3-D Minimum Energy Broadcasting*

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Abstract. The Minimum Energy Broadcast Routing problem was extensively studied during the last years. Given a sample space where wireless devices are distributed, the aim is to perform the broadcast pattern of communication from a given source while minimizing the total energy consumption. While many papers deal with the 2-dimensional case where the sample space is given by a flat area, few results are known about the more interesting and practical 3-dimensional case. In this paper we study such a case and we present a tighter analysis of the minimum spanning tree heuristic in order to considerably decrease its approximation factor from the known 26 to roughly 18.8. This decreases the gap with the known lower bound of 12 given by the so called kissing number.

1 Introduction

The study of a basic pattern of communication such as the Broadcast is of main interest in the context of Wireless Ad Hoc Networks. The broadcast can be in fact used to setup the network or to rapidly spread useful information. The wireless environment allows to all the devices in the range of a transmitter to receive the message. The range of a transmission basically depends by the environment in which the devices are distributed. According to the mostly used power attenuation model [1], for some constants $\alpha, \beta \in \mathbb{R}^+$, when a station s transmits with power P_s, a station r can receive its message if and only if

$$\frac{P_s}{\|s, r\|^\alpha} > \beta,$$

where $\|s, r\|$ is the Euclidean distance between s and r. Clearly in environments with obstacles the needed power α increases. Due to the nonlinear power attenuation, multi-hop transmission of messages through intermediate devices may result in energy saving. Thus, a naturally arising issue is that of supporting the broadcast with a minimum total energy consumption. The problem is called *Minimum Energy Broadcast Routing* ($MEBR$) and it is NP-hard, while if $\alpha = 1$ or $d = 1$ it is solvable in polynomial time [2, 3]. One of the most extensively studied cases concerns the 2-dimensional Euclidean space with $\alpha = 2$. Several papers

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progressively reduced the estimate of the approximation ratio of the fundamental Minimum Spanning Tree (MST) heuristic from 40 to 6 [4, 5, 6, 7, 8, 9, 10]. In [6] it was proven that for any considered dimension $d > 1$, the critical case to study is when $\alpha = d$ while for $\alpha > d$ any result can be easily extended to any power between d and α. Note that for $\alpha < d$ the ratios cannot be bounded by any function of α and d [4]. The MST and other heuristics have been presented in [1, 11] also for the multicasting variation of the problem. As already noted, the performance of the MST heuristic has been investigated by several authors and in the 2-dimensional Euclidean space, for $\alpha = 2$, the performed approximation ratio is 6 [5], and it is optimal [10]. Such a value coincides with the so called *kissing number* that was proven to be a lower bound for the approximation ratio of the MST heuristic for any dimension $d > 2$ and power $\alpha \geq d$ [4]. More precisely, the kissing number is the maximum number of d-spheres (or hyperspheres) of a given radius r that can simultaneously touch a d-sphere of radius r in the d-dimensional Euclidean space [12]. In the 3-dimensional Euclidean space the kissing number is 12 (see Figure 1) but the best known approximation ratio so far is 26 [6].

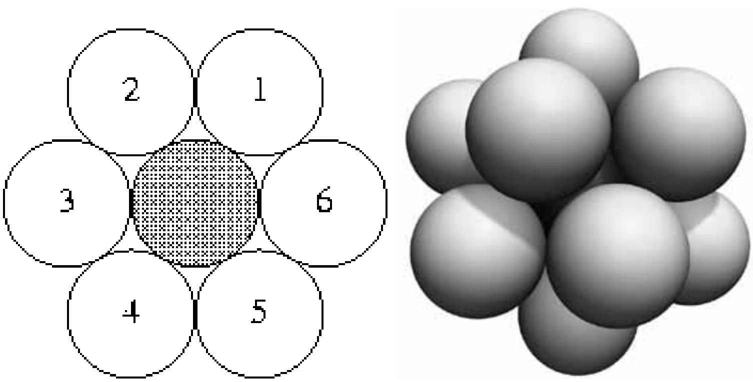

Fig. 1. The kissing number in the 2- and in the 3-dimensional case. It is given by 6 circles and 12 spheres respectively, simultaneously touching a central one.

In this paper we are interested in investigating more carefully this 3-dimensional case. We reduce the gap between upper and lower bound by decreasing the upper bound to roughly 18.8 (the exact obtained ratio is 18.802). Note that the 3-dimensional space better models practical environments since, in real life scenarios, radio stations are distributed over a 3-dimensional Euclidean space. Again the presence of obstacles can be overcome by the increasing of the power of transmission α. The main analysis is based on the study presented in [9] where a 6.33-approximation ratio of the MST heuristic for the 2-dimensional case was proven.

