Simultaneous Embedding with Two Bends per Edge in Polynomial Area

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Abstract. The simultaneous embedding problem is, given two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, to find planar embeddings $\varphi(G_1)$ and $\varphi(G_2)$ such that each vertex $v \in V$ is mapped to the same point in $\varphi(G_1)$ and in $\varphi(G_2)$. This article presents a linear-time algorithm for the simultaneous embedding problem such that edges are drawn as polygonal chains with at most two bends and all vertices and all bends of the edges are placed on a grid of polynomial size. An extension of this problem with so-called fixed edges is also considered.

A further linear-time algorithm of this article solves the following problem: Given a planar graph G and a set of distinct points, find a planar embedding for G that maps each vertex to one of the given points. The solution presented also uses at most two bends per edge and a grid whose size is polynomial in the size of the grid that includes all given points. An example shows two bends per edge to be optimal.

1 Introduction

The visualization of information has become very important in recent years. The information is often given in the form of graphs, which should at the same time aesthetically please and convey some meaning. Many aesthetic criteria exist, such as straight-line edges, few bends, a limited number of crossings, depiction of symmetry and a small area of the drawing given, e.g., a minimal distance between two vertices.

If graphs change over the course of time or if different relations among the same objects are presented in graphs, it is often useful to recognize the features of the graph that remain unchanged. If each graph is drawn in its own way, in other words if the graphs are embedded independently, there is probably only little correlation. Therefore, the embeddings of the graphs have to be constructed simultaneously to achieve that all or at least some features of the graph are fixed.

A viewer of a graph quickly develops a mental map consisting basically in the positions of the vertices. If k planar graphs with the same vertex set V are presented, it is desirable that the positions of all vertices in V remain fixed. This problem is called *simultaneous embedding*. An extension of the problem is the so-called *simultaneous embedding with fixed edges*: In addition to the k graphs, a set of edges F is given. A feasible solution is an embedding of the k graphs such that all vertices and all edges in F have fixed embeddings. An algorithm for the

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simultaneous embedding problem for k planar graphs with few bends per edge helps to find an embedding with few bends per edge for graphs of *thickness* k. The thickness of a graph G is the minimum number of planar subgraphs into which the edges of G can be partitioned. Since a graph of thickness k can be embedded in k layers without any edge crossings, thickness is an important concept in VLSI design. Additionally, an algorithm for the simultaneous embedding of k planar graphs with fixed edges helps to find an embedding of a graph of thickness k such that certain sets of edges are drawn straight-line as well as identically in all layers.

Definition 1. *A k-bend embedding of $G = (V, E)$ is an embedding such that each edge in E is drawn as a polygonal chain with $\leq k$ bends. Thus, an edge with l bends consists of $l + 1$ straight-line segments.*

Unless stated otherwise, the following embeddings place all vertices and all bends on a grid of size polynomial in the number of vertices. According to results of Pach and Wenger [9], for any number of planar graphs on the same vertex set of size n, an $O(n)$-bend simultaneous embedding is possible. Erten and Kobourov [6] show with a small example that a 0-bend simultaneous embedding does not always exist for two planar graphs. They show that three bends suffice to embed two planar graphs and that one bend is enough in the case of two trees. By using a new algorithm presented in Section 3, this article shows in Section 2 that the number of bends per edge in a simultaneous embedding of two planar graphs can be reduced to two.

Erten and Kobourov also examine simultaneous embeddings with fixed edges in the special case where one input graph is a tree and the other is a path. For special kinds of graphs (caterpillar and outerplanar graphs), Brass et al. [2] show how to embed simultaneously two of the special graphs such that all edges are fixed. For general graphs, the simultaneous embedding problem with fixed edges is considered in Section 4. However, if all edges are fixed, this problem is already for almost all instances of two planar graphs not solvable (Section 5)—even if the number of bends per edge is unbounded. Therefore, the algorithm presented in Section 4 works only with sets of fixed edges with certain properties.

Another variation of the simultaneous embedding problem is described in [1] by Bern and Gilbert: Given a straight-line planar embedding of a planar graph with convex and 4-sided faces, find a suitable location for dual vertices such that the edges of the dual graph are also straight-line segments and cross only their corresponding primal edges.

Kaufmann and Wiese [7] present an algorithm for the *vertices-to-points* problem, which computes an embedding of a planar graph such that the vertices are drawn on a grid at given points. If all vertices and all bends are placed on a grid whose size is polynomial in the size of the grid that includes all given points, their embedding requires up to three bends per edge, but via a similar algorithm as for the simultaneous embedding problem, a 2-bend embedding can be constructed (Sections 2 and 3). If an outer face is specified, Kaufmann and Wiese show that an 1-bend embedding for the vertices-to-points problem is not possible in general. In Section 5, a very short proof of the same lower bound is presented, but now no outer face must be specified.

