Strongly Complete Axiomatizations of "Knowing at Most" in Syntactic Structures

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Abstract. Syntactic structures based on standard syntactic assignments model knowledge directly rather than as truth in all possible worlds as in modal epistemic logic, by assigning arbitrary truth values to atomic epistemic formulae. This approach to epistemic logic is very general and is used in several logical frameworks modeling multi-agent systems, but has no interesting logical properties — partly because the standard logical language is too weak to express properties of such structures. In this paper we extend the logical language with a new operator used to represent the proposition that an agent "knows at most" a given finite set of formulae and study the problem of strongly complete axiomatization of syntactic structures in this language. Since the logic is not semantically compact, a strongly complete *finitary* axiomatization is impossible. Instead we present, first, a strongly complete *infinitary* system, and, second, a strongly complete finitary system for a slightly weaker variant of the language.

1 Introduction

Epistemic logic [1, 2] describe the knowledge of one or several agents. The by far most popular approach to epistemic logic has been to interpret knowledge as truth in all worlds considered possible. To this end, the formalisms of modal logic (see, e.g., [3]) are used: the logical language includes formulae of the form $K_i\phi$, and the semantics is defined by Kripke structures describing the possible worlds. While the modal approach to epistemic logic has been highly successful in many applications, in some contexts it is less applicable. An example of the latter is when we need to model the explicit knowledge an agent has computed, e.g., stored in his knowledge base, at a specific point in time. In modal epistemic logic, an agent necessarily knows all the logical consequences of his knowledge – *the logical omniscience problem* [4]. Furthermore, an agent cannot know a *contradiction* without knowing *everything*. Modal epistemic logic fails as a logic of the explicitly computed knowledge of real agents, because it assumes a very particular and extremely powerful mechanism for reasoning. In reality, different agents have different reasoning mechanisms (e.g. non-monotonic or resource-bounded) and representations of knowledge (e.g. as propositions or as syntactic formulae).

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In this paper we study a radically different approach to epistemic logic – the *syntactic approach*. In the syntactic approach, a formula $K_i\phi$ can be assigned a truth value independently of the truth value assigned to any other formula of the form $K_i\psi$. Thus the syntactic approach allows, e.g., an agent's knowledge to not be closed under logical consequence or other conditions, and to contain contradictions. Several logical frameworks modeling agents in general [5, 6, 7, 1, 8, 9, 10] and multi-agent systems in particular [11, 12, 13, 14, 15] are based on the syntactic approach. Of particular recent interest has been the body of work on *the Logic of General Awareness* [16, 17, 18, 11, 19, 20, 21], which combine an awareness operator with syntactic semantics with a traditional epistemic operator with possible worlds semantics.

We use the formalisation of the syntactic approach by [1], called *syntactic structures*, and present several new results about the axiomatisation of certain properties of such structures. A syntactic structure is an isolated abstraction of syntactic knowledge, but the results we obtain are also relevant for logics with, e.g., a combination of syntactic and semantic operators.

Knowledge can also be modeled directly by a semantic, rather than a syntactic, approach, by using, e.g., *Montague-Scott* structures [22, 23, 24]). Syntactic structures are generalizations of both Kripke structures and Montague-Scott structures. The literature contains numerous proposed solutions to the logical omniscience problem, see, e.g., [25, 26, 1] for reviews. Wansing [27] shows that many of these approaches can be modeled using Rantala models [28, 29], and that Rantala models can be seen as the most general models of knowledge. It is easy to see that syntactic structures are as general as Rantala models; any Rantala model can be simulated by a syntactic structure. However, syntactic structures are so general that they have no interesting logical properties that can be expressed in the traditional language of epistemic logic – indeed, they are completely axiomatized by propositional logic.

In this paper, in order to be able to express interesting properties of syntactic structures, we extend the logical language with an epistemic operator $\triangledown_i$ for each agent. $\triangledown_i X$, where X is a finite set of formulae, expresses the fact that agent i knows *at most* X. The main problem we consider is the construction of a strongly complete axiomatization of syntactic structures in this language. A consequence of the addition of the new operator is that semantic compactness is lost, and thus that a strongly complete finitary axiomatization is impossible. Instead we, first, present a strongly complete *infinitary* system, and, second, a strongly complete finitary system for syntactic structures for a slightly weaker variant of the epistemic operators.