The paper is organised as follows. In the next section, we introduce the $MEBR$ problem with notations and the necessary definitions. In Section 3, we describe the technique that was used in [6] to prove the mentioned upper bound of 26 and we explain how to modify it in order to obtain a tighter bound for the 3-dimensional case. In Section 4, we present our main contribution that leads to the 18.8-approximation ratio. Finally, in Section 5, we give some conclusive remarks and discuss some open questions.

2 Minimum Energy Broadcast Routing

Let us first provide a formal definition of the Minimum Energy Broadcast Routing problem. Given a set of points S in a d-dimensional Euclidean space that represents the set of radio stations, an integer $\alpha \geq 1$ and a constant $\beta \in \mathbb{R}^+$, let $G_\alpha(S)$ be the complete weighted graph obtained as follows. The nodes of $G_\alpha(S)$ represent the points of S and the weight of each edge $\{x, y\}$ is the power consumption needed for a correct communication between x and y, that is $\beta \cdot \|(x, y)\|^\alpha$. For any subset of stations $Q \subseteq S$, let $G_\alpha(Q)$ be the subgraph of $G_\alpha(S)$ induced by Q.

A range assignment for S is a function $r : S \rightarrow \mathbb{R}^+$ such that the range $r(x)$ of a station x denotes the maximal distance from x at which signals can be correctly received. The total cost of a range assignment is then

$$cost(r) = \sum_{x \in S} \beta \cdot r(x)^\alpha.$$

A range assignment r for S yields a directed communication graph $G^r = (S, A)$ such that, for each $(x, y) \in S^2$, the directed edge (x, y) belongs to A if and only if y is at distance at most $r(x)$ from x. In other words, (x, y) belongs to A if and only if the emission power of x is at least equal to the weight of $\{x, y\}$ in $G_\alpha(S)$. In order to perform the required $MEBR$ from a given source $s \in S$, G^r must contain a directed spanning tree rooted at s and must have a minimum cost, from now on denoted as $m^*_\alpha(S, s)$.

One fundamental algorithm, called the MST heuristic [1], is based on the idea of tuning ranges so as to include a spanning tree of minimum cost. Roughly speaking, the heuristic computes the directed minimum spanning tree from the given source to the leaves. Such a computation is made over the complete weighted graph obtained from the set of nodes in which weights are the power of α of the distances of the endpoints of the edges. For each node, then, the heuristic assigns a power of transmission equal to the weight of the longest outgoing edge.

More precisely, let $T_\alpha(S)$ be a minimum spanning tree of $G_\alpha(S)$ and $MST(G_\alpha(S))$ its cost. Considering $T_\alpha(S)$ rooted at the source station s, the heuristic directs the edges of $T_\alpha(S)$ toward the leaves and sets the range $r(x)$ of every internal station x of $T_\alpha(S)$ with k children $x_1, \ldots, x_k$ in such a way that $r(x) = \beta \cdot max_{i=1,\ldots,k} \|x, x_i\|^\alpha$. In other words, r is the range assignment of minimum cost inducing the directed tree derived from $T_\alpha(S)$ and is such that $cost(r) \leq MST(G_\alpha(S))$. Therefore, in order to bound the approximation ratio of the heuristic, it is sufficient to bound the ratio between the cost $MST(G_\alpha(S))$ of a minimum spanning tree of $G_\alpha(S)$ and the optimal cost $m^*_\alpha(S, s)$.

Starting from the definition of minimum spanning tree given in [13], in [6] an interesting way to evaluate the cost of the heuristic is provided. For any subset of stations $Q \subseteq S$, let $G_\alpha(Q, r)$ be the graph obtained by considering only the edges of $G_\alpha(Q)$ of length at most r (that clearly have cost at most βr^α) and let $CC(Q, r)$ be the set of the connected components of $G_\alpha(Q, r)$. Let $n(Q, r) = |CC(Q, r)|$ be the number of connected components in $G_\alpha(Q, r)$ and $r_{max}(Q)$ be the minimum r such that $G_\alpha(Q, r)$ is connected (i.e. $n(Q, r_{max}) = 1$).

Corollary 1. *[6] For any subset of stations $Q \subseteq S$,*

$$MST(G_\alpha(Q)) = \alpha\beta \int_0^{r_{max}(Q)} (n(Q, r) - 1)r^{\alpha-1}\partial r.$$

For any set of stations Q let $e(Q) = \min_{x \in Q} \max_{y \in Q} \|x, y\|$ be the eccentricity of Q. Hence, there exists a station $x \in Q$ such that $\|x, y\| \leq e(Q)$ for every other $y \in Q$. Once chosen such a station x, let $c(Q)$ be the sphere of radius $e(Q)$ centered at x. The following general lemma is useful in the estimation of the approximation ratio of the MST heuristic.

