Linear Convergence of Tatônnement in a Bertrand Oligopoly*

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Abstract. We show the linear convergence of the tatônnement scheme in a Bertrand oligopoly price competition game using a possibly asymmetric attraction demand model with convex costs. To demonstrate this, we also show the existence of the equilibrium.

1 Introduction and Model

In the Bertrand oligopoly price competition model for differentiated products, a variety of demand models and cost models have been used. The choice of these models affects the profit of each firm.

We let n be the number of firms, which are indexed by $i = 1, \ldots, n$. The demand for each firm is specified as a function of prices. Let p_i denote the price of firm i, and define the price vector of competing firms by $\mathbf{p}_{-i}$, which is a shorthand for $(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$. Also denote the vector of all prices by $\mathbf{p} = (p_1, \ldots, p_n) = (p_i, \mathbf{p}_{-i})$. The demand for each firm i is given by $d_i = d_i(\mathbf{p})$. We assume that firm i's demand is strictly decreasing in its price (i.e., $\partial d_i / \partial p_i < 0$), and that products are gross substitutes (i.e., $\partial d_i / \partial p_j \geq 0$ whenever $j \neq i$).

In this paper we consider a generalization of the logit demand model called the *attraction demand model*:

$$d_i(\mathbf{p}) \ := \ \frac{a_i(p_i)}{\sum_j a_j(p_j) + \kappa} \tag{1}$$

where κ is either 0 or strictly positive. As discussed in [2] and [14], the attraction model has successfully been used in estimating demand in econometric studies, and is increasingly accepted in marketing, e.g., [5]. For its applications in the operations management community, see [21], [4] and the references therein. Without any loss of generality, we normalize the total demand of the market to 1. The *attraction function* $a_i(\cdot)$ of firm i is a positive and strictly decreasing function of its price.

For the cost model, we assume that cost is not a function of price, but of demand alone. We denote firm i's cost function by $C_i(d_i)$ defined on $[0, 1]$ and

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assume C_i is increasing and convex. The profit of firm i is the difference between its revenue and cost, given by

$$\pi_i \ := \ \pi_i(\mathbf{p}) \ := \ p_i \cdot d_i(\mathbf{p}) - C_i(d_i(\mathbf{p})). \tag{2}$$

Each firm's objective is to maximize π_i. We impose technical conditions on the attraction demand and cost models as outlined in Section 2.

The study of oligopolistic interaction is a classical problem in economics. In the model proposed by Cournot, firms compete on production output quantities, which in turn determine the market price. In Bertrand's model, however, competition is based on prices instead of production quantities. In the price competition models by Edgeworth, each firm decides how much of its demand is satisfied, in which case an equilibrium solution may or may not exist. Price competition with product differentiation has also been studied by [12], [20] and [8]. An extensive treatment of the subject is found in [24]. We provide a brief summary of results regarding the existence, and stability of equilibrium, followed by their application in the operations management literature.

Existence. There are two common methods to show existence of an equilibrium in price competition games. The first method is to obtain existence through the quasi-concavity of π_i in p_i. See [7]. The second method shows existence through *supermodular* games. See [22] for the existence of an equilibrium in supermodular games, and [16], for monotone transformation of supermodular games. Thus, if the price competition game is supermodular, it has at least one equilibrium. Similarly, [17] shows the existence of a Nash equilibrium for a generalization of supermodular games, called games with strategic complementarities. Such games include instances of price competition. These however are not applicable to our model.

Stability. By definition, a set of actions at equilibrium is a fixed point of the best response mapping. A simultaneous discrete *tatônnement* is a sequence of actions in which the current action of each firm is the best response to the previous actions of other firms. An equilibrium is *globally stable* if the tatônnement converges to this equilibrium regardless of the initial set of actions. [23] shows that if a supermodular game with continuous payoffs has a unique equilibrium, it is globally stable. To our knowledge, there is no known result regarding the provable convergence rate of the tatônnement in the price competition game.

Operations Management Literature. There has been a growing interest in oligopolistic price competition in the operations literature. To predict and study market outcomes, the existence and the uniqueness of equilibrium are often required. Stability and convergence rate indicate both the robustness of equilibrium and the efficiency of computational algorithms. For example, see [4], [3], [1], and [6].

2 Assumptions

This section lists our assumptions on the attraction function $a_i(\cdot)$ in (1), and the profit function π_i and the cost function $C_i(\cdot)$ in (2).

