Intermediate Concepts in Normative Systems

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Abstract. In legal theory, a well-known idea is that an intermediate concept like "ownership" joins a set of legal consequences to a set of legal grounds. In our paper, we attempt to make the idea of a joining between grounds and consequences more precise by using an algebraic representation of normative systems earlier developed by the authors. In the first main part, the idea of intermediate concepts is presented and earlier discussions of the subjects are outlined. Subsequently, in the second main part, we introduce a more rigorous framework and develop the formal theory. In the third part, the formal framework is applied to examples and some remarks on a methodology of intermediate concepts are given.

1 The Problem of Intermediaries

1.1 Introduction

The role played by concept formation in philosophy and science has been varying. After some decades of rather low interest, there are signs indicating that the situation is changing. The aim of the present paper is to contribute to the study of this field. More specifically, our contribution aims at presenting a framework for analysing the role of what we call "intermediaries" as links between conceptual structures.

In [5], we presented a first working model for analysing the notion of intermediary. The present paper is different in several respects. The framework to be developed is based on the theory of Boolean algebra instead of lattice theory. The structures dealt with are not necessarily finite. The basic kind of relations considered are quasi-orderings rather than partial orderings as was the case in our previous paper, where partial orderings were introduced by a transition to equivalence classes. The framework is abstract in the sense that the main results are not tied to a specific interpretation in terms of conditions as was the case in the earlier paper.[1] Thus, the case where the domains of the orderings have conditions, or equivalence classes of conditions, as their members only plays the part of one of several models for the theory.

The first part of the paper presents the background of the idea of intermediaries. The second part introduces the formal framework. In the third part, the formal tools are used to clarify different types of intermediaries in concept formation.

[1] For our previous development of the abstract theory, see, in particular, [6] with further references. Cf. [8].

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1.2 Legal Concepts as Intermediaries

Facts, Deontic Positions and Intermediaries. Legal rules attach obligations, rights, deontic positions to facts, i.e., actions, events, circumstances. Deontic positions are, so we might say, legal consequences of these facts:

Facts	*Deontic positions*
Events, actions, circumstances	Obligations, claims, powers etc.

Facts and deontic positions are objects of two different sorts; we might call them Is-objects and Ought-objects. In a legal system, when Ought-objects are said to be "attached to" or to be "consequences of" Is-objects, there is sense of direction. In a legal system, inferences and arguments go from Is-objects to Ought-objects, not vice versa.

In the scheme just shown, something very essential is missing, namely the great bulk of more specific legal concepts. A few examples are: property, tort, contract, trust, possession, guardianship, matrimony, citizenship, crime, responsibility, punishment. These concepts are links between grounds on the left hand side and normative consequences on the right hand side of the scheme just given:

Facts	*Links*	*Deontic positions*
Events	Ownership	Obligations
Actions	Valid contract	Claims
Circumstances	Citizenship (etc.)	Powers (etc.)

Using this three-column scheme, we might say that ownership, valid contract, citizenship etc. are attached to certain facts, and that deontic positions, in turn, are attached to these legal positions.

To exemplify: Among the facts justifying an assertion that there is a valid contract between two parties are: that the parties have made an agreement, that they were in a sane state of mind when agreeing, that no force or deceit was used by any of them in the process, and so on. The deontic positions attached to there being a valid contract between them depend on what they have agreed on but are formulated in terms of claims and duties, legal powers etc. In the example, the facts are stated in terms of communicative acts, mental states and other descriptive notions, while the deontic positions are stated in normative or deontic terms.

Wedberg and Ross on Ownership. In the 1950's, each of the two Scandinavians Wedberg and Ross proposed the idea that a legal term such as "ownership", or "x is the owner of y at time t" is a syntactical tool serving the purpose of economy of expression of a set of legal rules. In the same year 1951, when Ross published his well-known essay "Tû-Tû" in a Danish Festschrift [10][2], Wedberg published an essay on the same theme in the Swedish journal *Theoria*. Possibly, the two authors arrived at these ideas independently of each other.[3] In any case no priority can be established.

[2] English translation [11].
[3] Cf [12] at p. 266, footnote 15, and [11] at p. 822, footnote 6.

As an example, the function of the term "ownership" is illustrated as follows by Ross [10], [11]:

$$\left.\begin{array}{l} F_1 \rightarrow \\ F_2 \rightarrow \\ F_3 \rightarrow \\ \vdots \\ F_p \rightarrow \end{array}\right\} \quad O \rightarrow \quad \left\{\begin{array}{l} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{array}\right.$$

Ross's scheme is aimed at representing a set of legal rules concerning ownership in a particular legal system (for example the rules on ownership in Danish law at a specific time). In the picture, the letters are to be interpreted as follows:

$F_1 - F_p$ **for:** x has lawfully purchased y, x has inherited y, x has acquired y by prescription, and so on.

$C_1 - C_n$ **for:** judgment for recovery shall be given in favor of x against other persons retaining y in their possession, judgment for damages shall be given in favor of x against other persons who culpably damage y, if x has raised a loan from z that it is not repaid at the proper time, z shall be given judgment for satisfaction out of y, and so on.