Lemma 1. *[6] If $MST(G_\alpha(Q)) \leq \rho\beta e(Q)^\alpha$ for any subset of stations $Q \subseteq S$, then the MST heuristic is a ρ-approximation algorithm for the $MEBR$ problem.*

In the following we will concentrate on the $MEBR$ problem with $\alpha = 3$ in the 3-dimensional case. Thus, the cost of each edge of the weighted complete graph $G_3(S)$ representing the input network is proportional to the cube of the distance between its endpoints. For ease of notation, for any set of stations Q we will denote $G_3(Q)$ simply as $G(Q)$. Moreover, for the sake of simplicity, without loss of generality we assume $\beta = 1$ and $e(Q) = 1$, as all the results provided under this assumption can be directly extended to the general case [6].

3 Description of the Approach

In this section we firstly describe the general technique presented in [6]. Such a technique leads to the $(3^d - 1)$-approximation ratio of the MST heuristic for the $MEBR$ problem for any $d > 1$ and any $\alpha \geq d$. In our specific case, that is $d = 3$, $\alpha = 3$, the obtained approximation is 26. Secondly, by following the ideas in [9], we describe how to modify the previous technique hence leading to a new and tighter estimation of the upper bound, that is, of roughly 18.8.

For the general case the technique was based on a growing process (from now on called *basic*) in which d-spheres of equal radii centered in the stations of the subset Q are synchronously grown (see for instance Figure 2). The process starts by setting the radius $r = 0$ and ends when $r = \frac{r_{max}(Q)}{2} \leq \frac{1}{2}$, that is, when $G(Q, 2r)$ becomes connected. This is accomplished by increasing at any infinitesimal step the current radii, all equal to a given r, by ∂r.

Starting from the equality established in Corollary 1 on the cost $MST(G(Q))$ of any minimum spanning tree of $G(Q)$, the idea was to provide suitable lower

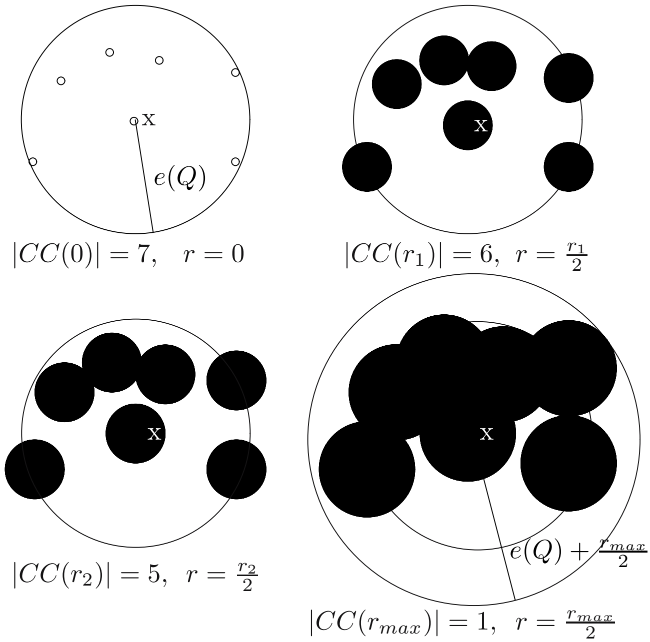

Fig. 2. The growing process of circles around the radio stations of the set Q in the 2-dimensional case

and upper bounds on the overall volume covered by the union of all the d-spheres at the end of the described process. In [6] the bound $MST(G(Q)) \leq 3^d - 1$ was proven, that by Lemma 1 implies the 26-approximability of the MST heuristic in the 3-dimensional case. Note that the lower bound is instead 12 and it is given by the kissing number [4, 12].

We now show how to improve the 26-approximation ratio by means of a new technique. The new analysis is based on the method presented in [9] where the 2-dimensional case was considered. The idea is to slightly change the shapes that are grown around stations at each infinitesimal step of the previously described basic growing process. More precisely, being in the 3-dimensional case we consider $c(Q)$ as the spherical place inside which the radio stations are thrown uniformly at random. While before each station was wrapped by a sphere, now things remain the same inside $c(Q)$, but the volume is thinned when growing outside $c(Q)$. Informally speaking, this allows to maintain the lower bound on the covered volume at the end of the growing process. On the other hand, the upper bound decreases since all the volume can be now included in a smaller sphere with respect to [6], thus improving the bound on the cost of the returned solution.

For the sake of clarity from now on we often drop Q from the notation, thus for instance writing G, $G(r)$, $CC(r)$, $n(r)$ and r_{max} instead of $G(Q)$, $G(Q, r)$, $CC(Q, r)$, $n(Q, r)$ and $r_{max}(Q)$, respectively.

