rough sets based on binary relations, and generalized rough sets based on coverings.

Keywords: Rough set, Covering, Granular computing, Data mining.

1 Introduction

Across a wide variety of fields, data are being collected and accumulated at a dramatic pace, especially at the age of Internet. Much useful information is hidden in the accumulated voluminous data, but it is very hard for us to obtain it. In order to mine knowledge from the rapidly growing volumes of digital data, researchers have proposed many methods other than classical logic, for example, fuzzy set theory [1], rough set theory [2], computing with words [3,4], granular computing [5], computational theory for linguistic dynamic systems [6], etc.

Rough set theory was originally proposed by Pawlak [2]. It provides a systematic approach for classification of objects through an indiscernability relation. An equivalence relation is the simplest formulization of the indiscernability. However, it cannot deal with some granularity problems we face in real information systems, thus many interesting and meaningful extensions have been made to tolerance relations [7,8], similarity relations [9], coverings [10,11,12,13,14,15,16], etc.

* The first author is in part supported by the New Economy Research Fund of New Zealand and this work is also in part supported by two 973 projects (2004CB318103) and (2002CB312200) from the Ministry of Science and Technology of China.

G. Wang et al. (Eds.): RSKT 2006, LNAI 4062, pp. 216–221, 2006.

In this paper, we summarize axiomatic systems for classical rough sets, binary relation based rough sets, and covering based rough sets. The remainder of this paper is structured as follows. Section 2 is devoted to axiomatic systems for classical rough sets from various points of view. In Section 3, we formulate axiomatic systems for generalized rough sets based on general binary relations, reflexive relations, symmetric relations, and transitive relations. Section 4 defines a new type of generalized rough sets based on coverings and establishes axiomatic systems for its lower and upper approximation operations. This paper concludes in Section 5.

2 Axiomization of Classical Rough Sets

People use algebraic, topological, logical and constructive methods to study rough sets and try to formulate axiomatic systems for classical rough sets from different views [18,19,20,21].

Lin and Liu obtained the following axiom system for rough sets [18] through topological methods.

Theorem 1. *For an operator $L : P(U) \to P(U)$, if it satisfies the following properties:*
1. $L(U) = U$, 2. $L(X) \subseteq X$, 3. $L(X \cap Y) = L(X) \cup L(Y)$,
4. $L(L(X)) = L(X)$, 5. $-L(X) = L(-L(X))$,
then there is an equivalence relation R such that L is the lower approximation operator induced by R.

Yao established the following result about axiomatic systems for classical rough sets [22,23].

Theorem 2. *For a pair of dual operators $H : P(U) \to P(U)$, if H satisfies the following five properties:*
1. $H(\phi) = \phi$, 2. $X \subseteq H(X)$, 3. $H(X \cup Y) = H(X) \cup H(Y)$,
4. $H(H(X)) = H(X)$, 5. $X \subseteq -H(-H(X))$,
then there is an equivalence relation R such that H is the upper approximation operator induced by R.

The following two axiomatic systems for rough sets belong to Zhu and He [19]. They discussed the redundancy problems in axiomatic systems.

Theorem 3. *For an operator $L : P(U) \to P(U)$, if it satisfies the following properties:*
1. $L(X) \subseteq X$, 2. $L(X) \cap L(Y) = L(X \cup Y)$, 3. $-L(X) = L(-L(X))$,
then there is an equivalence relation R such that L is the lower approximation operator induced by R.

Theorem 4. *For an operator $L : P(U) \to P(U)$, if it satisfies the following properties:*
1. $L(U) = U$, 2. $L(X) \subseteq X$, 3. $L(L(X) \cap Y) = L(X) \cup L(Y)$,
then there is an equivalence relation R such that L is the lower approximation operator induced by R.

Sun, Liu and Li established the following three axiomatic systems for rough sets [20]. They focused on replacing equalities with inequalities to achieve certain minimal property.