2 Finding an Embedding

Since the same ideas as already described in [7, 2, 6] are used, these will only be sketched. Many parts of these ideas help to find a 2-bend embedding for both of the two problems below. Assume for the time being that for all planar graphs $G = (V, E)$ considered in the following, a Hamilton cycle C exists and is known. Moreover, let f_G be a bijective function that maps each vertex to a number in $\{1, \ldots, |V|\}$ such that consecutive vertices in C have consecutive numbers modulo $|V|$. The knowledge of the Hamilton cycle C is useful because in a planar embedding of G, each edge not part of C is either completely inside or completely outside C. In the following two problems are defined and their solutions are presented subsequently.

Definition 2. *The simultaneous embedding problem is, given two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, to find planar embeddings $\varphi(G_1)$ and $\varphi(G_2)$ such that all vertices are fixed, i.e. $\forall v \in V : \varphi_1(v) = \varphi_2(v)$.*

As a first step to find a simultaneous embedding for G_1 and G_2, associate each vertex v with two numbers x, y, where $x = f_{G_1}(v)$ and $y = f_{G_2}(v)$. Use the two numbers of each vertex as its coordinates. Embed the edges in G_1 and G_2 by applying the procedure described below the following definition once for G_1 with $direction = horizontal$ and once for G_2 with $direction = vertical$.

Definition 3. *Let $G = (V, E)$ be a planar graph and let P be a set of distinct points in the plane. The vertices-to-points problem is to find a planar embedding φ such that $\forall v \in V : \varphi(v) \in P$.*

For an embedding, sort the given points according to their x-coordinates. Map the vertex v with number $i = f_G(v)$ to the point with the i'th smallest x-coordinate. Continue the embedding of the edges with $direction = horizontal$.

In the following the procedure to embed the edges is described:

Denote the graph under consideration by $G = (V, E)$ and the edge $\{f_G^{-1}(1), f_G^{-1}(|V|)\}$ by $\hat{e}$. W.l.o.g. assume that $direction = horizontal$. Otherwise turn around the construction by 90 degree.

First, embed the edges of the Hamilton path $P = C \setminus \{\hat{e}\}$ as straight lines. For each edge $e \in P$ let x_e and y_e be the absolute values of the differences of the x- and y-coordinates of the endpoints of e. Set $\alpha = \min_{e \in P} \tan(x_e/y_e)$. For each vertex v, let l_v be the vertical line through v. Using a combinatorial embedding of G, partition the edges not part of C in linear time into two sets E_1 and E_2 such that each set can be embedded inside (or outside) the Hamilton cycle without edge intersections. Add the edge $\hat{e}$ to E_1, say. Embed each edge $\{u, v\}$ in E_1 below P and in E_2 above P as part of two rays starting from vertex u to the right of l_u and from vertex v to the left of l_v, if $f_G(u) < f_G(v)$. Draw each ray in such a way that the angle between the ray and the corresponding vertical line is α and cut off the two rays at their point of intersection. If a vertex has several incident edges embedded on the same side of P or if the point of intersection is not on the grid, modify the angle slightly such that planarity is preserved. This yields a 1-bend embedding of G.

However one problem remains: How to find a Hamilton cycle and what to do if no Hamilton cycle exists. The solution is to modify G. According to Chiba and Nishizeki [3], G can be made 4-connected preserving planarity by repeated applying

Operation 1: adding an auxiliary edge and
Operation 2: splitting an original edge of G once and adding a new vertex between the two parts of the split edge.

Denote this modified graph by G'. In [4], Chiba et al. show that every 4-connected graph has a Hamilton cycle that can be found in linear time. Use an embedding for G' to obtain an embedding for G by removing the new edges, merging the embeddings of the two parts of each split edge and replacing each new vertex by a bend for the corresponding edge.

Observe that an edge $e = \{v_1, v_2\}$ in G corresponds to at most two split edges $e_1 = \{v_1, v_{\text{new}}\}$ and $e_2 = \{v_{\text{new}}, v_2\}$ in G'. If both edges e_1, e_2 are embedded with one bend and there is a further bend between the edges e_1, e_2 at v_{new}, the edge e is embedded with three bends. As we see later, one part of the two split edges is inside and the other part is outside the Hamilton cycle used. Thus, this third bend at v_{new} exists only if v_{new} does not appear between v_1 and v_2 in the Hamilton path used for the embedding.