The letter "O" is a link between the left hand side and the right hand side. It can be read "x is the owner of y".

In Ross's scheme, the number of implications to ownership from the grounds for ownership is p (since the grounds are $F_1, \ldots, F_p$); similarly the number of implications from ownership to consequences of ownership is n (since there are n consequences). Therefore, the total number of implications in the scheme is $p + n$. On the other hand, if the rules were formulated by attaching each C_j among the consequences to each F_i among the grounds, the number of rules would be $p \cdot n$. Consequently, by the formulation in the scheme, the number of rules is reduced from $p \cdot n$ to $p + n$, a number that is much smaller.[4] In this way, economy of expression is obtained.

The similarities between Wedberg's and Ross's ideas are striking. Both use the example of ownership. Central ideas propounded by both of them are: By use of the linking term, the number $p \cdot n$ of rules is reduced to $p + n$, and, the linking term has no independent meaning (Wedberg) or has no semantical reference (Ross).

In our view, there is a great difference between speaking of an expression like "O is the property of P at t" as meaningless and speaking of it as being without independent meaning. The latter way of speaking goes well together with the view that the term has meaning but that this meaning consists precisely in its occurrence and use in inference rules linking the term to facts, on one hand, and to deontic consequences on the other.

[4] [12] pp. 273 f.

1.3 Intermediaries in Non-legal Contexts

Michael Dummett's Example. Dummett distinguishes between the conditions for applying a term and the consequences of its application. According to Dummett both are part of the meaning. Dummett exemplifies by the use of the term "Boche" as a pejorative term.

> The condition for applying the term to someone is that he is of German nationality; the consequences of its application are that he is barbarous and more prone to cruelty than other Europeans. We should envisage the minimal joinings in both directions as sufficiently tight as to be involved in the very meaning of the word: neither could be severed without altering its meaning. Someone who rejects the word does so because he does not want to permit a transition from the grounds for applying the term to the consequences of doing so. The addition of the term 'Boche' to a language which did not previously contain it would produce a non-conservative extension, i.e., one in which certain statements which did not contain the term were inferable from other statements not containing it which were not previously inferable. [1] at p. 454.[5]

Dummett's example illustrates how the use of a word is determined by two rules (I) and (II):[6]

(I) Rule linking a concept a to an intermediary m:

$$\text{For all } x, y: \text{ If } a(x, y) \text{ then } m(x, y).$$

(II) Rule linking intermediary m to a concept b:

$$\text{For all } x, y: \text{ If } m(x, y) \text{ then } b(x, y).$$

If the standpoint "meaning is use" is adopted, it can be held that the meaning of m is given by two rules (I) and (II) together. To understand the meaning of an intermediary m is to know how it is used in such a pair of rules.

Dummett's example is not concerned with a legal system and with an inference from facts to deontic positions. We note, however, that the antecedent "being of German nationality" in (I) and the consequent "being more prone ... etc" in (II) are conditions of "different kinds".

[5] Since the example is interesting from a philosophical point of view, we use it even though it has the disagreeable feature of being offensive to German nationals.

[6] The rules (I) and (II) can be compared to the rules of introduction and rules of elimination, respectively, in Gentzen's theory of natural deduction in [2]. If this comparison is made, (I) is regarded as an introduction rule and (II) as an elimination rule for m. An obvious difference is that while Gentzen's introduction rules and elimination rules are rules of inference, the rules (I) and (II) are formulated in "if, then" sentences of predicate logic. A reason for the difference is, of course, that Gentzen aims at providing a theory for predicate logic, and, therefore, the language of predicate logic itself is not admissible within his theory.

Dummett intends his example to illustrate a non-conservative extension. In Section 3, where applications of our formal framework is discussed, we will indicate how this idea is expressed within our framework.

Other well-known examples, outside the area of connections from descriptive to normative, are the connection from physical to mental and the connection from chemical to biological. At a very general level, in empirical science, there is the problem of the connection from observable to theoretical.[7]

In some of the cases where the connection of different kinds is problematic, the notion of supervenience is used for clarifying the nature of the connection. Existing theories of supervenience, seem to us, however, to yield at best a very partial insight into the nature of the relation in view. In particular, they do not provide much information about the specific interrelations between parts of the two different structures.

2 The Formal Framework

2.1 Introduction

As stated in Section 1.1 above, we distinguish between the abstract level of formal analysis (to be dealt with in the present section), where a general algebraic framework is developed, and the level of applications where the abstract theory is used as a tool for analysing different conceptual structures (Section 3).

At the abstract algebraic level, the notion "intermediary" will not be used. In the algebraic theory, however, a technical notion "intervenient" will be defined. In Section 3, the notion "intervenient" will be used as a tool for analysis of what, informally, is called "intermediaries". More precisely, in Section 3, we will distinguish different types of intermediaries and indicate how intermediaries can be interrelated.

The algebraic theory contains a number of definitions of technical terms. Before going into this theory, it is appropriate briefly to suggest how the algebraic theory can be used for analysing a normative system with intermediaries.