We let $\rho_i := \inf\{p : a_i(p) = 0\}$ be the upper bound on price p_i. Firm i's action space for price is an open interval $(0, \rho_i)$. Let $\mathcal{P} := (0, \rho_1) \times \cdots \times (0, \rho_n)$. Let $\eta_i(p) := -p \cdot a_i'(p)/a_i(p)$ be the *elasticity* of firm i's attraction function. We adopt the following simplifying notation: $f(x+) := \lim_{h \downarrow x} f(h)$, $f(x-) := \lim_{h \uparrow x} f(h)$, $\inf \emptyset = \infty$, and $\frac{y}{y+k} = 1$ if $y = \infty$ and k is finite.

Condition A: (A1) $a_i(\cdot)$ is positive, strictly decreasing and continuously differentiable, i.e., $a_i(p) > 0$ and $a_i'(p) < 0$ for all $p \in (0, \rho_i)$.

(A2) The elasticity $\eta_i(\cdot)$ is nondecreasing.

(A3) If $a_i(0+) < \infty$, then $a_i'(0+) > -\infty$.

Condition B: (B1) $C_i(\cdot)$ is strictly increasing, continuously differentiable, and convex on $[0, 1]$, i.e., $c_i(\cdot) := C_i'(\cdot)$ is positive and increasing, and satisfies $c_i(0+) > 0$.

Condition C: (C1) $c_i(0) < \rho_i \cdot (1 - 1/\eta_i(\rho_i))$, i.e., the Lerner index $[p_i - c_i(d_i)]/p_i$ at price $p_i = \rho_i$ and demand $d_i = 0$ is strictly bounded below by $1/\eta_i(\rho_i)$.

(C2) If $\kappa = 0$, then $c_i(1) < \rho_i$.

(C3) If $\kappa = 0$, the following technical condition holds:

$$\sum_{i=1}^{n} \left(1 - \frac{1}{\eta_i(\rho_i) \cdot (1 - c_i(1)/\rho_i)}\right) > 1 \ .$$

Examples of $C_i(\cdot)$ include the linear function and exponential function. More examples are provided in Section 3. We remark that attraction functions do not need to be identical. Furthermore, even the form of the attraction function may not be same among firms. Analogously, the cost functions need not have the same form.

For the rest of this paper, we assume Conditions A, B and C hold. In Section 5, we introduce an additional assumption that both $C_i(\cdot)$ and $a_i(\cdot)$ are twice continuously differentiable.

3 Examples

In this section, we list price competition models in which the convex cost model is applicable. We present some examples from an inventory-capacity systems and a service system based on queues.

Inventory-Capacity System: In the first example, consider the pricing problem in the stochastic inventory system with exogenously determined stocking levels. We denote stochastic demand of firm i by $D_i(\mathbf{p})$, and its expected demand by $d_i(\mathbf{p})$. Demand is a function of the price vector $\mathbf{p} = (p_1, \ldots, p_n)$. We represent firm i's stochastic demand by $D_i(p_1, \ldots, p_n) = \varphi(d_i(\mathbf{p}), \varepsilon_i)$, where ε_i is a random variable. (We can allow φ to be dependent on i). We suppose the continuous density function $f_i(\cdot)$ for ε_i exists, and let $F_i(\cdot)$ denote its cumulative density function.

Let y_i be the exogenously fixed stocking level of firm i. For the first y_i units, the per-unit materials cost is w_i. If realized demand is at most y_i, the per-unit salvage value of $w_i - h_i > 0$ is obtained. Otherwise, the excess demand is met through an emergency supply at the cost of $w_i + b_i$ per unit, where $b_i \geq 0$. The profit of firm i is the difference between its revenue and costs, and the expected profit is $\pi_i(\mathbf{p}|y_i) = p_i \cdot d_i(\mathbf{p}) - C_i(d_i(\mathbf{p}), y_i)$, where

$$C_i(d_i, y_i) = w_i d_i + h_i E[y_i - \varphi(d_i, \varepsilon_i)]^+ + b_i E[\varphi(d_i, \varepsilon_i) - y_i]^+ \ ,$$

and h_i and b_i are the per-unit inventory overage and underage costs, respectively.

Our goal is to show that for fixed y_i, this function satisfies condition (B1). We achieve this goal with two common demand uncertainty models.