In order to better explain the new reshaping technique we describe it in two phases. For any given radius r, the shape of radius r associated to a given

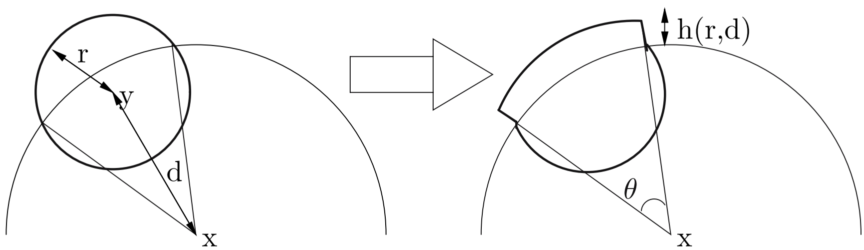

Fig. 3. Section of the new associated growing shape to each radio station

station y inside $c(Q)$ having distance d from the central station x is such that its intersection with $c(Q)$ coincides with the circular intersection of $c(Q)$ with a sphere of radius r centered at y. In other words, the intersection with $c(Q)$ of the new shape coincides with the *basic shape* given by the sphere of [6]. Outside $c(Q)$, the remaining portion of the sphere of radius r, if any, is reshaped as a kind of cylinder of suitable height $h(r, d)$ wrapping the outside spherical surface of $c(Q)$. In Figure 3 it is showed a cut section of the sphere $c(Q)$ centered at x and of the new shape. The height $h(r, d)$ is evaluated in such a way that its volume coincides with the volume of the corresponding portion of the basic shape outside $c(Q)$. This implies that the total volume remains $\frac{4}{3}\pi r^3$. With $\theta(r, d)$ we identify a conic angle obtained by connecting the center x with the circular intersection of the shape with $c(Q)$ (see Figure 3).

At each infinitesimal step in which the radius r grows by ∂r, given any function g depending on r, we denote by $\partial g(r) = g(r+\partial r)-g(r)$ the infinitesimal variation of $g(r)$.

At each infinitesimal step, while the growth of the spherical part inside $c(Q)$ is the same as in the basic case, the angle $\theta(r, d)$ of the outside part augments by a given quantity $\partial\theta(r, d)$. This is done according to the intersection of the increased sphere of radius $r+\partial r$ with $c(Q)$. About the height $h(r, d)$, it augments by $\partial h(r, d)$ in such a way that the total volume added to the shape is $4\pi r^2 \partial r$ as in the basic case.

Clearly, two shapes corresponding to a given radius r overlap if and only if the corresponding centers are at distance at most $2r$, as in the basic case. Starting from the observation that the shapes never meet at the circular intersections with the spherical surface of $c(Q)$,[1] it is possible to slightly enlarge the outside part of each shape.

This introduces the second phase of our shape modification by which enlarging the shape outside $c(Q)$ decreases its height. This must be done by increasing the angles $\theta(r, d)$ without violating the constraint that two shapes never meet outside $c(Q)$ before they meet inside. This allows to decrease the maximum

[1] The only exception is given when such intersections are subtended by the biggest section of the current sphere that they represent. To better explain this concept, in the 2-dimensional Euclidean space, this happen when the intersections are the endpoints of the diameter of the corresponding growing circle.

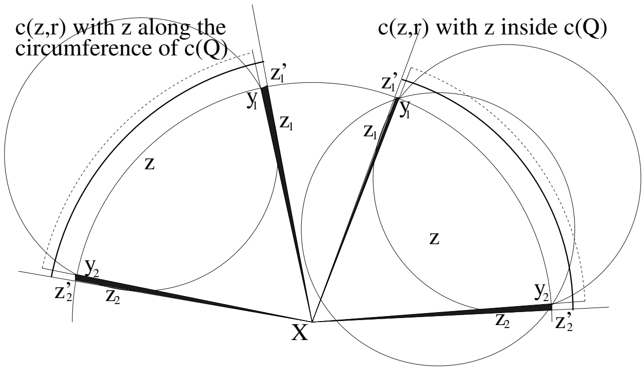

Fig. 4. Section of the new shape given by the increase of the angle θ by the black portions, yielding the new angle θ' and the decrease of the height from the dotted lines to the bold ones

height of the outside part of the shapes, thus yielding a further improvement on the approximation ratio. In other words the new shape will be larger but lower and it is defined as follows. Consider any point z inside $c(Q)$. Let $c(z, r)$ be the sphere of radius r centered at z and let $I(z, r)$ be the circular intersection of $c(z, r)$ with the spherical surface of $c(Q)$. Consider the sphere $c(z', r')$ centered on the border of $c(Q)$ and having the same intersection with the spherical surface of $c(Q)$, i.e., $I(z, r) \equiv I(z', r')$. The conic angle associated to z is now defined by the vertex x and the cone tangent to $c(z', r')$ (see the cut section of the conic angle in the right of Figure 4). Note that, in the case in which z lies on the spherical surface of $c(Q)$, $c(z, r)$ and $c(z', r')$ coincide (see the cut section of the new conic angle on the left of Figure 4). Indeed their angle does not, since, as already described, it is given by the tangent cone to the internal spherical shape and not, as before, by the cone wrapping the intersection with the surface of $c(Q)$.