Our motivation for pursuing the syntactic approach is not that we view it as an *alternative* to the modal approach for all purposes. Rather, we view it as a *complementary approach*, which can be more suitable than the modal approach in some circumstances. A disadvantage of the syntactic approach is that it does not explain knowledge in terms of more fundamental concepts such as possible worlds. But on the other hand, in some cases knowledge of formulae *is* the fundamental concept, for example when an agent stores its knowledge as syntactic strings in a database. Advantages of the syntactic approach include the fact that it can be used to model certain types of agents and certain types of situations which are difficult if not impossible to model with the modal approach; e.g., non-ideal – rather than ideal – agents, and situations where we are interested in explicit – rather than implicit – knowledge. As a concrete example, consider the explicitly computed knowledge of a (non-ideal) agent at a point in time at which it has computed $p \to q$ and p but not (yet) q. The formulae $K(p \to q)$, Kp and $\neg Kq$ can never be true at the same time in modal epistemic logic, but they can in the syntactic approach.

Rather than dictating the properties of knowledge, the syntactic approach is a general framework in which different properties can be explored. In this paper we are interested in logical systems describing syntactic knowledge which are strongly complete. If these systems are extended with a set of axioms, the resulting systems are automatically strongly complete with respect to the models of the axioms. For example, if we want to include the assumption that an agent cannot know both a formula and its negation at the same time, we can add the axiom schema $K_i\alpha \to \neg K_i\neg\alpha$ to one of the systems we discuss, and the resulting system will again be strongly complete with respect to syntactic structures with the mentioned property.

In Section 2 syntactic structures based on standard syntactic assignments and their use in epistemic logic are introduced, before the "at most" operator $\bigtriangledown_i$ and its interpretation in syntactic structures are presented in Section 3. The completeness results are presented in Section 4, and we discuss some related work and conclude in Sections 5 and 6. We presently define some logical concepts and terminology used in the remainder.

1.1 Logic

By "a logic" we henceforth mean a language of formulae together with a class of semantic structures and a satisfiability relation $\models$. The semantic structures considered in this paper each have a set of *states*, and satisfiability relations relate a formula to a pair consisting of a structure M and a state s of M. A formula ϕ is *satisfiable* if there is a model M with a state s such that $(M, s) \models \phi$. A formula ϕ is a (local) *logical consequence* of a theory (set of formulae) Γ, $\Gamma \models \phi$, iff $(M, s) \models \psi$ for all $\psi \in \Gamma$ implies that $(M, s) \models \phi$. The usual terminology and notation for Hilbert-style proof systems are used: $\Gamma \vdash_S \phi$ means that formula ϕ is derivable from theory Γ in system S, and when Δ is a set of formulae, $\Gamma \vdash_S \Delta$ means that $\Gamma \vdash_S \delta$ for each $\delta \in \Delta$. We use the following definition of maximality: a theory in a language L is maximal if it contains either ϕ or $\neg\phi$ for each $\phi \in L$. A logical system is *weakly complete*, or just *complete*, if $\models \phi$ (i.e. $\emptyset \models \phi$, ϕ is *valid*) implies $\vdash_S \phi$ (i.e. $\emptyset \vdash_S \phi$) for all formulae ϕ, and *strongly complete* if $\Gamma \models \phi$ implies $\Gamma \vdash_S \phi$ for all formulae ϕ and theories Γ. If a logic has a (strongly) complete logical system, we say that the logic *is* (strongly) complete. A logic is semantically *compact* if for every theory Γ, if every finite subset of Γ is satisfiable then Γ is satisfiable. It is easy to see that under the definitions used above:

Fact 1. A weakly complete logic has a sound and strongly complete finitary axiomatization iff it is compact.

2 The Epistemic Logic of Syntactic Structures

Syntactic structures are defined, and used to interpret the standard epistemic language, as follows. Given a number of agents n we write Σ for the set $\{1, \ldots, n\}$. The standard epistemic language:

Definition 2 ($\mathcal{L}$). Given a set of primitive propositions Θ and a number of agents n, $\mathcal{L}(\Theta, n)$ (or just $\mathcal{L}$) is the least set such that:

- $\Theta \subseteq \mathcal{L}$
- If $\phi, \psi \in \mathcal{L}$ then $\neg\phi, (\phi \wedge \psi) \in \mathcal{L}$
- If $\phi \in \mathcal{L}$ and $i \in \Sigma$ then $K_i\phi \in \mathcal{L}$ □

The set of *epistemic atoms* is $\mathcal{L}^{At} = \{K_i\phi : \phi \in \mathcal{L}, i \in \Sigma\}$. An epistemic formula is a propositional combination of epistemic atoms. A syntactic structure [1] assigns a truth value to the primitive propositions and epistemic atoms.