Theorem 5. *For an operator $L : P(U) \rightarrow P(U)$, if it satisfies the following properties:*
1. $L(X) \subseteq X$, 2. $L(X \cap Y) \subseteq L(X) \cup L(Y)$, 3. $-L(X) \subseteq L(-L(X))$,
then there is an equivalence relation R such that L is the lower approximation operator induced by R.

Theorem 6. *For an operator $L : P(U) \rightarrow P(U)$, if it satisfies the following properties:*
1. $L(X) \subseteq X$, 2. $L(X) \cup L(Y) \subseteq L(L(X) \cup Y)$, 3. $-L(X) \subseteq L(-L(X))$,
then there is an equivalence relation R such that L is the lower approximation operator induced by R.

Theorem 7. *For an operator $L : P(U) \rightarrow P(U)$, if it satisfies the following properties:*
1. $L(X) \subseteq X$, 2. $L(-X \cup Y) \subseteq -L(X) \cup L(Y)$, 3. $-L(-X) \subseteq L(-L(-X))$,
then there is an equivalence relation R such that L is the lower approximation operator induced by R.

3 Axiomization of Binary Relation Based Rough Sets

Paper [21,23,24,25] have done an extensive research on algebraic properties of rough sets based on binary relations. They proved the existence of a certain binary relation for an algebraic operator with special properties, but they did consider the uniqueness of such a binary relation. We proved the uniqueness of the existence of such binary relations in [26].

3.1 Basic Concepts and Properties

Definition 1 (Rough set based on a relation [23]). *Suppose R is a binary relation on a universe U. A pair of approximation operators, $L(R), H(R) : P(U) \rightarrow P(U)$, are defined by:*

$$L(R)(X) = \{x | \forall y,\ xRy \Rightarrow y \in X\},\ \text{and}\ H(R)(X) = \{x | \exists y \in X,\ s.t.\ xRy\}.$$

They are called the lower approximation operator and the upper approximation operator respectively. The system $(P(U), \cap, \cup, -, L(R), H(R))$ is called a rough set algebra, where $\cap, \cup$, and - are set intersection, union, and complement.

Theorem 8 (Basic properties of lower and upper approximation operators [23]). *Let R be a relation on U. $L(R)$ and $H(R)$ satisfy the following properties: $\forall X, Y \subseteq U$,*
(1) $L(R)(U) = U$ (2) $L(R)(X \cap Y) = L(R)(X) \cap L(R)(Y)$
(3) $H(R)(\phi) = \phi$ (4) $H(R)(X \cup Y) = H(R)(X) \cup H(R)(Y)$
(5) $L(R)(-X) = -H(R)(X)$

3.2 Axiomatic Systems of Generalized Rough Sets Based on Relations

Theorem 9. *[23,26] Let U be a set. If an operator $L : P(U) \to P(U)$ satisfies the following properties:*
(1)$L(U) = U$ (2)$L(X \cap Y) = L(X) \cap (Y)$
then there exists one and only one relation R on U such that $L = L(R)$.

Theorem 10. *[23,26] Let U be a set. If an operator $H : P(U) \to P(U)$ satisfies the following properties:*
(1)$H(\phi) = \phi$ (2)$H(X \cup Y) = H(X) \cup H(Y)$
then there exists one and only one relation R on U such that $H = H(R)$.

Theorem 11. *[23,26] Let U be a set. An operator $L : P(U) \to P(U)$ satisfies the following properties:*
(1)$L(U) = U$ (2)$L(X \cap Y) = L(X) \cap (Y)$,
then,
(A) L also satisfies $L(X) \subseteq X$ if and only if there exists one and only one reflexive relation R on U such that $L = L(R)$.
(B) L also satisfies $L(X) \subseteq L(-L(-X))$ if and only if there exists one and only one symmetric relation R on U such that $L = L(R)$.
(C) L also satisfies $L(X) \subseteq L(L(X))$ if and only if there exists one and only one relation R on U such that $L = L(R)$.