- Additive Demand Uncertainty Model: $\varphi(d_i, \varepsilon_i) = d_i + \varepsilon_i$ where $E[\varepsilon_i] = 0$. Then,

$$\begin{aligned} \frac{\partial C_i(d_i, y_i)}{\partial d_i} &= w_i - h_i P[y_i \geq d_i + \varepsilon_i] + b_i P[y_i \leq d_i + \varepsilon_i] \\ &= w_i - h_i F_i(y_i - d_i) + b_i(1 - F_i(y_i - d_i)). \end{aligned}$$

- Multiplicative Demand Uncertainty Model: $\varphi(d_i, \varepsilon_i) = d_i \cdot \varepsilon_i$ where ε_i is positive and $E[\varepsilon_i] = 1$. Then,

$$\begin{aligned} \frac{\partial C_i(d_i, y_i)}{\partial d_i} &= w_i - h_i \int_0^{y_i/d_i} \varepsilon dF_i(\varepsilon) + b_i \int_{y_i/d_i}^{\infty} \varepsilon dF_i(\varepsilon) \\ &= w_i - h_i + (h_i + b_i) \int_{y_i/d_i}^{\infty} \varepsilon dF_i(\varepsilon). \end{aligned}$$

In both cases, $\partial C_i(d_i, y_i)/\partial d_i$ is positive since $w_i > h_i$ and nondecreasing in d_i. We conclude that for fixed y_i, $C_i(d_i, y_i)$ is strictly increasing, twice continuously differentiable, and convex in d_i. Furthermore, $\partial C_i(d_i, y_i)/\partial d_i > 0$ at $d_i = 0$.

Service System: In the second example, we model each firm as a single server queue with finite buffer, where the firms' buffer sizes are given exogenously. Let κ_i denote the size of firm i's buffer; no more than κ_i customers are allowed to the system. We assume exponential service times and the Poisson arrival process. The rate μ_i of service times are exogenously determined, and the rate d_i of Poisson arrival is an output of the price competition. In the queueing theory notation, each firm i is a $M/M/1/\kappa_i$ system.

We assume that the materials cost is $w_i > 0$ per served customer, and the diverted customers' demand due to buffer overflow is met by an emergence supply at the cost of $w_i + b_i$ unit per customer, where $b_i > 0$. The demand arrival rate $d_i = d_i(\mathbf{p})$ is determined as a function of the price vector $\mathbf{p}$. It follows that firm i's time-average revenue is $p_i \cdot d_i - C_i(d_i)$, where $C_i(d_i)$ is the sum of $w_i \cdot d_i$ and the time-average number of customers diverted from the system is multiplied by b_i. Thus, according to elementary queueing theory (see, for example, [15]),

$$\begin{aligned} C_i(d_i) &= w_i \cdot d_i + b_i \cdot \frac{d_i \cdot (1 - d_i/\mu_i)(d_i/\mu_i)^{\kappa_i}}{1 - (d_i/\mu_i)^{\kappa_i + 1}}, \qquad \text{if } d_i \neq \mu_i \\ &= w_i \cdot d_i + b_i \cdot \frac{d_i}{\kappa_i + 1}, \qquad \text{if } d_i = \mu_i. \end{aligned}$$

Algebraic manipulation shows that $C_i(\cdot)$ is convex and continuously twice differentiable, satisfying $c_i(0) = w_i > 0$.

4 Existence of Equilibrium

In this section, we show that the oligopoly price competition has a unique equilibrium. Given the price vector, each firm's profit function is given by expression (2), where its demand is determined by (1). We first show that the first order condition $\partial\pi_i/\partial p_i = 0$ is sufficient for the Nash equilibrium (Proposition 1). For each value of a suitably defined aggregate attraction δ, we show that there is at most one candidate for the solution of the first order condition (Proposition 2). Then, we demonstrate that there exists a unique value δ of the aggregator such that this candidate indeed solves the first order condition (Propositions 3 and 4). We proceed by assuming both Conditions A, B and C. Let $\varsigma_i(\mathbf{p}) := \eta_i(p_i)\cdot(1-d_i(\mathbf{p}))$.

Proposition 1. *Firm i's profit function π_i is strictly quasi-concave in $p_i \in (0, \rho_i)$. If $\mathbf{p}^* = (p_1^*, \ldots, p_n^*) \in \mathcal{P}$ satisfies $\partial\pi_i(\mathbf{p}^*)/\partial p_i = 0$ for all i, $\mathbf{p}^*$ is the Nash equilibrium in $\mathcal{P}$, and $p_i^* > c_i(0)$ for each i. Furthermore, the condition $\partial\pi_i/\partial p_i = 0$ is equivalent to*

$$c_i(d_i(\mathbf{p}))/p_i \;=\; 1 - 1/\varsigma_i(\mathbf{p}) \; . \tag{3}$$

Given a price vector, let $\delta := \sum_{j=1}^{n} a_j(p_j)$ be the aggregate attraction. The support of δ is $\Delta := \Big(0,\ \sum_{j=1}^{n} a_j(0+)\Big)$. From (A1), it follows that $\delta \in \Delta$. Then, $d_i = a_i(p_i)/(\delta+\kappa)$. Since a_i^{-1} is well-defined by (A1), we get $p_i = {a_i}^{-1}((\delta+\kappa)d_i)$. Thus, (3) is equivalent to

$$\frac{c_i(d_i)}{{a_i}^{-1}((\delta+\kappa)d_i)} = 1 - \frac{1}{\eta_i \circ {a_i}^{-1}((\delta+\kappa)d_i)\cdot(1-d_i)}. \tag{4}$$