Definition 3 (Syntactic Structure). A *syntactic structure* is a tuple

$$(S, \sigma)$$

where S is a set of states and

$$\sigma(s) : \Theta \cup \mathcal{L}^{At} \rightarrow \{\mathbf{true}, \mathbf{false}\}$$

for each $s \in S$. The function σ is called a *standard syntactic assignment.* □

Satisfaction of an $\mathcal{L}$ formula ϕ by a state s of a syntactic structure M, written $(M, s) \models \phi$, is defined as follows:

$$\begin{array}{lcl} (M,s) \models p & \Leftrightarrow & \sigma(s)(p) = \mathbf{true} \\ (M,s) \models \neg\phi & \Leftrightarrow & (M,s) \not\models \phi \\ (M,s) \models (\phi \wedge \psi) & \Leftrightarrow & (M,s) \models \phi \text{ and } (M,s) \models \psi \\ (M,s) \models K_i\phi & \Leftrightarrow & \sigma(s)(K_i\phi) = \mathbf{true} \end{array}$$

We note that although [1] define syntactic structures in a possible worlds framework, the question of satisfaction of ϕ in a state s does not depend on any other state $(((S,\sigma), s) \models \phi \Leftrightarrow ((\{s\}, \sigma), s) \models \phi)$. We nevertheless keep the possible worlds framework in this paper, while pointing out that it does not play any significant role, for easier comparison with the standard formalisation. A consequence of this independence of states is the following: if a system is strongly complete with respect to all syntactic structures, then the system extended with a set of axioms Γ is strongly complete with respect to the models of Γ. For example, a strongly complete system extended with the axiom schema $K_i\alpha \rightarrow \neg K_i\neg\alpha$

will be strongly complete with respect to syntactic structures never assigning **true** to both α and $\neg\alpha$ for any formula α in any state.

Syntactic structures are very general descriptions of knowledge – in fact so general that no epistemic properties of the class of all syntactic structures can be described by the standard epistemic language:

Theorem 4. Propositional logic, with substitution instances for the language $\mathcal{L}$, is sound and complete with respect to syntactic structures. □

In the next section we increase the expressiveness of the epistemic language.

3 Knowing at Most

The formula $K_i\phi$ denotes that fact that i knows *at least* ϕ – he knows ϕ but he may know more. We can generalize this to finite sets $X \subseteq \mathcal{L}$ of formulae:

$$\triangle_i X \equiv \bigwedge\{K_i\phi : \phi \in X\}$$

representing the fact that i knows at least X. The new operator we introduce here[1] is a dual to $\triangle_i$, denoting the fact that i knows *at most* X:

$$\triangledown_i X$$

denotes the fact that every formula an agent knows is included in X, but he may not know all the formulae in X. If $\mathcal{L}$ was finite, the operator $\triangledown_i$ could (like $\triangle_i$) be defined in terms of K_i:

$$\triangledown_i X = \bigwedge\{\neg K_i\phi : \phi \in \mathcal{L} \setminus X\}$$

But since $\mathcal{L}$ is not finite (regardless of whether or not Θ is finite), $\triangledown_i$ is not definable by K_i. We also use a third, derived, epistemic operator: $\Diamond_i X \equiv \triangle_i X \wedge \triangledown_i X$ meaning that the agent knows exactly X. The extended language is called $\mathcal{L}_\triangledown$.

Definition 5 ($\mathcal{L}_\triangledown$). Given a set of primitive propositions Θ, and a number of agents n, $\mathcal{L}_\triangledown(\Theta, n)$ (or just $\mathcal{L}_\triangledown$) is the least set such that:

- $\Theta \subseteq \mathcal{L}_\triangledown$
- If $\phi, \psi \in \mathcal{L}_\triangledown$ then $\neg\phi, (\phi \wedge \psi) \in \mathcal{L}_\triangledown$
- If $\phi \in \mathcal{L}$ and $i \in \Sigma$ then $K_i\phi \in \mathcal{L}_\triangledown$
- If $X \in \wp^{fin}(\mathcal{L})$ and $i \in \Sigma$ then $\triangledown_i X \in \mathcal{L}_\triangledown$ □

The language $\mathcal{L}_\triangledown(\Theta, n)$ is defined to express properties of syntactic structures over the language $\mathcal{L}(\Theta, n)$ (introduced in Section 2), and thus the epistemic

[1] The $\triangledown_i X$ operator was also used in a similar logic for the special case of agents who can know only *finitely* many formulae at one time in [30]. The results in the current paper has been used to further investigate the case with the finiteness assumption [31].

operators K_i and ∇_i operate on formulae from $\mathcal{L}(\Theta, n)$. We assume that Θ is countable, and will make use of the fact that it follows that $\mathcal{L}_{\nabla}(\Theta, n)$ is (infinitely) countable.