When two new shapes are centered along the spherical surface of $c(Q)$ at distance $2r$, by construction, they meet outside at the same moment they meet inside, that is, when the radius grows till r. If we move one or both the corresponding centers more inside $c(Q)$ and leaving their distance at $2r$, the corresponding reshaped outside volumes remain disjoint.

4 18.8-Approximation Analysis of the MST Heuristic

In this section we formalise what was previously described. We provide a set of lemmata that describe a corresponding set of properties of the defined new shape that are useful in order to prove the 18.8-approximation claimed in the concluding theorem. The new shape must guarantee some properties that were true by means of the standard sphere. One of those properties is that two shapes growing according to a given radius r, touch each other only when the corresponding centers are at distance at most $2r$. Note that this is the fundamental property without which Corollary 1 cannot be applied for the estimation of the cost of the MST heuristic.

Lemma 2. *Given any subset of stations $Q \subseteq S$, for any $r < \frac{r_{max}}{2}$, two shapes overlap if and only if the corresponding points are at distance at most $2r$.*

Proof. If two shapes meet inside $c(Q)$, the property easily holds since the shape has the same behavior of spheres. In order to prove the claim we have to show that two shape never meet outside if they do not meet inside also. By construction, the external part of a shape is more extended (in terms of occupancy of the outer spherical surface) when the center resides along the spherical surface of $c(Q)$. In such a case, if two shapes touch each other, they do exactly at their intersection with the spherical surface of $c(Q)$ (see Figure 4). If one them has the center more inside, its growing outside part is less extended hence it cannot touch any other outer part of another shape. □

The following two lemmata consider more carefully the structure of the new shape by considering the conic angle and the outside growing height, respectively. About the angle, it is proven that the more a station, whose associated shape grows also outside, is closer to x, the more its angle grows at each infinitesimal step.

Lemma 3. *Given any subset of stations $Q \subseteq S$, for any $r < \frac{r_{max}}{2}$ and any $d_1 \leq 1$ and $d_2 \leq 1$ such that $1 - r \leq d_1 \leq d_2$, $\partial\theta(r, d_1) \geq \partial\theta(r, d_2)$.*

The following lemma, instead, proves that the further a station is from x, the more its height outside $c(Q)$ grows. Moreover it gives also a very useful lower bound to the infinitesimal growth of the height and its maximum extension. The new shape, in fact, grows in height, outside $c(Q)$ as at least $\frac{3}{5}$ the growth of the basic shape at any infinitesimal step. This guarantees that the growth of such a shape is quite uniform during the whole process hence it is still suitable for bounding the MST heuristic cost. Moreover the maximal height outside $c(Q)$ is bounded by .3527 hence decreasing the maximal extension of the basic shape that was of .5.

Lemma 4. *Given any subset of stations $Q \subseteq S$, for any $r < \frac{r_{max}}{2}$ and any $d_1 \leq 1$ and $d_2 \leq 1$ such that $1 - r \leq d_1 \leq d_2$, $h(r, d_1) \leq h(r, d_2)$. Moreover for any $d \leq 1$, $h(r, d) \leq .3527...$ and $\partial h(r, d) \geq \frac{3}{5}\partial r$.*

Note that Lemma 3 and Lemma 4 follow directly from the corresponding lemmata of the 2-dimensional case [9]. In order to better understand this, it is sufficient to consider the new shape as the 2-dimensional one rotated along the line passing through its center and the center of $c(Q)$ (see Figure 5). In this way it is clear that what was true about the angle of the 2-dimensional case is now straightforward for the new conic angle θ (Lemma 3). And the same happens for the height h that remains unchanged (Lemma 4).

With the last lemma we ensure that the new shape guarantees an infinitesimal growth, for each connected component equal to at least the same growth of one