Let C be a non-empty set. We say that $\mathcal{N} = \langle B, \wedge, ', \rho \rangle$ is a *supplemented Boolean algebra freely generated by* C if $\langle B, \wedge, ' \rangle$ is a Boolean algebra freely

[7] An interesting approach to the problem of intermediate terms in mechanics was outlined in the nineteenth century by Henri Poincaré. Poincaré pointed out that a proposition like (1) "the stars obey Newton's laws" can be broken up into two others, namely (2) "gravitation obeys Newton's laws" and (3) "gravitation is the only force acting on the stars". Among these, proposition (2) is a definition and not subject to the test of experiment, while (1) is subject to such a test. "Gravitation", according to Poincaré, is an intermediary. Poincaré maintains that in science, when there is a relation between two facts A and B, an intermediary C is often introduced by the formulation of one relationship between A and C, and another between C and B. The relation between A and C, then, is often elevated to a principle, not subject to revision, while the relation between C and B is a law, subject to such revision. See [9], pp. 124 f., in the chapter "Is science artificial?"

generated by C and ρ is a binary relation on B.[8] The partial ordering determined by the Boolean algebra $\langle B, \wedge, ' \rangle$ is a subset of ρ. An application can be that $\mathcal{N}$ is a normative system expressed in terms of a set of conditions B and a relation ρ such that, for $a, b \in B$, $a\rho b$ holds if and only if a implies b in the normative system $\mathcal{N}$.

Next, let $\langle B_1, \wedge, ' \rho/B_1 \rangle$ and $\langle B_2, \wedge, ', \rho/B_2 \rangle$ be two substructures of $\langle B, \wedge, ', \rho \rangle$ where B_1 and B_2 are disjoint, except for the zero and unit constants $\bot$ and $\top$. In the application where $\mathcal{N}$ is a normative system, we can think of B_1 as a set of descriptive conditions and B_2 as a set of normative conditions. If B is a set of conditions, $\bot$ stands for the absurd condition and $\top$ for the trivial condition.

Of special interest is where B contains a subset M, disjoint from $B_1 \cup B_2$, where, for $m \in M$, there is $a \in B_1$ and $b \in B_2$ such that $a\rho m$ and $m\rho b$. In this case, given certain further requirements, m will be called an "intervenient". In the application where $\mathcal{N}$ is a normative system, we can conceive of a case where a condition m belongs neither to the set B_1 of descriptive conditions nor to the set B_2 of normative conditions but where, in $\mathcal{N}$, m is implied by a descriptive condition and implies a normative condition.

2.2 The Basic Formal Framework

Boolean Quasi-orderings, Fragments and Joinings. One formal structure that will be used in our investigation of how subsystems of different kinds are linked is that of a *Boolean quasi-ordering* (*Bqo*). Technical concepts related to *Bqo*'s, defined in previous papers are: *fragments* of *Bqo*'s, and *joinings* of elements of *Bqo*'s. For formal definitions of a *Bqo* and of these related notions, the reader is referred to [6]. A short recapitulation is as follows.

The relational structure $\langle B, \wedge, ', R \rangle$ is a *Boolean quasi-ordering (Bqo)* if $\langle B, \wedge, ' \rangle$ is a Boolean algebra and R is a binary, reflexive and transitive relation on B (i.e. R is a quasi-ordering), $\bot$ is the zero element, $\top$ is the unit element, and where R satisfies some additional requirements.[9] If $\mathcal{B} = \langle B, \wedge, ', R \rangle$ is a Boolean quasi-ordering, and $\langle B_i, \wedge, ' \rangle$ is a subalgebra of $\langle B, \wedge, ' \rangle$, and $R_i = R/B_i$, then the structure $\mathcal{B}_i = \langle B_i, \wedge, ', R_i \rangle$ is a *fragment* of $\mathcal{B}$. Let $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{B}_2$ be *Bqo*'s such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of $\mathcal{B}$. A *joining* from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ is a pair $\langle b_1, b_2 \rangle$ in $\mathcal{B}$ such that $b_1 \in B_1$, $b_2 \in B_2$, $b_1 R b_2$, not $b_1 R \bot$ and not $\top R b_2$.

Narrowness and Minimal Elements. The *narrowness-relation determined by* two quasi-orderings $\langle B_1, R_1 \rangle$ and $\langle B_2, R_2 \rangle$ is the binary relation $\trianglelefteq$ on $B_1 \times B_2$ such that $\langle a_1, a_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle$ if and only if $b_1 R_1 a_1$ and $a_2 R_2 b_2$. $\langle a_1, a_2 \rangle$ is a *minimal element* in $X \subseteq B_1 \times B_2$ with respect to $\langle B_1, R_1 \rangle$ and $\langle B_2, R_2 \rangle$ if there is no $\langle x_1, x_2 \rangle \in X$ such that $\langle x_1, x_2 \rangle \vartriangleleft \langle a_1, a_2 \rangle$. The set of minimal elements in X

[8] For the notion of freely generated Boolean algebras, see for example [4] p.131. Instead of *freely generated* one can say *independently generated.*

[9] (1) aRb and aRc implies $aR(b \wedge c)$, (2) aRb implies $b'Ra'$, (3) $(a \wedge b)Ra$, (4) not $\top R \bot$.(Requirement (4) excludes the possibility that $R = B_1 \times B_2$, which holds for inconsistent systems.)