To see this, consider the next two examples.

If v_{new} is behind v_1 and v_2 on the Hamilton path, the edge e is drawn from v_1 rightwards to v_{new} and then leftwards to v_2. But if v_{new} is between v_1 and v_2, the two rays at v_{new} are drawn as one line from the bend point of e_1 through v_{new} to the bend point of e_2.

Using a shrinking angle during the process of embedding instead of an almost fixed angle α, Kaufmann and Wiese described in [7] how to remove the bend point at v_{new}, but this solution requires a grid of exponential size to place the bends of the edges.

Since it is essential where the numbering along the Hamilton cycle starts, let us consider the problem of finding a so-called *closable* Hamilton path. A Hamilton path is closable if it is contained in a Hamilton cycle. A closable Hamilton cycle makes it more explicit which part of the Hamilton cycle is used to number the vertices.

Definition 4. *An edge-extension of a planar graph G is a planar graph G^+ obtained from G by adding auxiliary edges or by splitting edges, i.e. replacing each such edge by a path of length two whose edges are split edges and whose midpoint is a so-called* new *vertex of degree 2. Thus, each edge in G corresponds to a unique path in G^+ of arbitrary length.*

Given a planar graph, an edge-extension is constructed in linear-time in the next section such that each edge in G corresponds to a path of length ≤ 2 in G^+. Moreover, a closable Hamilton path in G^+ is found at the same time that has the *between property*:

Definition 5. *Let G^+ be an edge-extension of $G = (V, E)$ and let P be a Hamilton path in G^+. P has the between property (in G^+ with respect to G) if each new vertex that was inserted between the two split parts of an edge $\{u, v\}$ is between u and v on the Hamilton path P.*

From the considerations, we can conclude the following.

Theorem 6. *Given two planar graphs G_1 and G_2 based on the same vertex set of size n, a 2-bend simultaneous embedding of G_1 and G_2 can be found in $O(n)$ time such that all vertices and all bends can be placed on a grid whose bounding box is of size $n^{O(1)}$.*

Theorem 7. *Given a planar graph $G = (V, E)$ and a set of at least $|V|$ distinct points P on a grid, a 2-bend embedding of G can be found in linear time such that each vertex is embedded on a point in P and such that the area of the embedding of G is polynomial in the size of the grid.*

3 Finding a Closable Hamilton Path

An extension H of G is first constructed. Although H will not be planar, a closable Hamilton path in H will help to construct a closable Hamilton path in a planar extension of G. Obtain $G' = (V, E)$ by triangulating G. Denote by $\varphi(G')$ a combinatorial embedding of G' and choose an arbitrary face of φ to be the outer face. Let $G'_D = (W, F)$ be the dual graph of G', but without a vertex (and its edges) for the outer face. For each vertex $w \in W$ representing a face A of $\varphi(G')$, denote by $\Delta(w)$ the set of the three vertices on the boundary of A. Define $D = \{(u, v) \mid u \in W \wedge v \in \Delta(u)\}$ and $H = (V \cup W, E \cup F \cup D)$. See Fig. 1 for an example, but for the time being ignore the distinction between vertices inside and outside the set A_{i-1}. Define an area as the union of some faces of $\varphi(G')$ and their boundaries. For an area A, let $V_A \subseteq V \cup W$ be the set of vertices in A, let $V_A^- \subseteq V \cap V_A$ be the set of vertices on the border of A adjacent to a vertex in $V \setminus V_A$, and let $E_A^- \subseteq E$ be the set of edges on the border of A. Choose $\hat{e} = \{u_1, u_2\} \in E$ as an arbitrary edge incident to the outer face of $\varphi(G')$. W.l.o.g. assume that u_1 is visited just before u_2 on a clockwise travel on the border of the outer face. Let $w \in W$ be the vertex of the dual graph that corresponds to the inner face of $\varphi(G')$ incident to $\hat{e}$. Moreover, denote the area of this inner face by A_0 and let u_3 be the third vertex incident to this inner face (i.e. $\Delta(w) = \{u_1, u_2, u_3\}$). Thus $V_{A_0} = \{u_1, u_2, u_3, w\}$, $V_{A_0}^- \subseteq \{u_1, u_2, u_3\}$ and $E_{A_0}^- = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_1, u_3\}\}$.