4 Axiomization of Covering Based Rough Sets

In this section, we present basic concepts for a new type of covering generalized rough sets and formulate axiomatic systems for them. As for their properties, please refer to [27,11,12,13,28,29].

4.1 A New Type of Covering Generalized Rough Sets

Paper [30] introduced a new definition for binary relation based rough sets. The core concept for this definition is the neighborhood of a point. As we can see from [13,31,23], binary relation based rough sets are different from covering based rough sets, thus we introduce the neighborhood concept into covering based rough sets [27].

Definition 2 (Neighborhood). *Let U be a set, C a covering of U. For any $x \in U$, we define the neighborhood of x as $Neighbor(x) = \cap\{K \in C | x \in K\}$.*

Definition 3 (Lower and upper approximations). *$\forall X \subseteq U$, the fourth type of lower approximation of X is defined as $X_+ = \cup\{K | K \in \mathbf{C}$ and $K \subseteq X\}$ and the fourth type of upper approximation of X is defined as $X^+ = X_+ \cup \{Neighbor(x) | x \in X - X_+\}$.*

Operations IL and IH on $P(U)$ defined as $IL_{\mathbf{C}}(X) = X_+$, $IH_{\mathbf{C}}(X) = X^+$ are called fourth type of lower and upper approximation operations, coupled with the covering $\mathbf{C}$, respectively. When the covering is clear, we omit the lowercase $\mathbf{C}$ for the two operations.

4.2 Axiomatic Systems of Generalized Rough Sets Based on Coverings

We present axiomatic systems for lower and upper approximation operations.

Theorem 12 (An axiomatic system for lower approximation operations [12,13]). *Let U be a non-empty set. If an operator $L : P(U) \rightarrow P(U)$ satisfies the following properties: for any $X, Y \subseteq U$,*
(1) $L(U) = U$ *(2) $X \subseteq Y \Rightarrow L(X) \subseteq L(Y)$*
(3) $L(X) \subseteq X$ *(4) $L(L(X)) = L(X)$*
then there exists a covering $\mathbf{C}$ *of U such that the lower approximation operation IL generated by* $\mathbf{C}$ *equals to L.*

Furthermore, the above four properties are independent.

Theorem 13 (An axiomatic system for upper approximation operations [27]). *Let U be a non-empty set. If an operation $H : P(U) \rightarrow P(U)$ is a closure operator, e. g., H satisfies the following properties: for any $X, Y \subseteq U$,*
(cl1) $H(X \cup Y) = H(X) \cup H(Y)$ *(cl2) $X \subseteq H(X)$*
(cl3) $H(\phi) = \phi$ *(cl4) $H(H(X)) = H(X)$*
then there exists a covering $\mathbf{C}$ *of U such that the fourth type of upper approximation operation IH generated by* $\mathbf{C}$ *equals to H.*

Furthermore, the above four properties are independent.

5 Conclusions

This paper is devoted to reviewing and summarizing axiomatic systems for classical rough sets, generalized rough sets based on binary relations, and generalized rough sets based on coverings.

References

1. Zadeh, L. A.: Fuzzy sets. *Information and Control* **8** (1965) 338–353.
2. Pawlak, Z.: Rough sets. *Internat. J. Comput. Inform. Sci.* **11** (1982) 341–356.
3. Wang, F. Y.: Outline of a computational theory for linguistic dynamic systems: Toward computing with words. *International Journal of Intelligent Control and Systems* **2** (1998) 211–224.
4. Zadeh, L.: Fuzzy logic = computing with words. *IEEE Transactions on Fuzzy Systems* **4** (1996) 103–111.
5. Lin, T. Y.: Granular computing - structures, representations, and applications. In: LNAI. Volume 2639. (2003) 16–24.
6. Wang, F. Y.: On the abstraction of conventional dynamic systems: from numerical analysis to linguistic analysis. *Information Sciences* **171** (2005) 233–259.
7. Cattaneo, G.: Abstract approximation spaces for rough theories. In: Polkowski L. and Skowron A., ed., *Rough Sets in Knowledge Discovery 1: Methodology and Applications.* (1998) 59–98.
8. Skowron, A., J. S.: Tolerance approximation spaces. *Fundamenta Informaticae* **27** (1996) 245–253.