is denoted $\min_{R_1}^{R_2} X$. When there is no risk of ambiguity we write just $\min X$. We note that $\trianglelefteq$ is a quasi-ordering. We let $\backsimeq$ denote the equality part of $\trianglelefteq$ and $\triangleleft$ the strict part of $\trianglelefteq$. The equality part $\backsimeq$ is an equivalence relation and we denote the equivalence class determined by $\langle b_1, b_2 \rangle \in B_1 \times B_2$ by $[b_1, b_2] \backsimeq$.[10]

Boolean Joining Systems (*Bjs*). Another important structure is that of a *Boolean joining-system* (*Bjs*), see [7]. A Boolean joining-system is an ordered triple $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ such that $\mathcal{B}_1 = \langle B_1, \wedge, ', R_1 \rangle$ and $\mathcal{B}_2 = \langle B_2, \wedge, ', R_2 \rangle$ are Boolean quasi-orderings and $J \subseteq B_1 \times B_2$, $J \neq \varnothing$ and three specific requirements are satisfied.[11]

If $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are Boolean quasi-orderings such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of $\mathcal{B}$ and J is the set of joinings from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$, then $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ is a *Bjs*. Also, if $a_1, b_1 \in B_1$, $a_2, b_2 \in B_2$, and $\langle a_1, a_2 \rangle \in J$, then $\langle a_1, a_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle$ implies $\langle b_1, b_2 \rangle \in J$.

Generating of Joining-Spaces. We note that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are *Bqo*'s and

$$\mathcal{J} = \{ J \subseteq B_1 \times B_2 | \, \langle \mathcal{B}_1, \mathcal{B}_2, J \rangle \text{ is a } Bjs \} ,$$

then $\mathcal{J}$ is a closure system.

If $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ is a Boolean joining-system, we call J the *joining-space* from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$. $\mathcal{J}$ is the family of all joining-spaces from $\mathcal{B}_1$ to $\mathcal{B}_2$. If $K \subseteq B_1 \times B_2$ let

$$[K]_{\mathcal{J}} = \cap \{ \, J \mid J \in \mathcal{J}, J \supseteq K \, \} .$$

$[K]_{\mathcal{J}}$ is the joining-space over $\mathcal{B}_1$ and $\mathcal{B}_2$ *generated by* K.[12]

If J is the joining-space from $\mathcal{B}_1$ to $\mathcal{B}_2$ generated by K but J is not generated by any proper subset of K, then we say that J is *non-redundantly generated* by K.

Connectivity. A *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ satisfies *connectivity* if whenever $\langle c_1, c_2 \rangle \in J$ there is $\langle b_1, b_2 \rangle \in J$ such that $\langle b_1, b_2 \rangle$ is a minimal joining in $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ and $\langle b_1, b_2 \rangle \trianglelefteq \langle c_1, c_2 \rangle$.

Suppose that $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ is a *Bjs* that satisfies connectivity. Then

$$J = \{ \, \langle b_1, b_2 \rangle \in B_1 \times B_2 : (\exists \, \langle a_1, a_2 \rangle \in \min J : \langle a_1, a_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle) \, \} .$$

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[10] The sign $\backsimeq$ should be written as a subscript. The reason why this is not done is typograhical.

[11] The requirements are: (1) for all $b_1, c_1 \in B_1$ and $b_2, c_2 \in B_2$, $\langle b_1, b_2 \rangle \in J$ and $\langle b_1, b_2 \rangle \trianglelefteq \langle c_1, c_2 \rangle$ implies $\langle c_1, c_2 \rangle \in J$, (2) for any $C_1 \subseteq B_1$ and $b_2 \in B_2$, if $\langle c_1, b_2 \rangle \in J$ for all $c_1 \in C_1$, then $\langle a_1, b_2 \rangle \in J$ for all $a_1 \in lub_{R_1} C_1$, (3) for any $C_2 \subseteq B_2$ and $b_1 \in B_1$, if $\langle b_1, c_2 \rangle \in J$ for all $c_2 \in C_2$, then $\langle b_1, a_2 \rangle \in J$ for all $a_2 \in glb_{R_2} C_2$. (Note that the definitions of least upper bound (*lub*) and greatest lower bound (*glb*) for partial orderings are easily extended to quasi-orderings, but the *lub* or *glb* of a subset of a quasi-ordering is not necessarily unique but can consist of a set of elements.)

[12] For definition and results of closure systems, see for example [3] p. 23f.

If we use the notion of an image of a set under a relation, then we can say that J is the image of $\min J$ under $\trianglelefteq$.

It is easy to see that if $\langle \mathcal{B}_1, \mathcal{B}_2, J_1 \rangle$ and $\langle \mathcal{B}_1, \mathcal{B}_2, J_2 \rangle$ are *Bjs* which satisfy connectivity and $\min J_1 = \min J_2$, then $J_1 = J_2$. Note that if we "substantially reduce" $\min J$, then the image of the new set under $\trianglelefteq$ is not J. To be more precise: Suppose that $\langle a_1, a_2 \rangle \in \min J$ and $K \subset \min J$ such that if $\langle a_1, a_2 \rangle \simeq \langle b_1, b_2 \rangle$ then $\langle b_1, b_2 \rangle \notin K$. Then it follows that the image of K under $\trianglelefteq$ is a proper subset of J.