Using $P_0 = (\{u_2, u_3\}, \{u_3, w\}, \{w, u_1\}, \{u_1, u_2\})$ as a first simple path in H and A_0 as the processed area, the aim is to extend P_0 and A_0 stepwise such that the following invariants are true after each step i for the processed area A_i and the current path P_i:

Invariant 1: P_i is a simple path containing all vertices in V_{A_i}.
Invariant 2: For all edges $\{u, v\} \in E$ that are crossed by a dual edge e_D on P_i, the subpath of P_i between u and v contains e_D.
Invariant 3: The vertices in P_i occur in the same order in P_i and on the border of A_i, starting with u_2.
Invariant 4: For all edges $(u, v) \in E^-_{A_i}$ one of the following is true:
Property a: (u, v) is part of the current path P_i.
Property b: Let $w \in W$ be the dual vertex corresponding to the face of G' that is incident to (u, v) and inside the processed area A_i. Then either (u, w) or (v, w) is part of the current path P_i.

These invariants are all true for P_0 and A_0. Initially ($i = 0$) and in each step i, calculate the sets $V_{A_i}, V^-_{A_i}, E^-_{A_i}$ and for each vertex v the list $V^v_{A_i} = \{u \in V \mid \{v, u\} \in E \wedge |\{v, u\} \cap V_{A_i}| = 1\}$, ordered in counter clockwise order around v in $\varphi(G')$. This list contains all vertices adjacent to v that are relative to v on the opposite side of A_i. Begin each list with the vertex that is met first on a clockwise travel on the border of A_i starting with u_2. If step i adds a vertex $s \in V$ to the processed area, all these sets and lists can be updated in time *O(degree of s)*.

Step i is carried out as follows: Choose $s \in V^v_{A_{i-1}}$ for some vertex $v \in V^-_{A_{i-1}}$ on the border of A_{i-1}. While only one vertex of V is to be added to the processed area, test if the processed area together with the edges from s to vertices in $V^s_{A_{i-1}}$ encloses additional vertices $t \in V \setminus \big(V_{A_{i-1}} \cup \{s\}\big)$. If such a vertex t exists, put s on a stack and process t first.

The test of whether such a vertex t exists is easy: Let $v_0, \ldots, v_k$ be the vertices of the ordered list $V^s_{A_{i-1}}$. Consider also Fig. 1. These vertices are all adjacent to s and they appear in clockwise order on the border of A_{i-1}. Consider in $\varphi(G')$ the vertices adjacent to s in counter clockwise order from v_0 to v_k. If these vertices are all in $V^s_{A_{i-1}}$, no such vertex t exists. Otherwise choose t as the first vertex found that does not belong to $V^s_{A_{i-1}}$. After processing t, continue this check for s.

If no such vertex t exists (any more), the $k+1$ vertices in $V^s_{A_{i-1}}$ together with s define k faces $W_s = \{w_1, \ldots w_k\}$. Number these faces such that w_j is incident

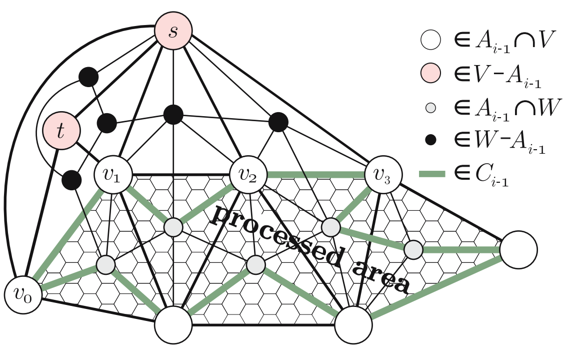


Fig. 1. Extended graph H of a graph $G' = (V, E)$

to v_{j-1} and v_j. In other words, each vertex $w \in W_s$ is adjacent in H to s and to two vertices in $V^-_{A_{i-1}}$. Extend the processed area A_{i-1} by the faces in W_s. For calculating the simple path P_i, two cases are considered. Figures 2 and 3 illustrate the cases 1 and 2, respectively.

Case 1. For some $j \in \{1, \ldots, k\}$, the edge $\{v_{j-1}, v_j\}$ lies on P_{i-1}. Set

$$\begin{aligned} P_i = &(P_{i-1} \setminus \{v_{j-1}, v_j\}) \\ &\cup \{\{v_j, w_{j+1}\}, \{w_{j+1}, w_{j+2}\}, \ldots, \{w_{k-1}, w_k\} \{w_k, s\}\} \\ &\cup \{\{s, w_1\}, \{w_1, w_2\}, \ldots, \{w_{j-1}, w_j\}, \{w_j, v_{j-1}\}\}. \end{aligned}$$