Observe that there is one-to-one correspondence between $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{d} = (d_1, \ldots, d_n)$, given δ (and of course, κ). Let $D_i(\delta)$ be the solution to (4) given δ (and κ). The existence and uniqueness of $D_i(\delta)$ are guaranteed by Proposition 2 below. The $D_i(\delta)$'s may not sum up to the "correct" value of $\delta/(\delta+\kappa)$ unless a set of conditions is satisfied (Propositions 3). Proposition 4 shows the existence of a unique δ such that the $D_i(\delta)$'s sum up to $\delta/(\delta+\kappa)$.

Let $\overline{d}_i(\delta) := \min\left\{\frac{a_i(0+)}{\delta+\kappa}, 1\right\}$ be an upper bound on the market share of firm i. For each fixed $\delta \in \Delta$, we define the following real-valued functions on $\big(0, \overline{d}_i(\delta)\big)$:

$$L_i(x_i|\delta) := \frac{c_i(x_i)}{{a_i}^{-1}((\delta+\kappa)x_i)} \;\text{ and }\; R_i(x_i|\delta) := 1 - \frac{1}{\eta_i \circ {a_i}^{-1}((\delta+\kappa)x_i)\ (1-x_i)}\,.$$

We remark that both $L_i(x_i|\delta)$ and $R_i(x_i|\delta)$ are continuous in x_i in $\big(0, \overline{d}_i(\delta)\big)$.

Proposition 2. *For each i and each $\delta \in \Delta$, $L_i(\cdot|\delta)$ is positive and strictly increasing, and $R_i(\cdot|\delta)$ is strictly decreasing. Furthermore, $L_i(x_i|\delta) = R_i(x_i|\delta)$ has a unique solution in $\big(0, \overline{d}_i(\delta)\big)$, i.e., $D_i(\delta)$ is a well-defined function of δ.*

For any aggregate attraction $\delta \in \Delta$, Proposition 2 shows that there is a unique solution x_i satisfying $L_i(x_i|\delta) = R_i(x_i|\delta)$, and this solution is $D_i(\delta)$. It represents demand that maximizes firm i's profit provided that the aggregate attraction remains at δ. Also, define $D(\delta) := D_1(\delta) + \cdots + D_n(\delta)$. Note that $D_i(\delta)$ is a strictly decreasing function since any increase in δ lifts the graph of $L_i(x_i|\delta)$ and drops that of $R_i(x_i|\delta)$. Therefore $D(\delta)$ is also a strictly decreasing function. Furthermore, $D_i(\delta)$ is a continuous function of δ. Thus, $D(\delta)$ is continuous.

Proposition 3. *For fixed $\delta \in \Delta$, $D(\delta) = \frac{\delta}{\delta+\kappa}$ holds if and only if there exist $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{d} = (d_1, \ldots, d_n)$ such that the following set of conditions hold: (i) $\delta = \sum_{j=1}^{n} a_j(p_j)$, (ii) $d_i = a_i(p_i)/(\delta + \kappa)$ for each i, and (iii) $L_i(d_i|\delta) = R_i(d_i|\delta)$ for each i. In such case, furthermore, the price vector corresponding to any δ satisfying $D(\delta) = \frac{\delta}{\delta+\kappa}$ is unique.*

If there is $\delta \in \Delta$ satisfying $D(\delta) = \frac{\delta}{\delta+\kappa}$, then by Proposition 3, the corresponding price vector satisfies $\partial \pi_i / \partial p_i = 0$ for all i. By Proposition 1, this price vector is a Nash equilibrium. For the unique existence of the equilibrium, it suffices to show the result of the following proposition.

Proposition 4. *There exists a unique $\delta \in \Delta$ such that $D(\delta) = \delta/(\delta + \kappa)$.*

Theorem 1. *There exists a unique positive pure strategy Nash equilibrium price vector $\mathbf{p}^* \in \mathcal{P}$. Furthermore, $\mathbf{p}^*$ satisfies $p_i^* > c_i(0)$ for all $i = 1, \cdots, n$.*

5 Convergence of Tatônnement Scheme

In this section, we show that the unique equilibrium is globally stable under the tatônnement scheme. Suppose each firm i chooses a best-response pricing strategy: choose p_i maximizing his profit $\pi_i(p_1, \ldots, p_n)$ while p_j's are fixed for all $j \neq i$.