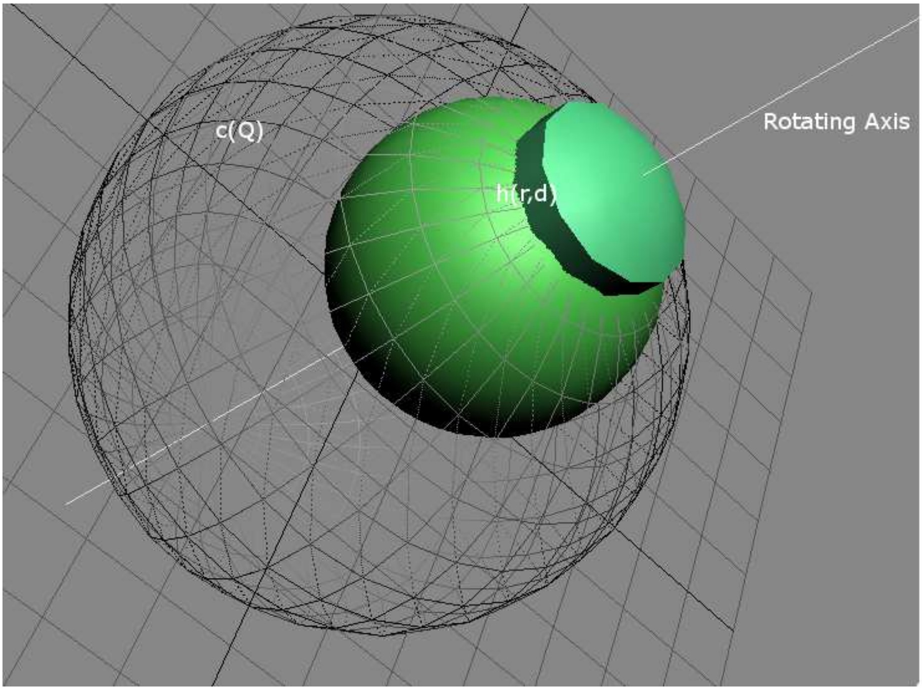

Fig. 5. The new shape obtained by means of a rotation of the 2-dimensional one along the line passing through its center and the center of $c(Q)$

sphere for each component. This was straightforward in the general case of d-spheres while it is quite complicated both in the 2- and the 3-dimensional case for the modified shapes.

Lemma 5. *The infinitesimal growth of the volume* $v(P, r)$ *of the region* $P(r)$ *covered by the shapes of a connected component* $P \in CC(2r)$ *of* $G(2r)$ *is* $\partial v(P, r) \geq 4\pi r^2 \partial r$.

Proof. (Sketch) If P contains just one station, then, by construction, the claim clearly holds. In fact, if the growth of the shape associated to such a station does not concern outside $c(Q)$ then it coincides with a growing sphere. Since the spherical surface is given by $4\pi r^2$, the infinitesimal growth is $\partial v(P, r) = 4\pi r^2 \partial r$. In the case the growth of the shape associated to the considered station goes outside $c(Q)$, then, by construction, the new shape is made in such a way that inside $c(Q)$ things do not change. Outside, the volume is maintained equal to the spherical case at every infinitesimal step, hence its growing too. When P contains more than one station, intuitively things can just go better, i.e., the growth of the union of the associated shapes is at least the growth of one sphere. This is given by the fact that both inside and outside $c(Q)$ when two shapes join in one connected component, their physical extension contains the shape corresponding to just one station. This suggest that at any infinitesimal step, its growth is bigger than the sphere. □

From all the above lemmata we can finally obtain the following theorem.

Theorem 1. *In the* 3*-dimensional Euclidean space the* MST *heuristic is a* 18.8*-approximation algorithm for the* $MEBR$ *problem.*

Proof. It is enough to prove that for any subset of stations $Q \subseteq S$, $MST(G(Q)) < 18.8$. The claim then follows by Lemma 1. Exploiting Lemma 5, we can easily provide a lower bound for the total region of the space covered by the union of all the shapes related to Q of radius $\frac{r_{max}}{2}$, that is $v(Q, \frac{r_{max}}{2})$, the covered volume at the end of the described growing process. In fact, recalling that by Corollary 1 $MST(G(Q)) = 3\int_0^{r_{max}(Q)}(n(Q,r)-1)r^2\partial r$,

$$v\left(Q, \frac{r_{max}}{2}\right) = \int_0^{\frac{r_{max}}{2}} \sum_{P \in CC(2r)} \partial v(P,r)\partial r \geq \int_0^{\frac{r_{max}}{2}} n(2r)4\pi r^2 \partial r =$$

$$= \frac{1}{8}4\pi \int_0^{r_{max}} n(r)r^2\partial r = \frac{1}{2}\pi \int_0^{r_{max}} (n(r)-1)r^2\partial r + \frac{1}{2}\pi \int_0^{r_{max}} r^2 \partial r =$$

$$= \frac{\pi}{6}MST(G) + \frac{\pi}{6}r_{max}^3.$$

Moreover, by Lemma 4, $v(Q, \frac{r_{max}}{2})$ is included in a sphere of radius $1 + h(\frac{r_{max}}{2}, 1)$ centered at the station x. Therefore, $v(Q, \frac{r_{max}}{2}) \leq \frac{4}{3}\pi(1+h(\frac{r_{max}}{2},1))^3$, so that

$$\frac{\pi}{6}MST(G) + \frac{\pi}{6}r_{max}^3 \leq v\left(Q, \frac{r_{max}}{2}\right) \leq \frac{4}{3}\pi\left(1 + h\left(\frac{r_{max}}{2}, 1\right)\right)^3,$$

hence,

$$MST(G) \leq 8\left(1 + h\left(\frac{r_{max}}{2}, 1\right)\right)^3 - r_{max}^3.$$

