Grayscale Watersheds on Perfect Fusion Graphs

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Abstract. In this paper, we study topological watersheds on perfect fusion graphs, an ideal framework for region merging. An important result is that contrarily to the general case, in this framework, any topological watershed is thin.

Then we investigate a new image transformation called C-watershed and we show that, on perfect fusion graphs, the segmentations obtained by C-watershed correspond to segmentations obtained by topological watersheds. Compared to topological watershed, a major advantage of this transformation is that, on perfect fusion graph, it can be computed thanks to a simple linear-time immersion-like algorithm. Finally, we derive characterizations of perfect fusion graphs based on thinness properties of both topological watersheds and C-watersheds.

1 Introduction

Region merging methods [1] consist of improving an initial segmentation by merging some pairs of neighboring regions. The watershed transform [2, 3, 4, 5] produces a set of connected regions separated by a divide. Therefore it has long been used as an entry point for region merging methods [6]. In [7], we developed a theoretical framework for the study of merging properties in graphs. A *(binary) watershed set* is a set of vertices which cannot be reduced without changing the number of connected components of its complementary. It models a frontier in a graph. In the general case such a watershed set can be thick and thus the induced region neighboring relationship, used by further merging procedures, can lack important properties.

An original approach to grayscale watershed [5, 8, 9, 10] consists of modifying the original image by lowering some points while preserving the connectivity of each lower section. Such a transformation (and its result) is called a W-thinning, a *topological (grayscale) watershed* being an "ultimate" W-thinning. In [8, 10], the authors prove that the only lowering transformation which preserves the connection value (a notion of contrast) between the minima of the original image is precisely the W-thinning. Due to this contrast preservation property, the divide (*e.g.*, the points not in any minimum) of a topological watershed is an interesting segmentation of the original image. Furthermore, this contrast preservation property is necessary for the correctness of many region merging methods based on watersheds (see [11, 12] for examples of such methods).

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An important result in [7] is that the class of all graphs in which any binary watershed set is thin is precisely the class of graphs in which any region can always be merged. Any element in this class is called a fusion graph. Surprisingly, the divides produced by watershed algorithms [2, 3, 4] and in particular by topological watershed algorithms [9], are not always binary watershed sets and can sometimes be thick, even on fusion graphs.

Therefore, in this paper, we consider a more restricted class of graphs called perfect fusion graph [7] which constitutes an ideal framework for region merging. An important result is that, on perfect fusion graphs, the divide of any topological watershed is a thin binary watershed set. The algorithms to compute topological watershed are not linear and require the computation of an auxiliary data structure called component tree [13]. Therefore, we investigate a new grayscale transformation: the *C-watershed*. Our main contributions concerning C-watersheds are the following:

1) We prove that, contrarily to the general case, on perfect fusion graphs, any C-watershed of a map is indeed a W-thinning and thus possesses the contrast preservation property, needed by morphological region merging methods.
2) On these graphs, the divide of any C-watershed is a thin binary watershed set. Consequently, we derive characterizations of perfect fusion graphs based on thinness properties of both C-watersheds and topological watershed functions.
3) We propose and prove the correctness of a new simple and linear-time algorithm to compute C-watershed on these graphs, while the correctness of such an algorithm cannot be guaranteed in the general case [10].

The proofs of the properties presented in this paper will be given in a forthcoming extended version.

2 Watersheds and Fusion Graphs

2.1 Basic Notions and Notations

Let E be a finite set, we denote by 2^E the set composed of all the subsets of E. We denote by $|E|$ the number of elements of E.

We define a graph as a pair (E, Γ) where E is a finite set and Γ is a binary relation on E (*i.e.,* $\Gamma \subseteq E \times E$), which is reflexive (for all x in E, $(x, x) \in \Gamma$) and symmetric (for all x, y in E, $(y, x) \in \Gamma$ whenever $(x, y) \in \Gamma$). Each element of E (resp. Γ) is called a *vertex* or a *point* (resp. *an edge*). We will also denote by Γ the map from E to 2^E such that, for all $x \in E$, $\Gamma(x) = \{y \in E \mid (x, y) \in \Gamma\}$. If $y \in \Gamma(x)$, we say that y is *adjacent to* x. Let $X \subseteq E$, we define $\Gamma(X) = \cup_{x \in X} \Gamma(x)$, and $\Gamma^\star(X) = \Gamma(X) \setminus X$. If $y \in \Gamma(X)$, we say that y *is adjacent to* X. If $X, Y \subseteq E$ and $\Gamma(X) \cap Y \neq \emptyset$, we say that Y *is adjacent to* X.