9. Slowinski, R., Vanderpooten, D.: A generalized definition of rough approximations based on similarity. *IEEE Trans. On Knowledge and Data Engineering* **12** (2000) 331–336.
10. Zakowski, W.: Approximations in the space (u, π). *Demonstratio Mathematica* **16** (1983) 761–769.
11. Bonikowski, Z., Bryniarski, E., Wybraniec-Skardowska, U.: Extensions and intentions in the rough set theory. *Information Sciences* **107** (1998) 149–167.
12. Zhu, F.: On covering generalized rough sets. Master's thesis, The Universite of Arizona, Tucson, Arizona, USA (2002).
13. Zhu, W., Wang, F. Y.: Reduction and axiomization of covering generalized rough sets. *Information Sciences* **152** (2003) 217–230.
14. Ma, J. M., Zhang, W. X., Li, T. J.: A covering model of granular computing. In: Proceedings of the Fourth International Conference on Machine Learning and Cybernetics. (2005) 1625–1630.
15. Wu, W. Z., Zhang, W. X.: Constructive and axiomatic approaches of fuzzy approximation operator. *Information Sciences* **159** (2004) 233–254.
16. Dai, J., Chen, W., Pan, Y.: A minimal axiom group for rough set based on quasi-ordering. *Journal of Zhejiang Unicersity Science* **5** (2004) 810–815.
17. Yao, Y., Wang, F. Y., Wang, J.: ″Rule + Exception″ strategies for knowledge management and discovery. In: RSFDGrC 2005. Volume 3641 of LNAI. (2005) 69–78.
18. Lin, T. Y., Liu, Q.: Rough approximate operators: axiomatic rough set theory. In: Ziarko, W., eds., *Rough Sets, Fuzzy Sets and Knowledge Discovery*, Springer (1994) 256–260.
19. Zhu, F., He, H. C.: The axiomization of the rough set. *Chinese Journal of Computers* **23** (2000) 330–333.
20. Sun, H., Liu, D. Y., Li, W.: The minimization of axiom groups of rough set. *Chinese Journal of Computers* **25** (2002) 202–209.
21. Thiele, H.: On axiomatic characterisations of crisp approximation operators. *Information Sciences* **129** (2000) 221–226.
22. Yao, Y.: Two views of the theory of rough sets in finite universes. *International Journal of Approximate Reasoning* **15** (1996) 291–317.
23. Yao, Y.: Constructive and algebraic methods of theory of rough sets. *Information Sciences* **109** (1998) 21–47.
24. Cattaneo, G., Ciucci, D.: Algebraic structures for rough sets. In: LNCS. Volume 3135. (2004) 208–252.
25. Yang, X. P., Li, T. J.: The minimization of axiom sets characterizing generalized approximation operators. *Information Sciences* **176** (2006) 887–899.
26. Zhu, W., Wang, F. Y.: Binary relation based rough set. manuscript (2006).
27. Zhu, W.: Topological approaches to covering rough sets. manuscript, submitted to *Information Sciences* (2006).
28. Bryniaski, E.: A calculus of rough sets of the first order. *Bull. Pol. Acad. Sci.* **36** (1989) 71–77.
29. Pomykala, J.: Approximation, similarity and rough constructions. In: ILLC Prepublication series, University of Amsterdam. Volume CT-93-07. (1993).
30. Allam, A., Bakeir, M., Abo-Tabl, E.: New approach for basic rough set concepts. In: RSFDGrC 2005. Volume 3641 of LNCS. (2005) 64–73.
31. Yao, Y.: Relational interpretations of neighborhood operators and rough set approximation operators. *Information Sciences* **101** (1998) 239–259.