If in a *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are complete (in a sense which is a straightforward generalization of the notion of completeness applied to Boolean algebras), then $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ satisfies connectivity.

Couplings and Pair Couplings. If $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ is a *Bjs* and the number of $\simeq$-equivalence classes defined by the elements in $\min J$ is exactly one, then the elements in $\min J$ are called *couplings*. If the number of equivalence classes defined by the elements in $\min J$ is exactly two, then sets consisting of one element from each equivalence class is called a *pair coupling*. Thus if $[b_1, b_2] \simeq$ is the only equivalence class, any $\langle a_1, a_2 \rangle \in J$ encompasses every element of $[b_1, b_2] \simeq$; similarly, if $[b_1, b_2] \simeq$ and $[c_1, c_2] \simeq$ are the only equivalence classes, any $\langle a_1, a_2 \rangle \in J$ encompasses every element of $[b_1, b_2] \simeq$ or every element of $[c_1, c_2] \simeq$.

Base of a Joining-Space and Counterparts. Note that if $\langle \mathcal{B}_1, \mathcal{B}_2\ J \rangle$ is a *Bjs* and J is generated by K, then J is also generated by $\min K$. If $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ is a *Bjs* and J is non-redundantly generated by K and $K \subseteq \min J$, then K is called a *base of* J in $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$.

Suppose that $K, L \subseteq B_1 \times B_2$ and that $K \simeq$ is the set of $\simeq$-equivalence classes defined by the elements in K and $L \simeq$ is the set of $\simeq$-equivalence classes defined by the elements in L. If there is a bijection φ between $K \simeq$ and $L \simeq$ such that $\varphi(x) = y$ if and only if there is $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in B_1 \times B_2$ such that $\langle b_1, b_2 \rangle \in x$ and $\langle a_1, a_2 \rangle \in y$ and $\langle a_1, a_2 \rangle \simeq \langle b_1, b_2 \rangle$, then we say that K and L are $\simeq$-*counterparts.*

If K and L are $\simeq$-counterparts, then the image of K under $\trianglelefteq$ is the same as the image of L under $\trianglelefteq$, and the sets of joinings generated by K and L are the same.

If, for a base K of J in $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$, K and L are $\simeq$-counterparts, then we say that L *up to* $\simeq$*-equivalence* is *the* base of J in $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$.

2.3 Intervenients

Weakest Grounds and Strongest Consequences. Suppose that $\langle B, \wedge, ', \rho \rangle$ is a supplemented Boolean algebra, $B_1, B_2 \subseteq B$ and $m \in B \backslash B_1$. Then $a_1 \in B_1$ is one of the *weakest grounds* in B_1 of m with respect to $\langle B, \wedge, ', \rho \rangle$ if $a_1 \rho m$, and it holds that if there is $b_1 \in B_1$ such that $b_1 \rho m$, then $b_1 \rho a_1$. Furthermore, $a_2 \in B_2$ is one of the *strongest consequences* of m in B_2 with respect to $\langle B, \wedge, ', \rho \rangle$ if $m \rho a_2$, and it holds that if there is $b_2 \in B_2$ such that $m \rho b_2$, then $a_2 \rho b_2$.

Definition of Intervenient. Suppose that C is a non-empty set and that $\langle B, \wedge, ' \rangle$ is the Boolean algebra freely generated by C. Suppose further that $\mathcal{N} = \langle B, \wedge, ', \rho \rangle$ where ρ is a binary relation over B, i.e. $\mathcal{N}$ is a supplemented Boolean algebra extended by the binary relation ρ (cf. above, Section 2.1). (Note that $\mathcal{N}$ is not necessarily a Boolean quasi-ordering.)

A *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ *lies within* a supplemented Boolean algebra $\langle B, \wedge, ', \rho \rangle$ if $\langle B_1, \wedge, ' \rangle$ and $\langle B_2, \wedge, ' \rangle$ are subalgebras of $\langle B, \wedge, ' \rangle$, $B_1 \cap B_2 = \{\top, \bot\}$, $\rho | B_1 = R_1$ and $\rho | B_2 = R_2$, and $\rho | (B_1 \times B_2) = J$.

Suppose that $\mathcal{N} = \langle B, \wedge, ', \rho \rangle$ is a supplemented Boolean algebra and that B_1 and B_2 are disjoint subsets of B such that $\langle B_1, \wedge, ' \rangle$ and $\langle B_2, \wedge, ' \rangle$ are subalgebras of $\langle B, \wedge, ' \rangle$. An element $m \in B \setminus (B_1 \cup B_2)$ is an *intervenient between* B_1 and B_2 in $\langle B, \wedge, ', \rho \rangle$ if there is $\langle a_1, a_2 \rangle \in \rho$ such that a_1 is a weakest ground in B_1 of m with respect to $\langle B, \wedge, ', \rho \rangle$ and a_2 is a strongest consequence in B_2 of m with respect to $\langle B, \wedge, ', \rho \rangle$. We say that the intervenient m *corresponds* to the joining $\langle a_1, a_2 \rangle$ from B_1 and B_2.