By Theorem 1, there exists a unique equilibrium vector, which is denoted by $\mathbf{p}^* = (p_1^*, \cdots, p_n^*) \in \mathcal{P}$. Define $\mathcal{Q} := (0, a_1(0+)) \times \cdots \times (0, a_n(0+))$. Let $\mathbf{q}^* = (q_1^*, \cdots, q_n^*) \in \mathcal{Q}$ be the corresponding attraction vector where $q_i^* := a_i(p_i^*)$. Let $\hat{q}_i := \sum_{j \neq i} q_j$ be the sum of attraction quantities of firms other than i. Set $\theta_i^* := q_i^*/(\hat{q}_i^* + \kappa)$ and $d_i^* := q_i^*/(q_i^* + \hat{q}_i^* + \kappa)$, which are both positive. Suppose we fix the price p_j for all $j \neq i$, and let $q_i := a_i(p_i)$ be the corresponding attraction. Since a_i is one-to-one and $\delta = q_i + \hat{q}_i$, condition (4) is equivalent to

$$\frac{c_i\left(\frac{q_i}{q_i+\hat{q}_i+\kappa}\right)}{{a_i}^{-1}(q_i)} = 1 - \frac{1}{\eta_i \circ {a_i}^{-1}(q_i)}\left(1 + \frac{q_i}{\hat{q}_i + \kappa}\right). \tag{5}$$

Using an argument similar to Proposition 2 and ensuing discussion, it can be shown that there is a unique solution q_i to (5) for each $\hat{q}_i$ given by any positive number less than $\sum_{j \neq i} a_i(0+)$. We call this solution q_i the *best response function* $\psi_i(\hat{q}_i)$ for firm i. The unique equilibrium satisfies $\psi_i(\hat{q}_i^*) = q_i^*$ where $\hat{q}_i^* = \sum_{j \neq i} q_j^*$. Furthermore, it is easy to show that $\psi_i(\cdot)$ is strictly increasing.

Proposition 5. *$\psi_i(\cdot)$ is a strictly increasing function.*

From the definition of θ_i^* and $\psi_i(\hat{q}_i^*) = q_i^*$, we know $\psi_i(\hat{q}_i)/(\hat{q}_i+\kappa) = \theta_i^*$ at $\hat{q}_i = \hat{q}_i^*$. The following proposition characterizes the relationship between $\psi_i(\hat{q}_i)/(\hat{q}_i+\kappa)$ and θ_i^*.

Proposition 6. *$\psi_i(\hat{q}_i)/(\hat{q}_i + \kappa)$ is strictly decreasing in $\hat{q}_i$, and satisfies $\psi_i(\hat{q}_i^*)/(\hat{q}_i^* + \kappa) = \theta_i^*$. Thus,*

$$\frac{\psi_i(\hat{q}_i)}{\hat{q}_i + \kappa} \begin{cases} > \ \theta_i^*, \textit{for } \hat{q}_i < \hat{q}_i^* \\ = \ \theta_i^*, \textit{for } \hat{q}_i = \hat{q}_i^* \\ < \ \theta_i^*, \textit{for } \hat{q}_i > \hat{q}_i^*. \end{cases}$$

Furthermore, $\psi_i'(\hat{q}_i)$ is continuous, and satisfies $0 < \psi_i'(\hat{q}_i^*) < \theta_i^*$.

Let $\mathbf{q} = (q_1, \ldots, q_n)$. We denote the vector of best response functions by $\Psi(\mathbf{q}) = (\psi_1(\hat{q}_1), \cdots, \psi_n(\hat{q}_n)) \in \mathcal{Q}$, where $\hat{q}_i = \sum_{j \neq i} q_j$. Note that $\mathbf{q}^* = (q_1^*, \cdots, q_n^*)$ is a fixed point of Ψ, i.e., $\Psi(\mathbf{q}^*) = \mathbf{q}^*$. By Proposition 5, we have $\Psi(\mathbf{q}^1) < \Psi(\mathbf{q}^2)$ whenever two vectors $\mathbf{q}^1$ and $\mathbf{q}^2$ satisfy $\mathbf{q}^1 < \mathbf{q}^2$. (The inequalities are component-wise.) We now show that best-response pricing converges to the unique equilibrium. We define the sequence $\{\mathbf{q}^{(0)}, \mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \cdots\} \subset \mathcal{Q}$ by $\mathbf{q}^{(k+1)} := \Psi(\mathbf{q}^{(k)})$ for $k \geq 0$.