Let (E, Γ) be a graph, let $X \subseteq E$, a *path in* X is a sequence $\pi = \langle x_0, ..., x_l \rangle$ such that $x_i \in X$, $i \in [0, l]$, and $x_i \in \Gamma(x_{i-1})$, $i \in [1, l]$. We also say that π is a *path from* x_0 *to* x_l *in* X and that x_0 and x_l are *linked for* X. We say that X *is connected* if any x and y in X are linked for X. In the sequel we will consider that (E, Γ) is a graph and we will assume that E is connected.

Let $X \subseteq E$ and $Y \subseteq X$. We say that Y *is a connected component of* X, or simply *a component of* X, if Y is connected and if Y is maximal for this property, *i.e.,* if $Z = Y$ whenever $Y \subseteq Z \subseteq X$ and Z connected. We denote by $\mathcal{C}(X)$ the set of all connected components of X.

Let $k_{\min}$ and $k_{\max}$ be two elements of $\mathbb{Z}$ such that $k_{\min} < k_{\max}$. We set $\mathbb{K} = \{k \in \mathbb{Z}; k_{\min} \leq k < k_{\max}\}$ and $\mathbb{K}^+ = \mathbb{K} \cup \{k_{max}\}$. We denote by $\mathcal{F}(E)$ the set composed of all functions from E to $\mathbb{K}$. Let $F \in \mathcal{F}(E)$, let $k \in \mathbb{K}^+$. We denote by $F[k]$ the set $\{x \in E; F(x) \geq k\}$ and by $\overline{F}[k]$ its complementary set; $F[k]$ is called an *upper section of* F and $\overline{F}[k]$, a *lower section of* F. A connected component of $\overline{F}[k]$ which does not contain a connected component of $\overline{F}[k-1]$ is a *(regional) minimum* of F. We denote by $M(F) \subseteq E$ the set of all points which are in a minimum of F. We say that $\overline{M}(F) = \overline{M(F)}$ is the *divide of* F.

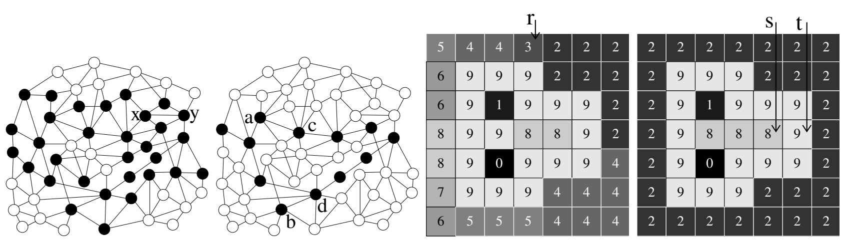
2.2 Watershed Set and Fusion Graphs

The notion of (binary) watershed set may be seen as a model of frontier in a graph. Many segmentation algorithms expect to compute such watershed sets.

In the following definitions "W-" stands for watershed.

Definition 1. *Let $X \subseteq E$ and let $p \in X$. We say that p is* W-simple for X *if p is adjacent to exactly one component of $\overline{X}$.*
The set X is a watershed if there is no W-simple point for X.

In Fig. 1a, y is W-simple for the set constituted by the black vertices. Observe that the set X of black points in Fig. 1b is a watershed set since it contains no W-simple point for X.



(a) (b) (c) (d)

Fig. 1. Illustration of watersheds. (a): A graph (E, Γ) and a subset X (black points) of E; (b): the set of black points is a watershed of X; (c): a graph corresponding to the 8-adjacency relation and a function F; (d): a topological watershed of F.

Let $X \subseteq E$ and let $p \in X$. We say that p is *an inner point for* X if p is not adjacent to $\overline{X}$. The interior of X is the set of all inner points for X, denoted $int(X)$. If $int(X) = \emptyset$, we say that X is thin.

For example, the point x in Fig. 1a is an inner point for the set of black vertices. In Fig. 1b, the set of black vertices is thin. The sets made of black and gray points in Fig. 3a and b are not thin: their interior, depicted in gray, are not empty. Observe also that they are watershed sets since they do not contain any W-simple points.

The theoretical framework set up in [7] allows to study the properties of region merging methods in graphs. In particular, one of the most striking theorems, allows to link the region merging properties with the thinness properties of watershed set.

In the following definition the prefix "F-" stands for fusion.

Let $X \subseteq E$. Let $x \in X$, we say that *x is F-simple (for X)*, if x is adjacent to exactly two components of $\overline{X}$. Let $S \subseteq X$. We say that *S is F-simple (for X)* if S is adjacent to exactly two components $A, B \in \mathcal{C}(\overline{X})$ such that $A \cup B \cup S$ is connected.

Let us look at Fig. 1b. The set X made of the black vertices separates its complementary set into four components. The points a and c are F-simple for X whereas b and d are not. The set $S = \{a, c\}$ is F-simple for X and $\{b, d\}$ is not. If we remove from X an F-simple set, S for instance, we obtain a set which separates its complementary into three components: we "merged two components of $\overline{X}$ through S". This operation may be seen as an elementary merging in the sense that only two components were merged.