We note that in a *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ lying within $\mathcal{N}$, an intervenient m between B_1 and B_2 can be used for inferring joinings from $\mathcal{B}_1$ to $\mathcal{B}_2$. That m is an intervenient in $\langle B, \wedge, ', \rho \rangle$ between B_1 and B_2 corresponding to the joining $\langle a_1, a_2 \rangle$ in $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ implies that $\langle a_1, a_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle$ if and only if $b_1 \rho m \rho b_2$.

The fact that, in the way shown, intervenients can be used for inferring joinings, makes it appropriate to speak of an intervenient as a "vehicle of inference".

***JoinM* and Systems of Intervenients.** Recalling the presuppositions concerning $\mathcal{N} = \langle B, \wedge, ', \rho \rangle$ above, let $M \subseteq B$ and $M \cap (B_1 \cup B_2) = \varnothing$. We say that M *produces* the set

$$K = \{ \langle b_1, b_2 \rangle \in B_1 \times B_2 \mid \exists m \in M : b_1 \rho m \rho b_2 \} .$$

The set of joinings corresponding to a set of intervenients M between B_1 and B_2 is denoted $JoinM$ where

$$JoinM = \{ \langle b_1, b_2 \rangle \in B_1 \times B_2 \mid \exists m \in M : \ m \text{ corresponds to } \langle b_1, b_2 \rangle \}$$

Note that M produces K iff K is the image of $JoinM$ under $\trianglelefteq$. We say that M *non-redundantly produces* K if M produces K but no proper subset of M produces K.

If M is a set of intervenients such that $JoinM$ is a base of J, we say that M is a base of intervenients for J. Of special interest is the case where M consists of a set of generators for the Boolean algebra $\langle B, \wedge, ' \rangle$ in $\mathcal{N}$.

Three Types of Intervenients. Suppose that m is an intervenient between B_1 and B_2 in $\langle B, \wedge, ', \rho \rangle$, corresponding to the joining $\langle a_1, a_2 \rangle$ in $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$. Then a classification can be made according to whether $\langle a_1, a_2 \rangle$ (1) is a joining that is not a minimal joining, (2) is a minimal joining that is not a pair coupling or coupling, or (3) is a pair coupling or coupling. In case (1), we say that m corresponds to a mere joining, in case (2), that m corresponds to a mere minimal joining, and, in case (3), that m corresponds to a pair coupling or coupling.

3 Applications

3.1 The cis Models

In what follows we shall be interested in a particular model of the abstract theory of quasi-orderings, Boolean quasi-orderings, and Boolean joining-systems. This model is the model of a *condition implication structure (cis)*.[13] A *cis* model of a *Bqo* $\langle B, \wedge, ', R\rangle$ is obtained if B is a domain of *conditions*, and aRb represents that *a implies b*. Similarly, a *cis* model of a *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ is obtained if $\mathcal{B}_1, \mathcal{B}_2$ are *cis* models of *Bqo's* and, for $a_1 \in B_1$ and $a_2 \in B_2$, $a_1 J a_2$ represents that a_1 implies a_2.

In simple cases, conditions can be denoted by expressions, using the sign of the infinitive, such as "to be of German nationality", "to be a citizen of the U.S.", "to be a child of", "to be entitled to inherit", or by corresponding expressions in the ing-form, like "being of German nationality" etc. Often, however, conditions should appropriately be expressed by open sentences, like "x's promises to pay \$ y to z", "x is a citizen of state y", "x is entitled to inherit y".

If a, b are conditions, we assume that a', b' are negations of a, b respectively, that $a \wedge b$ is the conjunction of a and b, and that $a \vee b$ is the disjunction of a and b.[14]

If a *Bjs* $\langle B_1, B_2, J\rangle$ represents a normative (mini-)system, a norm in this system is represented by $a_1 J a_2$, where $a_1 \in B_1$ is descriptive, while $a_2 \in B_2$ is normative.

Dummett's "Boche" Example Once More. In our formal framework, Dummett's Boche example can be represented as follows. Let $\mathcal{N} = \langle B, \wedge, ', \rho\rangle$ be a supplemented Boolean algebra freely generated by a set C of concepts, and let $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ be a *Bjs* which lies within $\mathcal{N}$. The set B_1 contains conditions expressing different nationalities and B_2 conditions expressing different psychological dispositions. Let $B^{(1)}$ be B extended with the term *Boche* and $\rho^{(1)}$ an extension of ρ such that *Boche* is an intervenient in $\mathcal{N}^{(1)} = \langle B^{(1)}, \wedge, ', \rho^{(1)}\rangle$ between B_1 and B_2. $J^{(1)}$ is the extension of J as an effect of the extension of ρ to $\rho^{(1)}$. Suppose that $\langle a_1, a_2\rangle$ is a joining in $\left\langle \mathcal{B}_1, \mathcal{B}_2, J^{(1)}\right\rangle$ but not a joining in $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$, and that the intervenient *Boche* corresponds to the joining $\langle a_1, a_2\rangle$ in $\left\langle \mathcal{B}_1, \mathcal{B}_2, J^{(1)}\right\rangle$. The question arises whether $\langle a_1, a_2\rangle$ is a mere joining or a minimal joining, perhaps a coupling or pair coupling. If Dummett's example is perceived to be such that in $\mathcal{N}^{(1)}$ *Boche* corresponds to a minimal joining, we can make an extension of the system $\mathcal{N}^{(1)}$ to a system $\mathcal{N}^{(2)} = \langle B^{(2)}, \wedge, ', \rho^{(2)}\rangle$ by adding the intervenient *Berserk* (See figure 1 below) corresponding to the joining $\langle b_1, a_2\rangle$ in $\left\langle \mathcal{B}_1, \mathcal{B}_2, J^{(2)}\right\rangle$. In $\mathcal{N}^{(2)}$ *Boche* corresponds to a mere joining, since $\langle c_1, a_2\rangle = \langle a_1 \vee b_1, a_2\rangle$ is a minimal joining in $J^{(2)}$.