Theorem 2. *If each firm employs the best response strategy based on the prices of other firms in the previous iteration, the sequence of price vectors converges to the unique equilibrium price vector.*

Proof. Let $\mathbf{q}^{(0)} \in \mathcal{Q}$ denote the attraction vector associated with the initial price vector. Choose $\underline{\mathbf{q}}^{(0)}, \overline{\mathbf{q}}^{(0)} \in \mathcal{Q}$ such that $\underline{\mathbf{q}}^{(0)} < \hat{b}^{(0)} < \overline{\mathbf{q}}^{(0)}$ and $\underline{\mathbf{q}}^{(0)} < \hat{b}^* < \overline{\mathbf{q}}^{(0)}$. Such $\underline{\mathbf{q}}^{(0)}$ and $\overline{\mathbf{q}}^{(0)}$ exist since $\mathcal{Q}$ is a box-shaped open set.

For each $k \geq 0$, we define $\underline{\mathbf{q}}^{(k+1)} := \Psi(\underline{\mathbf{q}}^{(k)})$ and $\overline{\mathbf{q}}^{(k+1)} := \Psi(\overline{\mathbf{q}}^{(k)})$. From the monotonicity of $\Psi(\cdot)$ (Proposition 5) and $\Psi(\hat{b}^*) = \hat{b}^*$, we get

$$\underline{\mathbf{q}}^{(k)} < \hat{b}^{(k)} < \overline{\mathbf{q}}^{(k)} \quad \text{and} \quad \underline{\mathbf{q}}^{(k)} < \hat{b}^* < \overline{\mathbf{q}}^{(k)}. \tag{6}$$

Let $u^{(k)} := \max_i \left\{ \hat{\overline{q}}_i^{(k)} \ / \ \hat{q}_i^* \right\}$. Clearly, $u^{(k)} > 1$ for all k by (6). We show that the sequence $\left\{ u^{(k)} \right\}_{k=0}^{\infty}$ is strictly decreasing. For each i,

$$\overline{q}_i^{(k+1)} = \psi_i(\hat{\overline{q}}_i^{(k)}) < \left(\hat{\overline{q}}_i^{(k)} + \kappa \right) \cdot \theta_i^* = \left(\hat{\overline{q}}_i^{(k)} + \kappa \right) \cdot \frac{q_i^*}{\hat{q}_i^* + \kappa} \leq \frac{\hat{\overline{q}}_i^{(k)}}{\hat{q}_i^*} \cdot q_i^* \leq u^{(k)} q_i^*,$$

where the first inequality comes from Proposition 6, the second one from (6), and the last one from the definition of $u^{(k)}$. Thus

$$\hat{\overline{q}}_i^{(k+1)} = \sum_{j \neq i} \overline{q}_j^{(k+1)} < \sum_{j \neq i} u^{(k)} q_j^* = u^{(k)} \hat{q}_i^* \ , \tag{7}$$

and $u^{(k+1)} = \max_i\{\hat{\overline{q}}_i^{(k+1)}/\hat{q}_i^*\} < u^{(k)}$. Since $\{u^{(k)}\}_{k=0}^{\infty}$ is a monotone and bounded sequence, it converges. Let $u^{\infty} := \lim_{k\to\infty} u^{(k)}$. We claim $u^{\infty} = 1$. Suppose, by way of contradiction, $u^{\infty} > 1$. By Proposition 6, $\psi_i(\hat{q}_i)/(\hat{q}_i + \kappa)$ is strictly decreasing in $\hat{q}_i$. Thus, for any $\hat{q}_i \geq \frac{1}{2}(1+u^{\infty})\cdot\hat{q}_i^*$, there exists $\epsilon \in (0,1)$ such that for each i, we have

$$\frac{\psi_i(\hat{q}_i)}{(\hat{q}_i+\kappa)} \;\leq\; (1-\epsilon)\cdot\frac{\psi_i(\hat{q}_i^*)}{\hat{q}_i^*+\kappa} \;=\; (1-\epsilon)\cdot\frac{q_i^*}{\hat{q}_i^*+\kappa}\;.$$

For any k, if $\hat{\overline{q}}_i^{(k)} \geq \frac{1}{2}(1+u^{\infty})\cdot\hat{q}_i^*$, then

$$\begin{aligned}\overline{q}_i^{(k+1)} = \psi_i\left(\hat{\overline{q}}_i^{(k)}\right) \;&\leq\; \left(\hat{\overline{q}}_i^{(k)}+\kappa\right)\cdot(1-\epsilon)\cdot\frac{q_i^*}{\hat{q}_i^*+\kappa}\\ &\leq \frac{\hat{\overline{q}}_i^{(k)}}{\hat{q}_i^*}\cdot q_i^*\cdot(1-\epsilon) \;\leq\; (1-\epsilon)\cdot u^{(k)}\cdot q_i^*\;.\end{aligned}$$