If X is a finite set of $\mathcal{L}_{\nabla}$ formulae, we write $\triangle_i X$ as discussed above (i.e., as a shorthand for $\bigwedge_{\phi \in X} K_i\phi$). In addition, we use $\Diamond_i X$ for $\triangle_i X \wedge \nabla_i X$, and the usual derived propositional connectives.

The interpretation of $\mathcal{L}_{\nabla}$ in a state s of a syntactic structure M is defined in the same way as the interpretation of $\mathcal{L}$, with the following clause for the new epistemic operator:

$$(M, s) \models \nabla_i X \quad \Leftrightarrow \quad \{\phi \in \mathcal{L} : \sigma(s)(K_i\phi) = \mathbf{true}\} \subseteq X$$

It is easy to see that

$$(M, s) \models \triangle_i X \quad \Leftrightarrow \quad \{\phi \in \mathcal{L} : \sigma(s)(K_i\phi) = \mathbf{true}\} \supseteq X$$
$$(M, s) \models \Diamond_i X \quad \Leftrightarrow \quad \{\phi \in \mathcal{L} : \sigma(s)(K_i\phi) = \mathbf{true}\} = X$$

3.1 Properties

The following schemata, where X, Y, Z range over finite sets of formulae and ϕ over single formulae, show some properties of syntactic structures, in the language $\mathcal{L}_{\nabla}$.

$\triangle_i \emptyset$		E1
$(\triangle_i X \wedge \triangle_i Y) \rightarrow \triangle_i(X \cup Y)$		E2
$(\nabla_i X \wedge \nabla_i Y) \rightarrow \nabla_i(X \cap Y)$		E3
$\neg(\triangle_i X \wedge \nabla_i Y)$	when $X \not\subseteq Y$	E4
$(\nabla_i(Y \cup \{\phi\}) \wedge \neg K_i\phi) \rightarrow \nabla_i Y$		E5
$\triangle_i X \rightarrow \triangle_i Y$	when $Y \subseteq X$	**KS**
$\nabla_i X \rightarrow \nabla_i Y$	when $X \subseteq Y$	**KG**

The properties are self-explanatory. **KS** and **KG** stands for knowledge *speciali-sation* and *generalisation*, respectively.

It is straightforward to prove the following.

Lemma 6. E1–E5, **KG**, **KS** are valid. □

4 Axiomatizations of Syntactic Structures

In this section we discuss axiomatizations of syntactic structures in the language $\mathcal{L}_{\nabla}$. The following lemma shows that the logic is not compact, and thus it does not have a strongly complete finitary axiomatization (Fact 1).

Lemma 7. The logic of syntactic structures in the language $\mathcal{L}_{\bigtriangledown}$ is not compact.□

PROOF. Let $p \in \Theta$ and let Γ_1 be the following $\mathcal{L}_{\bigtriangledown}$ theory:

$$\Gamma_1 = \{K_i p, \neg \bigtriangledown_i \{p\}\} \cup \{\neg K_i \phi : \phi \neq p\}$$

Let Γ' be a finite subset of Γ_1. Clearly, there exists a ϕ' such that $\neg K_i \phi' \notin \Gamma'$. Let $M = (\{s\}, \sigma)$ be such that $\sigma(s)(K_i\phi) = \mathbf{true}$ iff $\phi = p$ or $\phi = \phi'$. It is easy to see that $(M, s) \models \Gamma'$. If there was some (M', s') such that $(M', s') \models \Gamma_1$, then $(M', s') \models \neg \bigtriangledown_i \{p\}$ i.e. there must exist a $\phi \neq p$ such that $\sigma(s)(K_i\phi) = \mathbf{true}$ – which contradicts the fact that $(M', s') \models \neg K_i\phi$ for all $\phi \neq p$. Thus, every finite subset of Γ_1 is satisfiable, but Γ_1 is not. ■

We present a strongly complete *in*finitary axiomatization in Section 4.1. Then, in Section 4.2, a finitary axiomatization for a slightly weaker language than $\mathcal{L}_{\bigtriangledown}$ is proven strongly complete for syntactic structures.