Let $X \subset E$ and let A and B be two elements of $\mathcal{C}(\overline{X})$ with $A \neq B$. We say that *A and B can be merged (for X) through S* if S is F-simple and A and B are precisely the two components of $\overline{X}$ adjacent to S. We say that A *can be merged* (for X) if there exists $B \in \mathcal{C}(\overline{X})$ and $S \subseteq X$ such that A and B can be merged through S.

We say that (E, Γ) is a *fusion graph* if for any subset of vertices $X \subseteq E$ such that $|\mathcal{C}(\overline{X})| \geq 2$, any component of $\overline{X}$ can be merged.

Notice that all graphs are not fusion graphs. For instance, the graphs induced by the 4-adjacency relation depicted on Fig. 3a is not a fusion graph. On the other hand, the graph induced by the 8-adjacency depicted on Fig. 3c is an example of fusion graph.

The most striking theorem (33) in [7] states that the class of fusion graphs is precisely the class of graphs in which any watershed set is thin.

We set $\Gamma^\star(A, B) = \Gamma^\star(A) \cap \Gamma^\star(B)$ and if $\Gamma^\star(A, B) \neq \emptyset$, we say that *$A$ and B are neighbors.*

Definition 2. *We say that (E, Γ) is a perfect fusion graph if, for any $X \subseteq E$, any neighbors A and B in $\mathcal{C}(\overline{X})$ can be merged through $\Gamma^\star(A, B)$.*

In other words, the perfect fusion graphs are the graphs in which merging two neighboring regions can always be performed by removing from the frontier set all the points which are adjacent to both regions. This class of graphs allows, in particular, to rigorously define hierarchical schemes based on region merging and to implement them in a straightforward manner. It has been shown [7] that any perfect fusion graph is a fusion graph and that the converse is not true. For instance, the graphs induced by the 8-adjacency relation are not, in general, perfect fusion graphs (see counter-examples in Appendix A) whereas they are fusion graphs. In [7] the authors introduce a family of adjacency relations on $\mathbb{Z}^n$

that can be used in image processing and that induce perfect fusion graphs. See Appendix B for an illustration of these relations on $\mathbb{Z}^2$ and $\mathbb{Z}^3$.

The two following necessary and sufficient conditions for perfect fusion graphs show show the deep relation existing between perfect fusion graphs and thin watershed set.

Theorem 1 (from 41 in [7]). *The three following statements are equivalent:*

i) (E, Γ) *is a perfect fusion graph;*
ii) for any $x \in E$*, any* $X \subseteq \Gamma(x)$ *contains at most two connected components;*
iii) for any watershed Y *in* E *such that* $\mathcal{C}(\overline{Y}) \geq 2$*, each point* x *in* Y *is F-simple.*

To conclude this section, we recall the definition of line graphs. This class of graphs allows to make a strong link between the framework developed in this paper and the approaches of watershed and region merging based on edges rather than vertices.

Let (E, Γ) be a graph. The *line graph* of (E, Γ) is the graph (E', Γ') such that $E' = \Gamma$ and (u, v) belongs to Γ' whenever $u \in \Gamma$, $v \in \Gamma$ and u, v share a common vertex of E.

We say that the graph (E', Γ') is a line graph if there exists a graph (E, Γ) such that (E', Γ') is isomorphic to the line graph of (E, Γ).

It has been proved [7] that any line graph is a perfect fusion graph and that the converse is not true. We point out that the definitions, properties and algorithm for watershed on perfect fusion graph developed in Section 3 also holds for watershed approaches based on edges rather than vertices.

2.3 W-Thinnings and Topological Grayscale Watersheds

We now recall the notions of W-thinning and topological grayscale watershed which have been introduced and studied in [5, 8, 9, 10].

Let $F \in \mathcal{F}(E)$. We denote by $[F \setminus x]$ the map in $\mathcal{F}(E)$ such that $[F \setminus x](x) = F(x) - 1$, and $[F \setminus x](y) = F(y)$ for any $y \in E$, $y \neq x$.