Otherwise, we have $\hat{q}_i^* < \hat{\overline{q}}_i^{(k)} < \frac{1}{2}(1+u^{\infty})\cdot\hat{q}_i^*$. By Proposition 6,

$$\overline{q}_i^{(k+1)} = \psi_i\left(\hat{\overline{q}}_i^{(k)}\right) \;<\; \left(\hat{\overline{q}}_i^{(k)}+\kappa\right)\cdot\frac{q_i^*}{\hat{q}_i^*+\kappa} \leq \frac{\hat{\overline{q}}_i^{(k)}}{\hat{q}_i^*}\cdot q_i^* \;\leq\; \frac{1}{2}(1+u^{\infty})\cdot q_i^*\;.$$

Therefore, we conclude, using an argument similar to (7), $u^{(k+1)} \leq \max\{(1-\epsilon)\cdot u^{(k)}, (1+u^{\infty})/2\}$. From $u^{(k+1)} > u^{\infty} > (1+u^{\infty})/2$, we obtain $u^{(k+1)} \leq (1-\epsilon)\cdot u^{(k)}$, implying $u^{(k)} \to 1$ as $k \to \infty$. This is a contradiction. Similarly, we can show that $l^{(k)} := \min_i\left\{\hat{\underline{q}}_i^{(k)} \;/\; \hat{q}_i^*\right\}$ is a strictly increasing sequence converging to 1. □

The following proposition shows the linear convergence of tatônnement in the space of attraction values.

Proposition 7. *The sequence* $\{\mathbf{q}^{(k)}\}_{k\geq 0}$ *converges linearly.*

Proof. Consider $\{\underline{\mathbf{q}}^{(k)}\}_{k=0}^{\infty}$ and $\left\{\overline{\mathbf{q}}^{(k)}\right\}_{k=0}^{\infty}$ in the proof of Theorem 2. Recall $\underline{\mathbf{q}}^{(k)} < \hat{b}^{(k)} < \overline{\mathbf{q}}^{(k)}$ and $\underline{\mathbf{q}}^{(k)} < \hat{b}^* < \overline{\mathbf{q}}^{(k)}$. We will show that $\underline{\mathbf{q}}^{(k)}$ and $\overline{\mathbf{q}}^{(k)}$ converges to $\hat{b}^*$ linearly. Since $\mathcal{Q}$ is a box-shaped open set, there exists a convex compact set $\mathcal{B} \subset \mathcal{Q}$ containing all elements of $\{\underline{\mathbf{q}}^{(k)}\}_{k=0}^{\infty}$ and $\left\{\overline{\mathbf{q}}^{(k)}\right\}_{k=0}^{\infty}$. From the proof of Proposition 6, there exists $\delta > 0$ such that for any $\hat{b} \in \mathcal{B}$, we have

$$\frac{d}{d\hat{q}_i}\left(\frac{\psi(\hat{q}_i)}{\hat{q}_i+\kappa}\right) \leq -\delta.$$

From integrating both sides of the above expression from $\hat{q}_i^*$ to $\hat{\overline{q}}_i^{(k)}$,

$$\frac{\psi_i(\hat{\overline{q}}_i^{(k)})}{\hat{\overline{q}}_i^{(k)}+\kappa} - \frac{\psi(\hat{q}_i^*)}{\hat{q}_i^*+\kappa} \leq -\delta\left(\hat{\overline{q}}_i^{(k)} - \hat{q}_i^*\right)$$

since the line segment connecting $\hat{b}^*$ and $\overline{\mathbf{q}}^{(k)}$ lies within $\mathcal{B}$.

Define $\delta_1 := \delta \cdot \min_i \min_{\hat{b}\in\mathcal{B}} \{(\hat{q}_i + \kappa) \cdot \hat{q}_i^*/q_i^*\} > 0$. We choose $\delta > 0$ is sufficiently small such that $\delta_1 < 1$. Recall $\overline{q}_i^{(k+1)} = \psi_i(\hat{\overline{q}}_i^{(k)})$ and $q_i^* = \psi(\hat{q}_i^*)$. Rearranging the above inequality and multiplying it by $(\hat{\overline{q}}_i^{(k)} + \kappa)/q_i^*$,

$$\begin{aligned}\overline{q}_i^{(k+1)}/q_i^* &\leq (\hat{\overline{q}}_i^{(k)} + \kappa)/(\hat{q}_i^* + \kappa) \;-\; \delta \cdot (\hat{\overline{q}}_i^{(k)} - \hat{q}_i^*) \cdot (\hat{\overline{q}}_i^{(k)} + \kappa)/q_i^* \\ &\leq \hat{\overline{q}}_i^{(k)}/\hat{q}_i^* \;-\; \delta_1 \cdot (\hat{\overline{q}}_i^{(k)}/\hat{q}_i^* - 1) = (1-\delta_1) \cdot \hat{\overline{q}}_i^{(k)}/\hat{q}_i^* \;+\; \delta_1.\end{aligned}$$

where the second inequality comes from $\hat{\overline{q}}_i^{(k)} > \hat{q}_i^*$ and the definition of δ_1.