Case 2. Otherwise. Let $\hat{w} \in W \cap V_{A_{i-1}}$ be the vertex inside the processed area A_{i-1} adjacent to v_0 and v_1. Since property a of Invariant 4 does not hold, we know that $\{v_0, \hat{w}\} \in P_{i-1}$ or $\{v_1, \hat{w}\} \in P_{i-1}$. In the first case set $\hat{v} = v_0$ and $\hat{P} = \{\{\hat{v}, s\}, \{\hat{w}, w_1\}, \{w_1, w_2\}\}$; in the other case set $\hat{v} = v_1$ and $\hat{P} = \{\{\hat{w}, w_1\}, \{w_1, s\}, \{\hat{v}, w_2\}\}$. Then

$$\begin{aligned} P_i = &(P_{i-1} \setminus \{\hat{w}, \hat{v}\}) \cup \hat{P} \\ &\cup \{\{w_2, w_3\}, \ldots, \{w_{k-1}, w_k\}, \{w_k, s\}\}. \end{aligned}$$

By the construction of P_i and since Invariant 3 held before the i'th step, Invariants 1 and 2 are true after the i'th step. Since the border of A_i results from the border of A_{i-1} by a replacement of $v_1, \ldots, v_{k-1}$ by s and since the simple

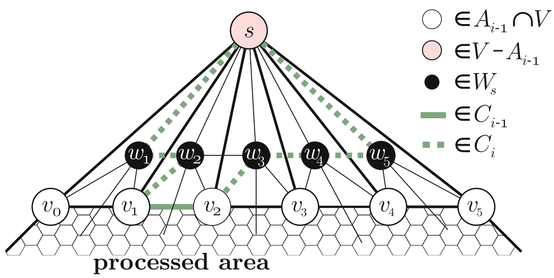


Fig. 2. Face w_2 is incident to an edge in P_{i-1}

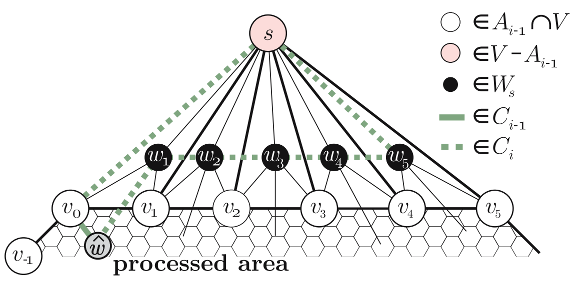


Fig. 3. No face in W_s is incident to an edge in P_{i-1}

path P_i is an extension of P_{i-1} such that s is inserted between some vertices in $\{v_0, \ldots, v_k\}$, Invariant 3 is preserved.

Observe that for each new edge e on the boarder of the processed area (i.e. $e \in E^-_{A_i} \setminus E^-_{A_{i-1}}$), either Property a or b of Invariant 4 is true. Furthermore, in Case 1, the edge $\{v_{j-1}, v_j\} \in P_{i-1} \setminus P_i$ is not in $E^-_{A_i}$ any more after step i. In Case 2, let $v_{-1} \in \Delta(\hat{w}) \setminus \{v_0, v_1\}$. If $\{v_{-1}, v_0\} \in E^-_{A_{i-1}}$, then v_0 is adjacent to only three vertices in A_{i-1} and thus $\{v_{-1}, v_0\} \in P_{i-1}$. Altogether, Invariant 4 is also true after the i'th step.

After $|V|-3$ steps, $A_{|V|-3}$ equals to the whole internal area of G'. Because of Invariant 1, a closable Hamilton path $P_{|V|-3}$ in H is found. It remains to show how to use the knowledge of a closable Hamilton path in H to find a closable Hamilton path P in a planar extension of G' that is also a planar extension of G. Let $v_{\sigma_1}, \ldots, v_{\sigma_{|V|}}$ be the order of the vertices of V as they appear on $P_{|V|-3}$.

The closable Hamilton path P in an edge-extension of G is constructed by connecting the vertices v_{σ_i} and $v_{\sigma_{i+1}}$ $(1 \leq i < |V|)$. If $\{v_{\sigma_i}, v_{\sigma_{i+1}}\} \in E$, add $\{v_{\sigma_i}, v_{\sigma_{i+1}}\}$ to P. Otherwise draw an edge p from v_{σ_i} to $v_{\sigma_{i+1}}$ such that only the faces are visited that are also visited by $P_{|V|-3}$ and such that each edge in E crossed by p is also crossed by $P_{|V|-3}$. Each time p crosses an edge $e \in E$, break e into two split edges and add a new vertex between them. Also replace p by a path of auxiliary edges that traverses all these new vertices and thus connects v_{σ_i} and $v_{\sigma_{i+1}}$. Add all these newly inserted auxiliary edges to P. Since $P_{|V|-3}$ is a simple path and each edge in E is crossed by only one edge in F, the construction of P breaks each edge $\{u, v\}$ in E into at most two split edges $\{u, v_{\text{new}}\}$ and $\{v_{\text{new}}, v\}$. Additionally, because of Invariant 2, the new vertex v_{new} is between u and v on P. Therefore P has the between property.