Definition 3. *Let* $x \in E$*. Let* $F \in \mathcal{F}(E)$ *and let* $k = F(p)$*. We say that* p *is* W-destructible for F *if* p *is W-simple for* $F[k]$*.*
If there is no W-destructible point for F *we say that* F *is a* (topological) watershed*.*
Let $G \in \mathcal{F}(E)$*.*
We say that G *is a* W-thinning of F *if* $G = F$ *or if there exists a W-thinning* $H \in \mathcal{F}(E)$ *of* F *and a point* $x \in E$*, which is W-destructible for* H*, such that* $G = [H \setminus x]$*.*
If G *is both a W-thinning of* F *and a watershed we say that* G *is a* (topological) watershed of F*.*

In Fig. 1c and d, assume that the graph is the one corresponding to the 8-adjacency relation. In both Fig. 1c and d, it may be seen that there are three minima which are the components with levels 0,1 and 2. In Fig. 1c, the point labeled r is W-destructible. In Fig. 1d, no point is W-destructible. The function depicted in Fig. 1d is a watershed of the function in Fig. 1c. Observe that in Fig. 1d the minima of Fig. 1c have been extended as much as possible while preserving the number of components of all the lower sections of Fig. 1c. The divide of a topological watershed constitutes an interesting segmentation [8, 10] which possesses important properties not guaranteed by most watershed algorithms [2, 3]. In particular, it preserves the connection value between the minima of the original function; the connection value (see [8, 10, 14, 15]) between two minima is the minimal altitude at which one need to climb in order to reach one minimum from the other. This contrast preservation property is a requirement for region merging method based on watershed [11, 12].

3 C-Watersheds: Definitions, Properties and Algorithm

In [7], we have shown that any subset of $\mathbb{Z}^2$ equipped with the 8-adjacency forms a fusion graph but not, in general, a perfect fusion graph. In particular, the graphs considered in Fig. 1c and d are fusion graphs but not perfect fusion graphs. Let us consider the function F depicted in Fig. 1d. We have seen that F is a topological watershed. If we examine the divide of F, it may be seen that the point labeled s is inner (in the binary sense) for the divide. Thus, on fusion graphs, there exist topological watersheds whose divides are not thin.

On the same figure, remark also that the point labeled t is W-simple for $\overline{M}(F)$, thus $\overline{M}(F)$ is not a binary watershed set. Thus, on fusion graphs, there exist topological grayscale watersheds whose divides are not binary watershed sets.

In the remaining of this paper, we study W-thinnings and topological grayscale watersheds on perfect fusion graphs and we show, among other properties, that the divide of any topological watershed is necessarily a thin watershed set.

Let us first define a type of points that we call M-cliff. Given a graph and a function, these points are the lowest points adjacent to a single minimum. We will show that if the graph is a perfect fusion graph, any M-cliff point is W-destructible (Th. 2).

Definition 4. *Let $F \in \mathcal{F}(E)$ and let $x \in E$. We say that x* is a cliff point (for F) *if $x \in \overline{M}(F)$ and if it is adjacent to a single minimum of F. We say that x* is M-cliff (for F) *if x is a cliff point with minimal altitude (*i.e., $F(x) = \min\{F(y) \mid y \in E$ *is a cliff point for* $F\}$*).*

Let us look at Fig. 2. Thanks to Th. 1.ii, it may be seen that the depicted graphs are perfect fusion graphs. In Fig. 2a, the points with level 3 are cliff points and the bold circled point is the only M-cliff point. In figure 2b and c, it can be seen that there is no M-cliff point and no cliff point.

Let $F \in \mathcal{F}(E)$ and $j \in \mathbb{K}$. The point x *is W-destructible with lowest value j (for F)* if for any $h \in \mathbb{K}$ such that $j < h \leq F(x)$, x is W-simple for $F[h]$ and if x is not W-simple for $F[j]$.

Let $h \in \mathbb{K}$ such that $h < F(x)$, we denote by $[F \setminus x \downarrow h]$ the function of $\mathcal{F}(E)$ such that $[F \setminus x \downarrow h](x) = h$ and $[F \setminus x \downarrow h](y) = F(y)$ for all $y \in E \setminus \{x\}$.

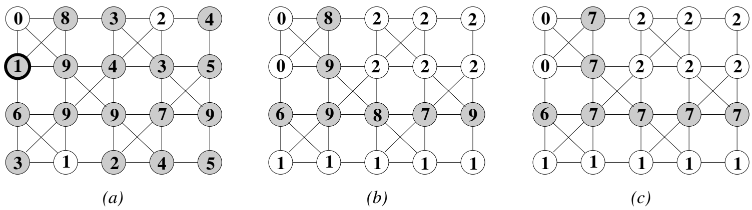


Fig. 2. Example of function on perfect fusion graphs, the minima are in white; (a): the bold circled vertex is M-cliff; (b): a C-watershed of (a); (c): a topological watershed of both (a) and (b)

Theorem 2. *Let $F \in \mathcal{F}(E)$. Let $x \in E$ be M-cliff for F and let $l \in \mathbb{K}$ be the level of the only minimum adjacent to x. If (E, Γ) is a perfect fusion graph then x is W-destructible with lowest value l for F.*

Remark that on non-perfect fusion graphs, the points which are M-cliff are not necessarily W-destructible. For example, the point labeled t in Fig. 1d is M-cliff whereas it is not W-destructible.