4.1 An Infinitary System

We define a proof system EC^ω for the language $\mathcal{L}_{\bigtriangledown}$ by using properties presented in Section 3 as axioms, in addition to propositional logic. In addition, EC^ω contains an infinitary derivation clause **R***. After presenting EC^ω, the rest of the section is concerned with proving its strong completeness with respect to the class of all syntactic structures. This is done by the commonly used strategy of proving satisfiability of maximal consistent theories. Thus we need an infinitary variant of the Lindenbaum lemma. However, the usual proof of the Lindenbaum lemma for finitary systems is not necessarily applicable to infinitary systems. In order to prove the Lindenbaum lemma for EC^ω, we use the same strategy as [32] who prove strong completeness of an infinitary axiomatization of PDL (there with canonical models). In particular, we use the same way of defining the derivability relation by using a weakening rule **W**, and we prove the deduction theorem in the same way by including a cut rule **Cut**.

Definition 8 (EC^ω). EC^ω is a logical system for the language $\mathcal{L}_{\bigtriangledown}$ having the following axiom schemata

All substitution instances of tautologies of propositional calculus		**Prop**
$\neg(\triangle_i X \wedge \bigtriangledown_i Y)$	when $X \not\subseteq Y$	E4
$(\bigtriangledown_i(Y \cup \{\gamma\}) \wedge \neg K_i\gamma) \rightarrow \bigtriangledown_i Y$		E5
$\bigtriangledown_i X \rightarrow \bigtriangledown_i Y$	when $X \subseteq Y$	**KG**

The derivation relation $\vdash_{EC^\omega}$ – written $\vdash_\omega$ for simplicity – between sets of $\mathcal{L}_{\bigtriangledown}$ formulae and single $\mathcal{L}_{\bigtriangledown}$ formulae is the smallest relation closed under the following conditions:

$$\vdash_\omega \phi \qquad \text{when } \phi \text{ is an axiom} \qquad \textbf{Ax}$$

$$\{\phi, \phi \to \psi\} \vdash_\omega \psi \qquad \textbf{MP}$$

$$\bigcup_{j \in J} \{\alpha_j \to \neg K_i \gamma : \gamma \notin X_j\} \vdash_\omega \bigwedge_{j \in J} \alpha_j \to \triangledown_i X \qquad \textbf{R*}$$

$$\text{when } X = \bigcap_{j \in J} X_j \text{ and } X \text{ and } J \text{ are finite}$$

$$\frac{\Gamma \vdash_\omega \phi}{\Gamma \cup \Delta \vdash_\omega \phi} \qquad \textbf{W}$$

$$\frac{\Gamma \vdash_\omega \Delta, \Gamma \cup \Delta \vdash_\omega \phi}{\Gamma \vdash_\omega \phi} \qquad \textbf{Cut}$$

In the above schemata, X, Y, Z, X_j range over sets of $\mathcal{L}$ formulae, γ over $\mathcal{L}$ formulae, Γ, Δ over sets of $\mathcal{L}_\triangledown$ formulae, ϕ, ψ, α_j over $\mathcal{L}_\triangledown$ formulae, i over agents, and J over sets of indices. □

It is easy to see that E1, E2, E3 and **KS** are derivable in EC^ω.

In order to understand the meaning of the **R*** rule, first consider the following instance, obtained by taking $J = \{1, \ldots, k\}$ and α_j to be a tautology for every $j \in J$, where $X_1, \ldots, X_k$ are arbitrary sets of $\mathcal{L}$ formulae and i an agent:

$$\{\neg K_i \gamma : \gamma \notin X_1\} \cup \cdots \cup \{\neg K_i \gamma : \gamma \notin X_k\} \vdash_\omega \triangledown_i \bigcap_{1 \le j \le k} X_j$$

This expression says that if it is the case that, for each X_j, the agent (i) does not know anything which is not in X_j, then the agent knows *at most* the intersection of $X_1, \ldots, X_k$. The general case when α_j is not necessarily an tautology is easily understood in light of this special case: if, for each X_j, α_j implies that i does not know any formula outside X_j, then the conjunction of $\alpha_1, \ldots, \alpha_k$ implies that i knows *at most* the intersection of $X_1, \ldots, X_k$.