Definition 8. *Call an edge-extension G^+ of G a good edge-extension if each new vertex is only incident to two auxiliary and to two split edges.*

Theorem 9. *A good edge-extension G^+ of a planar graph G and a closable Hamilton path P in G^+ can be found in linear time such that each edge in G corresponds to a path of length two in G^+ and P has the between property.*

As discussed in Section 2, this proves Theorems 6 and 7.

4 Simultaneous Embedding with Fixed Edges

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two planar graphs and let $F \subset E_1 \cup E_2$. The goal is to find a simultaneous embedding of G_1 and G_2 such that the edges in F can be drawn in both embeddings as straight lines; in particular, edges in $F \cap E_1 \cap E_2$ are drawn identically in the two embeddings. However, F must have some special properties. First, let F be a set such that no vertex is incident to more than one fixed edge. Later, this restriction is relaxed. Iterate the following once for $G = G_1$ and once for $G = G_2$.

Find a Hamilton cycle C in a good edge-extension of G in which no fixed edge is split, i.e. no fixed edge is crossed by C. Using a more difficult case distinction we can use the algorithm of Section 3 to find a Hamilton cycle in such an edge-extension. However, since we can later handle paths of fixed edges that are crossed several times by the Hamilton cycle, in particular, since we can handle a fixed edge crossed by C, details are omitted. Let φ be the used combinatorial embedding of the algorithm in Section 3. The edges of F are added now successively to C.

Consider the situation shown in Fig. 4. Let $\{\hat{u}, \hat{v}\} \in F$ be an edge that is not part of the Hamilton cycle. Since a Hamilton cycle contains all vertices, two other edges incident to $\hat{u}$ and $\hat{v}$, respectively, are part of the Hamilton cycle. For each vertex v and an incident edge e, denote by E_v^e the sequence of edges incident to v in clockwise order around v in φ starting with e. We add the edge $\{\hat{u}, \hat{v}\}$ to C in two steps.

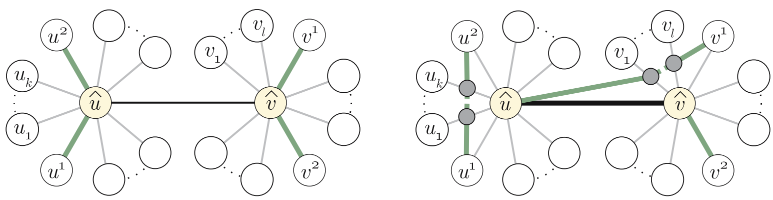


Fig. 4. A fixed edge f (black) and a part of H (bold)

Let $\{u^1, \hat{u}\}$ and $\{u^2, \hat{u}\}$ be the first and second edge in $E_{\hat{u}}^{\{\hat{u},\hat{v}\}}$, respectively, that is part of the Hamilton cycle. Replace successively each edge $\{u_i, \hat{u}\}$ in the list $E_{\hat{u}}^{\{u^1,\hat{u}\}}$ between $\{u^1, \hat{u}\}$ and $\{u^2, \hat{u}\}$—but not equal to one of these—by a new vertex u_i^{new} and the split edges $\{u_i, u_i^{\text{new}}\}$ and $\{u_i^{\text{new}}, \hat{u}\}$. Let $u_1^{\text{new}}, \ldots, u_k^{\text{new}}$ be the new vertices of this step. Replace the part $u^1, \hat{u}, u^2$ of the Hamilton cycle by $u^1, u_1^{\text{new}}, \ldots, u_k^{\text{new}}, u^2$ by the use of new auxiliary edges. Let $\{v^1, \hat{v}\}$ and $\{v^2, \hat{v}\}$ be the first and second edge in $E_{\hat{v}}^{(\hat{u},\hat{v})}$, respectively, that is part of the Hamilton cycle. Replace successively each edge $\{v_i, \hat{v}\}$ in the list $E_{\hat{v}}^{(\hat{u},\hat{v})}$ between $\{\hat{u}, \hat{v}\}$ and $\{v^1, \hat{v}\}$—but not equal to one of these—by a new vertex v_i^{new} and the split edges $\{v_i, v_i^{\text{new}}\}$ and $\{v_i^{\text{new}}, \hat{v}\}$. Let $v_1^{\text{new}}, \ldots, v_l^{\text{new}}$ be the new vertices of this step. Replace the part $v^1, \hat{v}, v^2$ of the Hamilton cycle by $v^1, v_1^{\text{new}}, \ldots, v_l^{\text{new}}, \hat{u}, \hat{v}$, v^2 by the use of new auxiliary edges.