[13] The present section on condition implication structures recapitulates ideas presented in earlier papers. See, in particular, [6].

[14] The procedure of forming compounds can be iterated. So, for example, $(a \wedge b) \vee c$ is a condition. A condition a is simple if it is not compound.

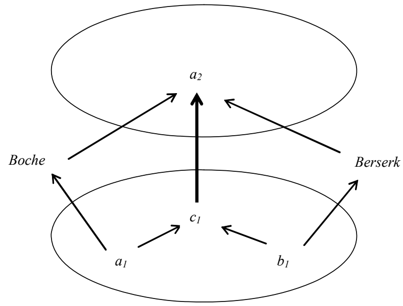

Fig. 1.

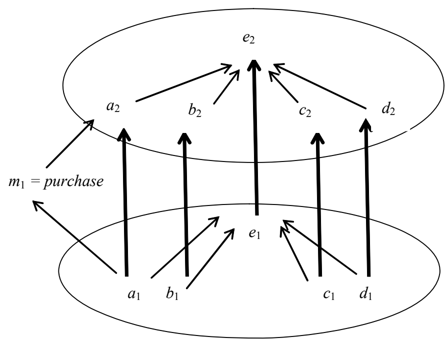

Fig. 2.

Minimal Joining and Modes for Acquiring Ownership. Next, we give a legal example concerning modes for ownership acquisition. The legal system we study is represented by the supplemented Boolean algebra $\langle B, \wedge, ', \rho\rangle$. The legal rules of ownership are expressed in terms of a set M of conditions: *purchase* m_1, *inheritance* m_2, *occupation* m_3, *specification* m_4, *ownership* m_5 (See figure 2 above). M is a subset of B. B_1 is a subset of B containing the following conditions: a_1 (making a contract etc.), b_1 (having particular kinship relationship), c_1 (appropriating something not owned), d_1 (creating a valuable thing out of worthless material), $e_1 = \langle a_1 \vee b_1 \vee c_1 \vee d_1\rangle$. The weakest grounds in B_1 of the conditions in M with respect to $\langle B, \wedge, ', \rho\rangle$ are described by the following set G of ordered pairs: $\langle a_1, m_1\rangle$, $\langle b_1, m_2\rangle$, $\langle c_1, m_3\rangle$, $\langle d_1, m_4\rangle$, $\langle e_1, m_5\rangle$. The strongest consequences in $B_2 \subseteq B$ of the conditions in M with respect to $\langle B, \wedge, ', \rho\rangle$ are described by the following set C of ordered pairs: $\langle m_1, a_2\rangle$, $\langle m_2, b_2\rangle$, $\langle m_3, c_2\rangle$, $\langle m_4, d_2\rangle$, $\langle m_5, e_2\rangle$, where $e_2 = \langle a_2 \vee b_2 \vee c_2 \vee d_2\rangle$. Note that $G \cup C \subseteq \rho$ and that M is a set of intervenients from B_1 to B_2 in $\langle B, \wedge, ', \rho\rangle$.

Let the joining-space J from $\mathcal{B}_1 = \langle B_1, \wedge, ', \rho|B_1\rangle$ to $\mathcal{B}_2 = \langle B_2, \wedge, ', \rho|B_2\rangle$ be characterized by G and C in the following sense: J is the the joining-space generated by $JoinM$. Then m_1 corresponds to $\langle a_1, a_2\rangle$, m_2 corresponds to $\langle b_1, b_2\rangle$, m_3 corresponds to $\langle c_1, c_2\rangle$, m_4 corresponds to $\langle d_1, d_2\rangle$ and m_5 corresponds to $\langle e_1, e_2\rangle$.

Each of $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle$, $\langle c_1, c_2\rangle$, $\langle d_1, d_2\rangle$ and $\langle e_1, e_2\rangle$ is a minimal joining in J. Note that M is not a base of intervenients for J but under plausible assumptions, it can be assumed that the subset $\{m_1, \ldots, m_4\}$ is such a base. Then the *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ can be described by the system $\langle \mathcal{B}_1, M, \mathcal{B}_2\rangle$ which can appropriately be called a *ground-intervenient-consequence-system*, abbreviated a *GIC-system*.