Standard maximization argument obtained for r_{max} ranging from 0 to 1 shows that the quantity $8(1+h(\frac{r_{max}}{2},1))^3 - r_{max}^3$ is maximised for $r_{max} = 1$, and since by Lemma 4, $h(r,d) \leq .3527...$, it finally results

$$MST(G) \leq 8\left(1 + h\left(\frac{1}{2}, 1\right)\right)^3 - 1 < 18.802.$$ □

5 Conclusion

In this paper we have investigated the Minimum Energy Broadcast Routing problem in the 3-dimensional Euclidean space. We have improved the previous known upper bound on the approximation ratio of the MST heuristic from 26 to 18.8, considerably decreasing the gap with the lower bound of 12 [4]. It is worth noting that, according to the considered method, such a new bound is not tight in terms of the associated volume outside $c(Q)$ as it was in the 2-dimensional case. Let us consider, in fact, the instance of the lower bound obtained by thirteen stations distributed like the centers of the spheres of the kissing number, i.e., everyone at distance at least $r_{max} = 1$ from each other inside $c(Q)$. The resulting associated volume of the new shapes does not fulfil neither $c(Q)$ as it was for the 2-dimensional case, nor the external volume in between the two spheres of radii 1 and $1 + h_{max} \approx 1.3527$ respectively, see Figure 6. Assuming the lower bound of 12 as the real bound of the MST heuristic in the 3-dimensional Euclidean space, the loss of 6.8 with respect to it must be found then in those "holes" inside and outside $c(Q)$, that is, the shaded volumes of Figure 6.

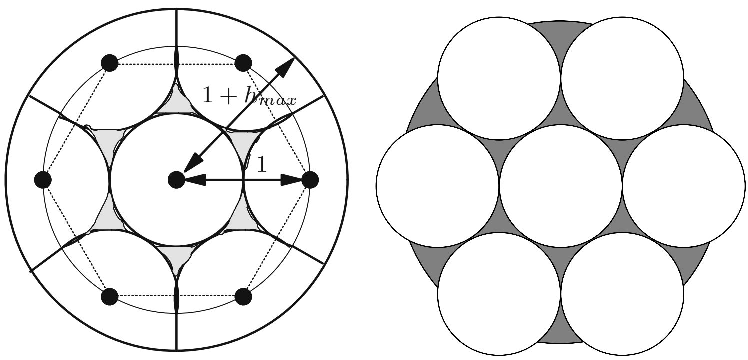

Fig. 6. On the right, a cut section of the lower bound case with the associated shapes. Shaded areas represent the mentioned holes inside the sphere $c(Q)$ of radius 1. On the left, a squeesed representation of what happens outside $c(Q)$. Again the shaded surfaces represent the mentioned holes outside $c(Q)$.

An interesting issue for a future work is of trying to apply the arguments of [5] in this 3-dimensional case and check whether they lead to anything better than the obtained 18.8 bound. The 3-D Delaunay triangulation is something known [14, 15] but it is not clear if the 2-dimensional arguments of [5] can be directly extended to the 3-dimensional case.

Another interesting case in the 3-dimensional environment is given for $2 \leq \alpha < d$. Since it can happen in practical application that the presence of obstacles can be both in contrast and in favor of communications, it depends on the desired directions. In the former case the given solution for the free 3-dimensional case is still valid since it is enough to suitably increase the value of α. In the latter, things become harder. In this case, in fact, it is not clear what the best solution may be. Moreover, the 18.8-approximation ratio does not hold for values of α smaller than d.

As last remark, from the experimental point of view, no results are known concerning the 3-dimensional case. All the experimental papers and the proposed heuristics start to investigate the 2-dimensional case (see for instance [16, 17, 18]). Is there any property not already captured that may lead to a better heuristic in the 3-dimensional case? In [7, 19], for instance, nice approaches to better understand the behavior of the MST heuristic in the 2-dimensional case are provided. The experiments have shown how good is the heuristic when applied on practical instances, like the high-density ones. It may be of deep interest to investigate in this direction for the 3-dimensional case as well.