Now, the edge $\{\hat{u}, \hat{v}\}$ is part of C. Observe that this edge is never removed by the subsequent steps. Moreover, no edge in F and no auxiliary edge is ever split. Calling the parts of a multiple split edge further on split edges, we can conclude the following.

Corollary 10. *Given a planar graph G and a set of fixed edges F such that no vertex is incident to ≥ 2 fixed edges, a good edge-extension G^+ of G and a Hamilton cycle C in G^+ can be found such that $F \subset C$.*

Property 11. We can always assume that both auxiliary edges of a new vertex v_{new} are part of the Hamilton cycle C. Otherwise remove v_{new}, its auxiliary edges and merge its split edges. Possibly reroute C.

We can use the ideas of Section 2 to obtain a simultaneous embedding and to draw all edges in F as straight lines. However, we do not know how many bends are necessary for an edge outside the Hamilton cycle. The following lemma helps us to limit the number of bends per edge. Let $V_1 = V$ and let V_2 be the set of new vertices of G^+. Use the following lemma iteratively for each path Q of length > 3 in G^+ corresponding to an edge in G. Observe that Q and C have no edges in common and all edges of Q are split edges. Since the edge-extension G^+ is good and because of Property 11, the application of the lemma below needs no edge splitting and the obtained edge-extension remains good.

Lemma 12. *Let $H = (V_1 \cup V_2, E)$ be a planar graph and let C be a cycle in H that visits all vertices of V_1. Additionally, let $Q = (v_1, v_2, \ldots, v_k)$ be a path in H whose endpoints belong to V_1 and whose remaining vertices all belong to V_2. H can be modified by adding edges and splitting some edges e neither part of C nor part of Q incident to an inner vertex at most two times such that a cycle $\hat{C}$ can be found that visits all vertices of V_1 and $\hat{C}$ crosses Q at most two times.*

Due to space limitations, a proof of Lemma 12 is omitted. Figure 5 sketches one iteration of the proof. Observe that a path Q that is crossed two times by C can be reduced to a path of length 3 (Property 11).

Corollary 13. *Let G be a planar graph, let F be a set of edges and let G^+ be a good edge-extension of G with a Hamilton cycle $C \supseteq F$. Another good edge-extension G^+_{new} of G with a Hamilton cycle C_{new} can be constructed such that C_{new} also contains all edges in F and each edge in G corresponds to a path of length ≤ 3 in G^+_{new}.*

In the following, we consider a generalized set of fixed edges. Moreover, the following algorithm works directly with the algorithm of Section 3.

Definition 14 (star-free). *For a given graph $G = (V, E)$, a set of edges $F \subseteq E$ is star-free if F does not contain three edges with a common endpoint.*

Definition 15 (cycle-free). *For a given graph $G = (V, E)$, a set of edges $F \subseteq E$ is cycle-free if each cycle spanned by F is a Hamilton cycle.*

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two planar graphs and let F be a set of edges that is star- and cycle-free with respect to G_1 and G_2. These graphs are handled now one after another. The set F can contain several paths of fixed edges. For the graph under consideration, let $Q_1, \ldots, Q_r$ denote the paths in F that can not be extended. Again, using the ideas of Section 2, we need a Hamilton cycle C in an edge-extension G^+ that contains all fixed edges.

This can be done iteratively by adding complete paths Q_i for $i = 1, \ldots, r$ to the Hamilton cycle. Construct an arbitrary Hamilton cycle C_0 with the algorithm of Section 3 and let C_i be the Hamilton cycle after step i that contains $Q_1, \ldots, Q_i$.

It remains to show how to add one path Q_i to C_{i-1}. First, use Lemma 12 to reduce the crossings of Q_i and C_{i-1}.

As shown in Fig. 6 by the dashed edges, reroute the ≤ 2 crossings of Q_i and C_{i-1} around one of the endpoints of Q_i. At the same time, handle the complete path Q_i of fixed edges similarly to one fixed edge: Add Q_i to C_{i-1} as shown in Fig. 6 by the dotted edges.