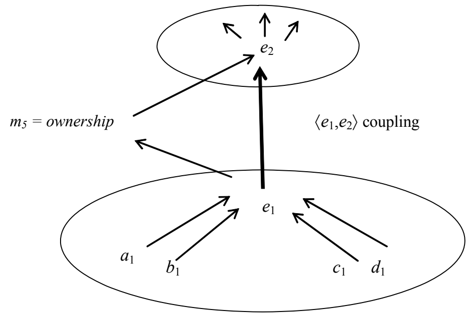

Fig. 3.

Ownership as Corresponding to a Coupling. Recalling the example of the previous subsection with $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ lying within $\mathcal{N} = \langle B, \wedge,', \rho\rangle$, let $\mathcal{N} = \langle B, \wedge,', \rho\rangle$ be exchanged for $\mathcal{N}^* = \langle B^*, \wedge,', \rho^*\rangle$, where m_5 (ownership) is a member of B^*, but where a_2, b_2, c_2, d_2 and $m_1, \ldots, m_4$ are not members of B^* and where ρ^* is restricted accordingly. Thus subset B_1^* of B^* is as B_1 in the previous example, but a_2, b_2, c_2, d_2 are not members of subset B_2^*. In $\mathcal{N}^*$, (like in $\mathcal{N}$), e_1 is a weakest ground for m_5 and e_2 is a strongest consequence of m_5. The set M of intervenients from B_1 to B_2^*, however, has m_5 as its only member.In $\langle \mathcal{B}_1^*, \mathcal{B}_2^*, J^*\rangle$, $\langle e_1, e_2\rangle$ is the only member of $\min J^*$. Therefore $\langle e_1, e_2\rangle$ is a *coupling* (see Section 2.2) and, in the example, m_5 (ownership) corresponds to a coupling (See figure 3 above). This system is strikingly similar to Ross's scheme, since Ross (like Wedberg) does not take into account such consequences that are specific to particular modes of acquisition such as purchase, inheritance, occupation etc.[15]

3.2 The Methodology of Intermediaries in Legal Systems

From the point of view of methodology, there is the task of formulating rational principles for constructing a system with concepts that, in a *Bjs* representation, are appropriately represented by intervenients. Three aspects to be taken into account are: (i) economy of expression, (ii) efficient inference, and (iii) adaptation to linguistic usage and commonly made distinctions.

A concept appropriately represented by an intervenient corresponding to a minimal joining, a pair coupling, or a coupling will serve the purpose of economy of expression and efficient inference. We recall the discussion concerning ownership as corresponding to a minimal joining or a coupling.

[15] Thus in the *Bjs* $\langle \mathcal{B}_1^*, \mathcal{B}_2^*, J^*\rangle$ lying within $\mathcal{N}^*$, B_2^* is generated by those simple conditions that are consequences of ownership regardless of mode of acquisition.

With regard to concepts represented by intervenients corresponding to mere joinings, considerations relating to economy of expression and efficient inference do not justify having these concepts in the system. What comes into focus is rather aspect (iii). Here, we can distinguish two situations:

One is the case where, in the appropriate representation of linguistic usage, several grounds $a_1, b_1, \ldots$ have the same strongest consequence a_2. If, in a *Bjs* representing linguistic usage, an intervenient corresponding to $\langle a_1 \vee b_1 \vee \ldots, a_2 \rangle$ does not appropriately represent linguistic usage and commonly made distinctions, this usage might sometimes be more appropriately represented by a *Bjs* with particular intervenient(s) corresponding to one or more of $\langle a_1, a_2 \rangle$, $\langle b_1, a_2 \rangle$ etc., even though these are mere joinings. Thus in Dummett's example (see above), where the intervenient *Boche* corresponds to the mere joining $\langle a_1, a_2 \rangle$.

The dual situation is where, in the representation of linguistic usage, a_1 is the weakest ground for several consequences $a_2, b_2, \ldots$ If, in the *Bjs* representation, an intervenient corresponding to $\langle a_1, a_2 \wedge b_2 \wedge \ldots \rangle$ is not an appropriate representation of usage, a better representation can sometimes be achieved with particular intervenient(s) corresponding to one or more of the mere joinings $\langle a_1, a_2 \rangle$, $\langle a_1, b_2 \rangle$ etc.

4 Conclusion

As exemplified in the foregoing, intermediate concepts (intermediaries) play an essential role in normative systems. In the paper, we have used an algebraic framework, previously developed by us, for representing normative systems. Within this framework, we have outlined a theory of intervenients including weakest grounds and strongest consequences and bases of intervenients. Also, we have taken a first step towards a typology of intervenients. This theory is intended as a means for analysing intermediate concepts, and we have sketched its application in a few cases. As a report on work in progress, we have focused on systems consisting of an algebra of grounds and an algebra of consequences and a system of intervenients between these algebras (*GIC-systems*). In further developments of the theory, we intend to extend the investigation to incorporate nets of *GIC*-systems, where the consequence-structure in one system can be the ground-structure in another, and the intervenients in one *GIC*-system can be grounds or consequences in another. Consequently, in more complex normative systems, there can be hierarchies of intervenients worth investigating.

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[16] The paper, as well as our earlier joint papers, are the result of wholly joint work where the order of appearance of our author names has no significance.

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