The use of the weakening rule instead of more general schemata makes inductive proofs easier, but particular derivations can sometimes be more cumbersome. For example:

Lemma 9.

$$\Gamma \cup \{\phi\} \vdash_\omega \phi \qquad \text{R1}$$

$$\frac{\vdash_\omega \psi \to \phi}{\Gamma \cup \{\psi\} \vdash_\omega \phi} \qquad \text{R2}$$

□

Proof.

R1: $\{\phi, \phi \to \phi\} \vdash_\omega \phi$ by **MP**; $\vdash_\omega \phi \to \phi$ by **Ax**; $\{\phi\} \vdash_\omega \phi \to \phi$ by **W**; $\{\phi\} \vdash_\omega \phi$ by **Cut** and $\Gamma \cup \{\phi\} \vdash_\omega \phi$ by **W**.

R2: Let $\vdash_\omega \psi \to \phi$. By **W**, $\{\psi\} \vdash_\omega \psi \to \phi$; by **MP** $\{\psi, \psi \to \phi\} \vdash_\omega \phi$ and thus $\{\psi\} \vdash_\omega \phi$ by **Cut**. By **W**, $\Gamma \cup \{\psi\} \vdash_\omega \phi$. ■

In order to prove the Lindenbaum lemma, we need the deduction theorem. The latter is shown by first proving the following rule.

Lemma 10. The following rule of *conditionalization* is admissible in EC^{ω}.

$$\frac{\Gamma \cup \Delta \vdash_{\omega} \phi}{\Gamma \cup \{\psi \rightarrow \delta : \delta \in \Delta\} \vdash_{\omega} \psi \rightarrow \phi} \quad \textbf{Cond}$$

□

PROOF. The proof is by infinitary induction over the derivation $\Gamma \cup \Delta \vdash_{\omega} \phi$ (derivations are well-founded). The base cases are **Ax**, **MP** and **R***, and the inductive steps are **W** and **Cut**.

Ax: $\Gamma = \Delta = \emptyset$. We must show that $\vdash_{\omega} \psi \rightarrow \phi$ when $\vdash_{\omega} \phi$. By **W** we get $\phi \rightarrow (\psi \rightarrow \phi) \vdash_{\omega} \phi$, then $\phi, \phi \rightarrow (\psi \rightarrow \phi) \vdash_{\omega} \psi \rightarrow \phi$ is an instance of **MP**, and by **Cut** we get $\phi \rightarrow (\psi \rightarrow \phi) \vdash_{\omega} \psi \rightarrow \phi$. By **Prop**, $\vdash_{\omega} \phi \rightarrow (\psi \rightarrow \phi)$, so by **Cut** once more we get $\vdash_{\omega} \psi \rightarrow \phi$.

MP: $\Gamma \cup \Delta = \{\phi', \phi' \rightarrow \phi\} \vdash_{\omega} \phi$. That $\Gamma \cup \{\psi \rightarrow \delta : \delta \in \Delta\} \vdash_{\omega} \psi \rightarrow \phi$ can be shown for each of the four possible combinations of Γ and Δ in a similar way to the **Ax** case.

R*: $\phi = \wedge_{j \in J} \alpha_j \rightarrow \bigtriangledown_i X$ and $\Gamma \cup \Delta = \cup_{j \in J} \{\alpha_j \rightarrow \neg K_i \phi' : \phi' \in \mathcal{L} \setminus X_j\}$ where J is finite and $X = \cap_{j \in J} X_j$ is finite, i.e. there exist for each $j \in J$ sets Y_j and Z_j such that $\mathcal{L} \setminus X_j = Y_j \uplus Z_j$ and

$$\Gamma = \bigcup_{j \in J} \{\alpha_j \rightarrow \neg K_i \phi' : \phi' \in Y_j\}$$
$$\Delta = \bigcup_{j \in J} \{\alpha_j \rightarrow \neg K_i \phi' : \phi' \in Z_j\}$$

Let

$$\Gamma' = \bigcup_{j \in J} \{(\psi \wedge \alpha_j) \rightarrow \neg K_i \phi' : \phi' \in Y_j\}$$
$$\Delta' = \bigcup_{j \in J} \{(\psi \wedge \alpha_j) \rightarrow \neg K_i \phi' : \phi' \in Z_j\}$$