Definition 5. *Let $F \in \mathcal{F}(E)$, we say that $G \in \mathcal{F}(E)$* is a C-thinning of F *if*

i) $G = F$, or if
ii) there exists a function H which is a C-thinning of F and there exists a point x M-cliff for H, with lowest value k such that $G = [H \setminus x \downarrow k]$.

We say that F is a C-watershed *if there is no M-cliff point for F. If G is both a C-thinning of F and a C-watershed we say that G* is a C-watershed of F.

The following property follows immediately from definition 5 and Th. 2.

Property 3. *Let (E, Γ) be a perfect fusion graph and let $F \in \mathcal{F}(E)$. If F is a topological watershed then F is a C-watershed. If G is a C-thinning of F then G is a W-thinning of F.*

The converses of the two propositions in Prop. 3 are not true. The function of Fig. 2b is a C-watershed of Fig. 2a but is not a topological watershed. Indeed, the points at altitude 9 are W-destructible. The function depicted in Fig. 2c is a W-thinning of Fig. 2a but not a C-thinning of 2a. Indeed some points at level 9 have been lowered down to 7, and 7 is not the altitude of any minimum.

Observe that, on perfect fusion graphs, since any C-thinning is a W-thinning, from the contrast preservation theorem presented in the introduction, we can immediately deduce that C-thinnings, and hence C-watershed, preserves the connection value between the minima of the original map.

It can be easily seen, that in a C-thinning sequence the points which are in a minimum at a given step become neither M-cliff, nor W-destructible further in the sequence. This observation leads us to the definition of Algorithm 1, a very simple algorithm for computing C-watersheds.

At each iteration of the main loop (line 6) of Algorithm 1, F is a C-thinning and a W-thinning of the input function.

Algorithm 1. C-watershed

Data: a perfect fusion graph (E, Γ), a function $F \in \mathcal{F}(E)$
Result: F

$L := \emptyset$; $K := \emptyset$;
Attribute distinct labels to all minima of F and label the points of $M(F)$ with the corresponding labels;
foreach $x \in E$ **do**
  if $x \in M(F)$ **then** $K := K \cup \{x\}$;
  else if *x is adjacent to $M(F)$* **then** $L := L \cup \{x\}$; $K := K \cup \{x\}$;
while $L \neq \emptyset$ **do**
  $x :=$ an element with minimal altitude for F in L;
  $L := L \setminus \{x\}$;
  if *x is adjacent to exactly one minimum of F* **then**
    Set $F[x]$ to the altitude of the only minimum of F adjacent to x;
    Label x with the corresponding label;
    foreach $y \in \Gamma^{\star}(x) \cap \overline{K}$ **do** $L := L \cup \{y\}$; $K := K \cup \{y\}$;

At the end of Algorithm 1, F is a C-watershed of the input function.

In Algorithm 1, the operations performed on the set L are the insertion of an element and the extraction of an element with minimal altitude. Thus L may be managed as a priority queue.

Lemma 4. *Let $F \in \mathcal{F}(E)$. Let $x \in E$ be M-cliff for F and let $k = F(x)$. If (E, Γ) is a perfect fusion graph, any $y \in E$ which is inner for $\overline{M}(F)$ is such that $F(y) \geq k$.*

On non-perfect fusion graphs, the previous lemma is in general not true.

From Lem. 4, we deduce that in Algorithm 1, when the function F is lowered at a point x with altitude k, any point inserted further in the set L has a level greater than or equal to k. Thus the set L may be managed by a monotone priority queue. Recently, M. Thorup [16] proved that if we can sort n-keys in time $n.s(n)$ then and only then there is a monotone queue with capacity n, supporting the *insert* and *extract-min* operations in $s(n)$ amortized time.

Property 5. *If the elements of E can be sorted according to F in $o(|E|)$, then Algorithm 1 terminates in linear time with respect to $(|E| + |\Gamma|)$.*

Since Algorithm 1 possesses the monotone property discussed above, it can be classified in the group of immersion algorithms (see [2, 3, 10] for examples). Moreover, it is the first immersion algorithm proved to compute W-thinnings in linear time with respect to the size of the graph.

Notice that computing a topological grayscale watershed from a C-watershed is not straightforward. For more details we refer to [9].

Let us now state some properties of C-watersheds on perfect fusion graphs. Let $G \in \mathcal{F}(E)$ and let $k = G(x)$. If x is F-simple for $G[k]$, we say that x is *F-simple for G*.

Theorem 6 (Grayscale characterizations of perfect fusion graphs). *The three following statements are equivalent:*

i) (E, Γ) *is a perfect fusion graph;*
ii) for any C-watershed $G \in \mathcal{F}(E)$*, any point of* $\overline{M}(G)$ *is F-simple for* $\overline{M}(G)$*;*
iii) for any topological grayscale watershed $G \in \mathcal{F}(E)$*, any point in* $\overline{M}(G)$ *is F-simple for* G*.*

Thanks to Th. 6, we immediately deduce the following theorem.