Each edge incident to a vertex on Q_i is split ≤ 2 times by Lemma 12, ≤ 2 times by the rerouting and ≤ 1 time by the step that adds Q_i to C_{i-1}. Altogether, such an edge is split ≤ 5 times. Since an edge in G can be incident only to two inner vertices of paths $Q_1, \ldots, Q_r$, an edge can be split $\leq 2 \cdot 5 = 10$ times after iterating over all $Q_1, \ldots, Q_r$. Again, use Lemma 12 to reduce the crossings of each edge and C_r to two without removing an edge of F from C_r. Use the algorithm of Section 2 to find a 5-bend simultaneous embedding of G_1 and G_2. With a similar argument as for Lemma 3.2 in [7], the number of bends per edge can be reduced to 3 at the expense of exponential area for the embedding.

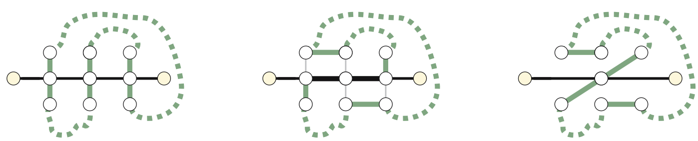


Fig. 5. Three crossings of Q and C can be reduced to one crossing

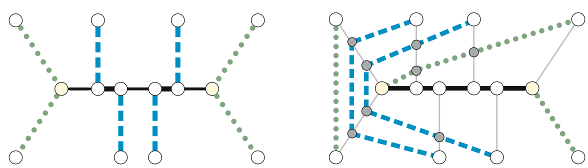


Fig. 6. A path of fixed edges Q (black) and some edges of a Hamilton path (dashed and dotted)

Corollary 16. *A 5-bend simultaneous embedding of two planar graphs with a star- and cycle-free set of fixed edges can be found in linear time. If the area may be arbitrary, three bends suffice.*

5 A Lower Bound and Other Restrictions

The graph shown in Fig. 7 clearly has no Hamilton path, since the white vertices outnumber the black ones by two, but form an independent set.

Lemma 17. *No* 1*-bend embedding for the vertices-to-points problem is possible in general.*

Proof. Let G be a planar, triangulated graph without a Hamilton path and let P be a set of vertices on a line. Since G has no Hamilton path, there must be two vertices embedded to consecutive points being not adjacent. Since G is triangulated, there is no face incident to these two vertices. Therefore, an edge $\{u, v\}$ with two bends has to exist that crosses the line between the two consecutive points. See Fig. 8(a).

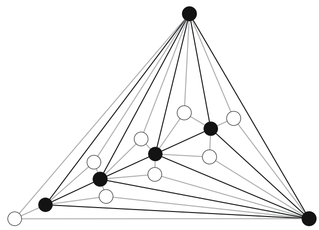
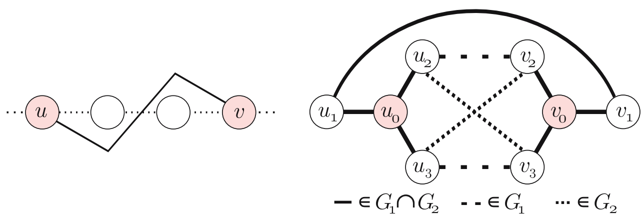


Fig. 7. A triangulated graph without a Hamilton path



(a) Vertices on a line (b) No sim. embedding

Fig. 8. Two counterexamples

The algorithm in the last section can only handle a star- and cycle-free set of fixed edges. The question arises whether this restriction is necessary or not. Consider first the case where two triangulated planar graphs and a not cycle-free set of fixed edges are given. Denote the cycle of fixed edges by $C \subseteq F$. If there are two vertices not part of C that are on the same side of the cycle in one of the two graphs and on different sides in the other graph, no simultaneous embedding is possible. Second, consider two triangulated planar graphs and two vertices u_0 and v_0 that are incident to at least three fixed edges $\{u_0, u_1\}, \{u_0, u_2\}, \{u_0, u_3\}$ and $\{v_0, v_1\}, \{v_0, v_2\}, \{u_0, v_3\}$, respectively. See Fig. 8(b). If in one graph the pairs of vertices $\{u_1, v_1\}$, $\{u_2, v_2\}$ and $\{u_3, v_3\}$, in the other graph the pairs of vertices $\{u_1, v_1\}$, $\{u_2, v_3\}$ and $\{u_3, v_2\}$ are connected by vertex-disjoint paths, respectively, again no simultaneous embedding is possible.

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