$\Gamma' \cup \Delta' = \cup_{j \in J} \{(\psi \wedge \alpha_j) \rightarrow \neg K_i \phi' : \phi' \in \mathcal{L} \setminus X_j\}$, and thus $\Gamma' \cup \Delta' \vdash_{\omega} \gamma'$, where $\gamma' = \wedge_{j \in J} (\psi \wedge \alpha_j) \rightarrow \bigtriangledown_i X$, by **R***. By **W**, $\Gamma' \cup \Delta' \cup \Gamma \vdash_{\omega} \gamma'$. By **Prop**, $\vdash_{\omega} (\alpha_j \rightarrow \neg K_i \phi') \rightarrow ((\psi \wedge \alpha_j) \rightarrow \neg K_i \phi')$ for each $\alpha_j \rightarrow \neg K_i \phi' \in \Gamma$, and by R2 (once for each formula in Γ) $\Delta' \cup \Gamma \vdash_{\omega} \Gamma'$. By **Cut**, $\Delta' \cup \Gamma \vdash_{\omega} \gamma'$, and it only remains to convert the conjunctions in Δ' and γ' to implications: $\Delta' \cup \Gamma \cup \{\gamma'\} \vdash_{\omega} \psi \rightarrow \phi$ by **Prop** and R2, and by **Cut** and **W** it follows that $\Delta' \cup \Gamma \cup \{\psi \rightarrow \delta : \delta \in \Delta\} \vdash_{\omega} \psi \rightarrow \phi$. By **Prop** and R2 (once of each formula in Δ), $\Gamma \cup \{\psi \rightarrow \delta : \delta \in \Delta\} \vdash_{\omega} \Delta'$, and by **Cut** $\Gamma \cup \{\psi \rightarrow \delta : \delta \in \Delta\} \vdash_{\omega} \psi \rightarrow \phi$, which is the desired conclusion.

W: $\Gamma' \cup \Delta' \vdash_{\omega} \phi$ for some $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. By the induction hypothesis we can use **Cond** to obtain $\Gamma' \cup \{\psi \rightarrow \delta : \delta \in \Delta'\} \vdash_{\omega} \psi \rightarrow \phi$, and thus $\Gamma \cup \{\psi \rightarrow \delta : \delta \in \Delta\} \vdash_{\omega} \psi \rightarrow \phi$ by **W**.

Cut: $\Gamma \cup \Delta \vdash_\omega \Delta'$ and $\Gamma \cup \Delta \cup \Delta' \vdash_\omega \phi$, for some Δ'. By the induction hypothesis on the first derivation (once for each $\delta' \in \Delta'$), $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \phi \to \delta'$ for each $\delta' \in \Delta'$. By the induction hypothesis on the second derivation, $\Gamma \cup \{\psi \to \delta : \delta \in \Delta \cup \Delta'\} \vdash_\omega \psi \to \phi$. By **Cut**, $\Gamma \cup \{\psi \to \delta : \delta \in \Delta\} \vdash_\omega \psi \to \phi$. ∎

Theorem 11 (Deduction Theorem). The rule

$$\frac{\Gamma \cup \{\phi\} \vdash_\omega \psi}{\Gamma \vdash_\omega \phi \to \psi} \qquad \textbf{DT}$$

is admissible in EC^ω. □

Proof. If $\Gamma \cup \{\phi\} \vdash_\omega \psi$, then $\Gamma \cup \{\phi \to \phi\} \vdash_\omega \phi \to \psi$ by **Cond**. $\Gamma \vdash_\omega \phi \to \phi$ by **Ax** and **W**, and thus $\Gamma \vdash_\omega \phi \to \psi$ by **Cut**. ∎

Now we are ready to show that consistent theories can be extended to maximal consistent theories. The proof relies on **DT**.

Lemma 12 (Lindenbaum lemma for EC^ω). If Γ is EC^ω-consistent, then there exists an $\mathcal{L}_\bigtriangledown$-maximal and EC^ω-consistent Γ' such that $\Gamma \subseteq \Gamma'$. □

Proof. Recall **R***:

$$\bigcup_{j \in J} \{\alpha_j \to \neg K_k \psi : \psi \not\in X_j\} \vdash_\omega \bigwedge_{j \in J} \alpha_j \to \bigtriangledown_k X.$$

Formulae which can appear on the right of $\vdash_\omega$ in its instances will be said to have **R***-form. A special case of this schema is when $\bigwedge_j \alpha_j$ is a tautology (i.e., each α_j is), from which

$$\bigcup_{j \in J} \{\neg K_k \phi : \psi \not\in X_j\} \vdash_\omega \bigtriangledown_k X.$$

can be obtained. Now, $\Gamma' \supset \Gamma$ is constructed as follows. $\mathcal{L}_\bigtriangledown$ is countable, so let $\phi_1, \phi_2, \ldots$ be an enumeration of $\mathcal{L}_\bigtriangledown$ respecting the subformula relation (i.e., when ϕ_i is a subformula of ϕ_j then $i < j$).