Theorem 7. *Let* (E, Γ) *be a perfect fusion graph and let* $F \in \mathcal{F}(E)$*. If* F *is a C-watershed then* $\overline{M}(F)$ *is a watershed.*

In other words, on a perfect fusion graph, the minima of a C-watershed cannot be further extended.

In this section, we have seen that:

1) on perfect fusion graphs, the C-watersheds preserves the connection value between the minima of the original map; and
2) in this framework, the divide of any C-watersheds is a thin binary watershed set.

Since perfect fusion graphs allow to rigorously define region merging procedure, the divide of C-watersheds on perfect fusion graphs is an ideal entry point for hierarchical methods based on watersheds.

On perfect fusion graph, any topological watershed is a C-watershed (Prop. 3), thus we may easily deduce from Th. 6 and 7 that:

i) a graph is a perfect fusion graph if and only if, for any topological watershed F, any point of the divide of F is adjacent to exactly two minima of F; and
ii) on a perfect fusion graph, the divide of any topological grayscale watershed is a binary watershed set.

4 Perspectives: Perfect Fusion Grids and Hierarchical Schemes

Following some properties given in [7] and the examples depicted in this paper (see Fig. 1cd, Fig. 6), it may be seen that there exist topological watersheds whose divides are not thin in 2D on the 4-, 6- and 8-connected grids, and in 3D on the 6- and 26-connected grids. In this paper, we have shown that, on perfect fusion graphs, the divide of any topological grayscale watershed is a thin binary watershed set. On these graphs, region merging schemes are easy to rigorously define and straightforward to implement. Thus, the framework of perfect fusion graph is adapted for region merging methods based on topological watersheds.

In [7], we introduced the family of perfect fusion grids over $\mathbb{Z}^n$, for any $n \in \mathbb{N}$. Any element of this family is indeed a perfect fusion graph. We proved that any of these grids is “between” the direct adjacency graph (which generalizes the 4-adjacency to $\mathbb{Z}^n$) and the indirect adjacency graph (which generalizes the

8-adjacency to $\mathbb{Z}^n$). These n-dimensional grids are all equivalent (up to a translation) and, in a forthcoming paper, we intend to prove that they are the only graphs that possess these two properties. Examples of (restrictions of) 2 and 3-dimensional perfect fusion grids are presented in Appendix B.

Perfect fusion grids constitute an interesting alternative for region merging methods based on watersheds. Future work will include revisiting hierarchical segmentation methods [11, 12] on perfect fusion grids.

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Appendix A: Counter-Examples of Merging Properties in Usual Grids

Let us first consider the 4-connected graph depicted in Fig. 3a. Since none of the components of the complementary of the black vertices can be merged, the depicted graph is not a fusion graph. As an illustration of the fusion graphs

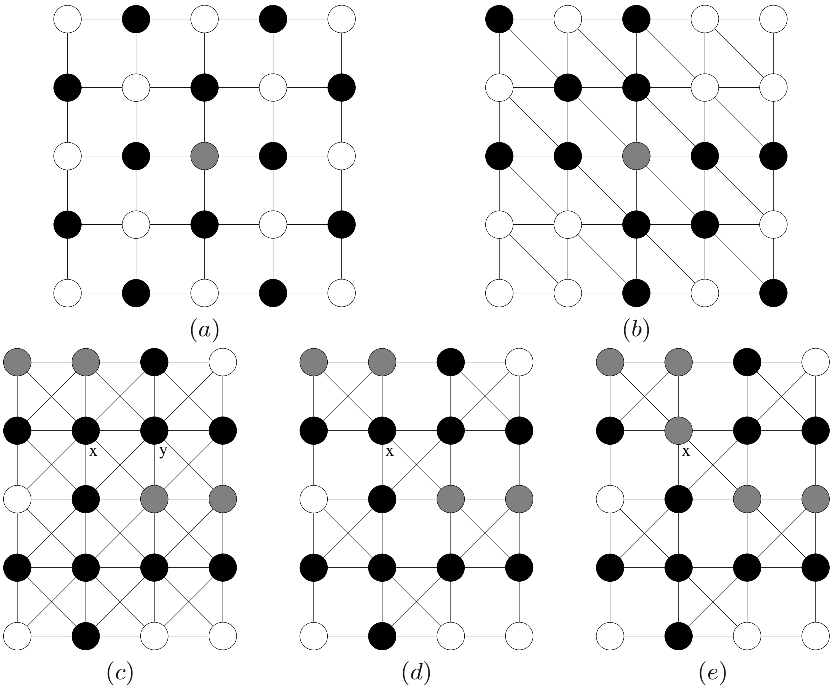

Fig. 3. Counter example of merging properties in usual grids $(a - c)$, and illustration of merging properties in a perfect fusion grid (d, e)