Let $\rho(k) := \max_i\{\overline{q}_i^{(k)}/q_i^*\}$. Thus, $\overline{q}_j^{(k)} \leq \rho(k) \cdot q_j^*$ holds for all j, and summing this inequality for all $j \neq i$, we get $\hat{\overline{q}}_i^{(k)} \leq \rho(k) \cdot \hat{q}_i^*$. Thus, $\overline{q}_i^{(k+1)}/q_i^*$ is bounded above by $(1-\delta_1)\cdot\rho(k)+\delta_1$ for each i, and we obtain $\rho(k+1) \leq (1-\delta_1)\cdot\rho(k)+\delta_1$. Using induction, it is easy to show

$$\rho(k) \leq (1-\delta_1)^k \cdot (\rho(0) - 1) + 1$$

Therefore, we obtain

$$\begin{aligned}\max_i \left\{(\overline{q}_i^{(k)} - q_i^*)/q_i^*\right\} &= \rho(k) - 1 \;\leq\; (1-\delta_1)^k \cdot (\rho(0) - 1) \\ &= (1-\delta_1)^k \cdot \max_i \left\{(\overline{q}_i^{(0)} - q_i^*)/q_i^*\right\} \qquad \text{and}\end{aligned}$$

$$\max_i \left\{\overline{q}_i^{(k)} - q_i^*\right\} \leq (1-\delta_1)^k \cdot \max_i\{q_i^*\} \cdot \max_i \left\{(\overline{q}_i^{(0)} - q_i^*)/q_i^*\right\} ,$$

showing the linear convergence of the upper bound sequence $\overline{\mathbf{q}}^{(k)}$. A similar argument shows the linear convergence of $\underline{\mathbf{q}}^{(k)}$. □

The linear convergence in the above theorem is not with respect to the price vector but with respect to the attraction vector, i.e. not in $\mathbf{p}$ but in $\mathbf{q}$. Yet, the following theorem also shows the linear convergence with respect to the price vector. Let $\{\mathbf{p}^{(k)}\}_{k\geq 0}$ be the sequence defined by $p_i^{(k)} := a_i^{-1}(q_i^{(k)})$.

Theorem 3. *The sequence $\{\mathbf{p}^{(k)}\}_{k\geq 0}$ converges linearly.*

Proof. Consider $\{\underline{\mathbf{q}}^{(k)}\}_{k=0}^{\infty}$ and $\left\{\overline{\mathbf{q}}^{(k)}\right\}_{k=0}^{\infty}$ in the proof of Proposition 7. Let $\{\underline{\mathbf{p}}^{(k)}\}_{k=0}^{\infty}$ and $\left\{\overline{\mathbf{p}}^{(k)}\right\}_{k=0}^{\infty}$ be the corresponding sequences of price vectors. Within the compact interval $[\overline{p}_i^{(0)}, \underline{p}_i^{(0)}]$, the derivative of a_i is continuous and its infimum is strictly negative. By the Inverse Function Theorem, the derivative of $a_i^{-1}(\cdot)$ is continuous in the compact domain of $[\underline{q}_i^{(0)}, \overline{q}_i^{(0)}]$. Recall $p_i^* = a_i^{-1}(q_i^*)$. There exists some bound $M > 0$ such that

$$|p_i - p_i^*| \;=\; |a_i^{-1}(q_i) - a_i^{-1}(q_i^*)| \;\leq\; M \cdot |q_i - q_i^*|$$

whenever $a_i(p_i) = q_i \in [\underline{q}_i^{(0)}, \overline{q}_i^{(0)}]$ for all i. From the proof of Proposition 7, we obtain $q_i^{(k)} \in (\underline{q}_i^{(k)}, \overline{q}_i^{(k)}) \subset [\underline{q}_i^{(0)}, \overline{q}_i^{(0)}]$. Therefore, the linear convergence of $\{\mathbf{q}^{(k)}\}_{k\geq 0}$ implies the linear convergence of $\{\mathbf{p}^{(k)}\}_{k\geq 0}$. □

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