References

1. Wieselthier, J.E., Nguyen, G.D., Ephremides, A.: On the construction of energy-efficient broadcast and multicast trees in wireless networks. In: Proceedings of the 19^{th} Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM), IEEE Computer Society (2000) 585–594
2. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P.: New results for energy-efficient broadcasting in wireless networks. In: Proceedings of the 13^{th} International Symposium on Algorithms and Computation (ISAAC), Springer-Verlag (2002) 332–343
3. Clementi, A.E.F., Ianni, M.D., Silvestri, R.: The minimum broadcast range assignment problem on linear multi-hop wireless networks. Theoretical Computer Science **299**(1-3) (2003) 751–761
4. Clementi, A., Crescenzi, P., Penna, P., Rossi, G., Vocca, P.: On the complexity of computing minimum energy consumption broadcast subgraph. In: Proceedings of the 18^{th} Annual Symposium on Theoretical Aspects of Computer Science (STACS). Volume 2010 of Lecture Notes in Computer Science., Springer-Verlag (2001) 121–131
5. Ambuehl, C.: An optimal bound for the mst algorithm to compute energy efficient broadcast trees in wireless networks. In: Proceedings of the 32^{nd} International Colloquium on Automata, Languages and Programming (ICALP). Volume 3580 of Lecture Notes in Computer Science., Springer Verlag (2005) 1139–1150
6. Flammini, M., Klasing, R., Navarra, A., Perennes, S.: Improved approximation results for the Minimum Energy Broadcasting Problem. In: Proceedings of ACM Joint Workshop on Foundations of Mobile Computing (DIALM-POMC). (2004) 85–91. To appear on the associated Special Issue of *Algorithmica*.
7. Flammini, M., Navarra, A., Perennes, S.: The "Real" approximation factor of the MST heuristic for the Minimum Energy Broadcasting. In: Proceedings of the 4^{th} International Workshop on Efficient and Experimental Algorithms (WEA). Volume 3503 of Lecture Notes in Computer Science., Springer Verlag (2005) 22–31. To appear on the associated Special Issue of *Journal of Experimental Algorithmics*.
8. Klasing, R., Navarra, A., Papadopoulos, A., Perennes, S.: Adaptive Broadcast Consumption (ABC), a new heuristic and new bounds for the minimum energy broadcast routing problem. In: Proceedings of the 3^{rd} IFIP-TC6 International Networking Conference. Volume 3042 of Lecture Notes in Computer Science., Springer Verlag (2004) 866–877
9. Navarra, A.: Tighter bounds for the Minimum Energy Broadcasting problem. In: Proceedings of the 3^{rd} International Symposium on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt). (2005) 313–322
10. Wan, P.J., Calinescu, G., Li, X., Frieder, O.: Minimum energy broadcasting in static ad hoc wireless networks. Wireless Networks **8**(6) (2002) 607–617
11. Liang, W.: Constructing minimum-energy broadcast trees in wireless ad hoc networks. In: Proceedings of the 3^{rd} ACM international symposium on Mobile ad hoc networking and computing (MOBIHOC). (2002) 112–122

12. Conway, J.H., Sloane, N.J.A.: "The Kissing Number Problem" and "Bounds on Kissing Numbers". Ch. 2.1 and Ch. 13 in: Sphere Packings, Lattices, and Groups. Springer-Verlag, New York (3rd edition, 1998)
13. Frieze, A.M., McDiarmid, C.J.H.: On Random Minimum Length Spanning Trees. Combinatorica **9** (1989) 363–374
14. Attali, D., Boissonnat, J.D.: A linear bound on the complexity of the delaunay triangulation of points on polyhedral surfaces. In: Proceedings of the 7^{th} ACM symposium on Solid modeling and applications (SMA). (2002) 139–146
15. Fang, T.P., Piegl, L.A.: Delaunay triangulation in three dimensions. IEEE Computer Graphics and Applications **15**(5) (1995) 62–69
16. Athanassopoulos, S., Caragiannis, I., Kaklamanis, C., Kanellopoulos, P.: Experimental Comparison of Algorithms for Energy-Efficient Multicasting in Ad Hoc Networks. In: Proceedings of the 3^{rd} International Conference on Ad-Hoc Networks and Wireless (ADHOC-NOW). Volume 3158 of Lecture Notes in Computer Science., Springer Verlag (2004) 183–196
17. Penna, P., Ventre, C.: Energy-efficient broadcasting in ad-hoc networks: combining msts with shortest-path trees. In: Proceedings of the 1^{st} ACM International Workshop on Performance Evaluation of Wireless, Ad Hoc, Sensor and Ubiquitous Networks (PE-WASUN). (2004) 61–68
18. Yuan, D.: Computing Optimal or Near-Optimal Trees for Minimum-Energy Broadcasting in Wireless Networks. In: Proceedings of the 3^{rd} International Symposium on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt). (2005) 323–331
19. Clementi, A., Huiban, G., Penna, P., Rossi, G., Verhoeven, Y.C.: On the approximation ratio of the mst-based heuristic for the energy-efficient broadcast problem in static ad-hoc radio networks. In: Proceedings of the 3^{rd} IEEE IPDPS Workshop on Wireless, Mobile and Ad Hoc Networks (WMAN). (2003) 222