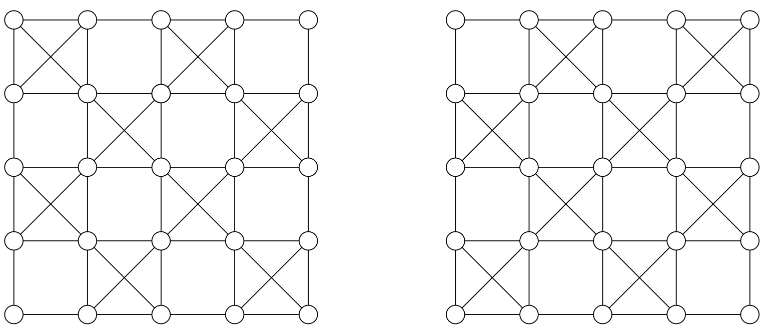

Fig. 4. Samples of the two perfect fusion grids on $\mathbb{Z}^2$

fundamental theorem (33 in [7]) recalled in Section 2, we can observe that the set of black and gray points is a non-thin binary watershed.

The graph of Fig. 3b, which is a 2D 6-connected graph, is not a fusion graph since the gray point which is a component of the complementary of the black vertices cannot be merged.

Since the graph, depicted on Fig. 3c, induced by the 8-adjacency relation is a fusion graph, any binary watershed on this graph is thin. This property can be verified, in particular, for the watershed made of the black points on Fig. 3c. Observe on the same figure that the two neighboring gray components cannot be merged through $\{x, y\}$ their common neighborhood. The black vertices is thus a counter-example of the perfect fusion property for the depicted graph.

In Fig. 3d, the same sets of black and gray points are considered on a perfect fusion grid. Observe that the two gray components can now be merged through their common neighborhood $\{x\}$. Remark also that the set obtained by removing $\{x\}$ from the black points (Fig. 3e) is still a watershed. This desirable property,

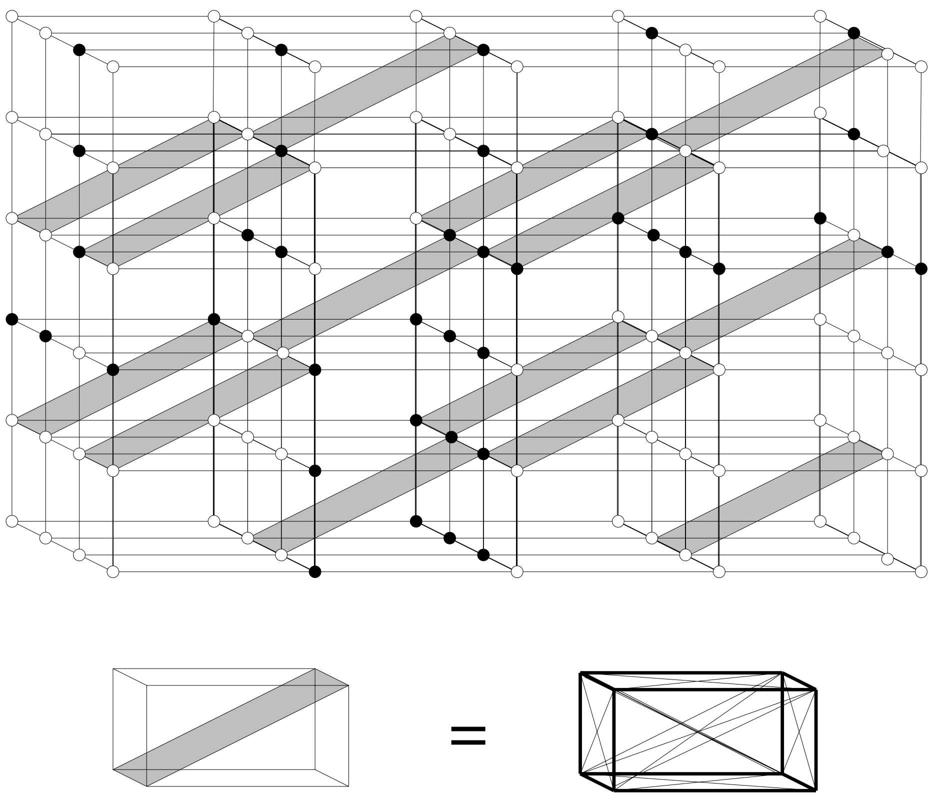

Fig. 5. A 3-dimensional perfect fusion grid. Black points constitute a set which is a watershed.

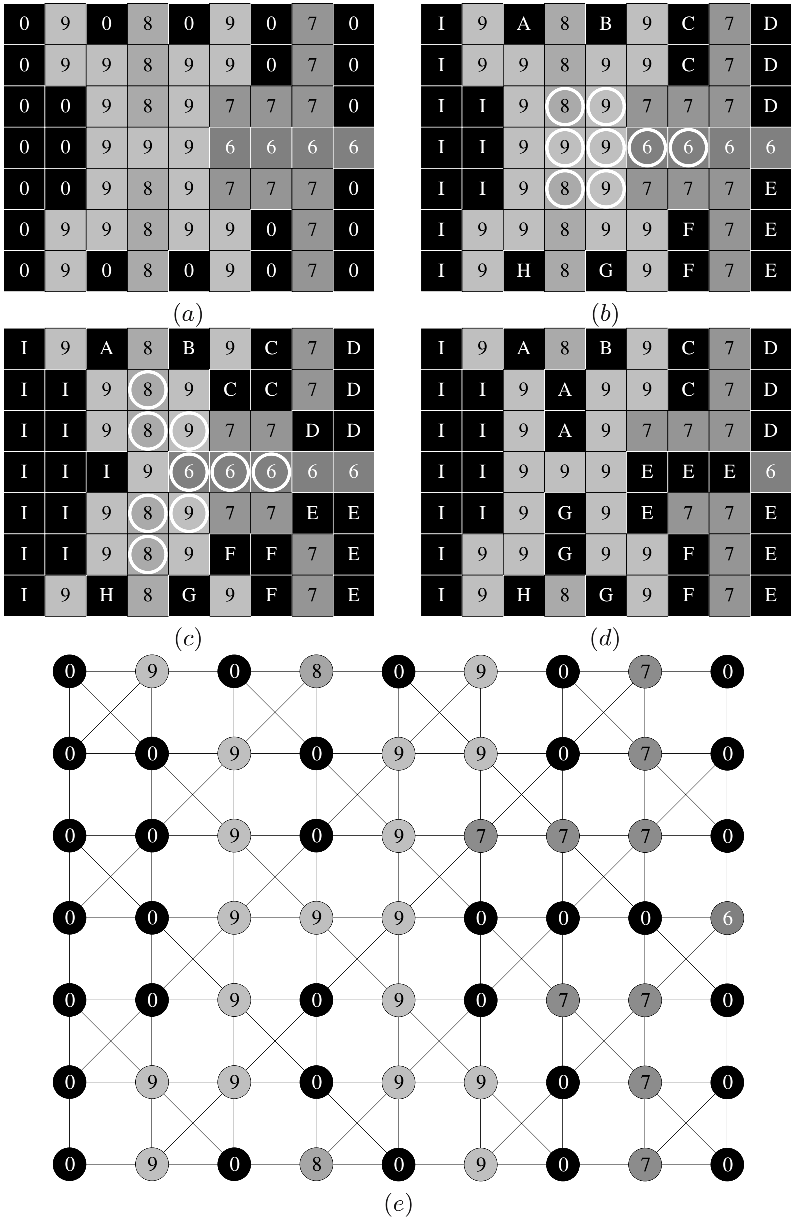

Fig. 6. Comparison of topological watershed using different grids. The minima, labeled by letters, are supposed to be at altitude 0; the circled points are inner for the divide of the depicted function with respect to the assumed adjacency; (a), an image; (b), a topological watershed of (a) when the 8-adjacency graph is assumed; (c), a topological watershed of (a) when the 4-adjacency graph is assumed; (d), a topological watershed of (a) when one of the perfect fusion grids is assumed; (e) same as (d) showing the assumed adjacency relation.

which does not hold in the general case, can be easily proved on perfect fusion graphs.

We finish this appendix section, with a table that sums up the status of the different graphs used in 2D and 3D image processing. See [7] for more details. For non trivial images, we have:

	fusion graph	perfect fusion graph
$2D$: 4-connected graph	is not a	is not a
$3D$: 6-connected graph	is not a	is not a
$2D$: 8-connected graph	is a	is not a
$3D$: 26-connected graph	is not a	is not a
$2D$: 6-connected graph	is not a	is not a

Appendix B: Perfect Fusion Grids: 2D and 3D Cases

A formal definition of perfect fusion grids can be found in [7]. In $\mathbb{Z}^2$, there are two distinct perfect fusion grids, in $\mathbb{Z}^3$ there are four. Actually it has been proved that, for any strictly positive integer n, there are exactly 2^{n-1} perfect fusion grids over $\mathbb{Z}^n$ which are all equivalent (up to a unit translation). Samples of the two perfect fusion grids on $\mathbb{Z}^2$ are depicted in Fig. 4. Fig. 5 shows a binary watershed (black points) on one of the 3D perfect fusion grids. To clarify the figure, we use the following convention: any two points belonging to a same cube marked by a gray stripe are adjacent to each other.