$$\Gamma_0 = \Gamma$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\phi_{i+1}\} & \text{if } \Gamma_i \vdash_\omega \phi_{i+1} \\ \Gamma_i \cup \{\neg\phi_{i+1}\} & \text{if } \Gamma_i \not\vdash_\omega \phi_{i+1} \text{ and } \phi_{i+1} \text{ does not have the } \textbf{R*}\text{-form} \\ \Gamma_i \cup \{\neg\phi_{i+1}, K_k\psi\} & \text{if } \Gamma_i \not\vdash_\omega \phi_{i+1} \text{ and } \phi_{i+1} \text{ has the } \textbf{R*}\text{-form, where } \psi \\ & \textit{is} \text{ arbitrary such that } \psi \not\in X \text{ and } \Gamma_i \not\vdash_\omega \neg K_k\psi \end{cases}$$

$$\Gamma' = \bigcup_{i=0}^{\omega} \Gamma_i$$

The existence of ψ in the last clause in the definition of Γ_{i+1} is verified as follows: since $\Gamma_i \not\vdash_\omega \phi_{i+1}$, there must be, to prevent an application of **R***, at least one

α_j and $\psi \notin X$ such that $\Gamma_i \nvdash_\omega \alpha_j \to \neg K_k\psi$. By construction (and the ordering of formulae), each α_j or its negation is included in Γ_i. If $\Gamma_i \vdash_\omega \neg\alpha_j$ then also $\Gamma_i \vdash_\omega \alpha_j \to \neg K_k\psi$, and this would be the case also if $\Gamma_i \vdash_\omega \neg K_k\psi$. So $\Gamma_i \vdash_\omega \alpha_j$ and $\Gamma_i \nvdash_\omega \neg K_k\psi$.

It is easy to see that Γ' is maximal.

We show that each Γ_i is consistent, by induction over i. For the base case, Γ_0 is consistent by assumption. For the inductive case, assume that Γ_i is consistent. Γ_{i+1} is constructed by one of the three cases in the definition:

1. Γ_{i+1} is obviously consistent.
2. If $\Gamma_{i+1} = \Gamma_i \cup \{\neg\phi_{i+1}\} \vdash_\omega \bot$, then $\Gamma_i \vdash_\omega \phi_{i+1}$ by **DT** and **Prop**, contradicting the assumption in this case.
3. Consider first the special case (when all α_j are tautologies). Assume that $\Gamma_{i+1} = \Gamma_i \cup \{\neg \bigtriangledown_k X, K_k\psi\} \vdash_\omega \bot$. Then $\Gamma_i \vdash_\omega K_k\psi \to \bigtriangledown_k X$ by **DT** and **Prop** and by E4, since $\psi \notin X$, $\Gamma_i \vdash_\omega K_k\psi \to \neg \bigtriangledown_k X$, and thus $\Gamma_i \vdash_\omega \neg K_k\psi$ contradicting the assumption in this case.
   In the general case, assume that $\Gamma_{i+1} = \Gamma_i \cup \{\neg(\bigwedge_j \alpha_j \to \bigtriangledown_k X), K_k\psi\} \vdash_\omega \bot$:
   i Then $\Gamma_i \vdash_\omega K_k\psi \to (\neg(\bigwedge_j \alpha_j \to \bigtriangledown_k X) \to \bot)$, i.e., $\Gamma_i \vdash_\omega K_k\psi \to (\bigwedge_j \alpha_j \to \bigtriangledown_k X)$, i.e., $\Gamma_i \vdash_\omega \bigwedge_j \alpha_j \to (K_k\psi \to \bigtriangledown_k X)$.
   ii By assumption in the construction, $\Gamma_i \nvdash_\omega \neg(\bigwedge_j \alpha_j)$ (for otherwise it would prove $\bigwedge_j \alpha_j \to \bigtriangledown_k X$), but since $\bigwedge_j \alpha_j$ (as well as each α_j) is a subformula of ϕ_{i+1}, it or its negation is already included in Γ_i. But this means that $\Gamma_i \vdash_\omega \bigwedge_j \alpha_j$. Combined with (i), this gives $\Gamma_i \vdash_\omega K_k\psi \to \bigtriangledown_k X$, i.e., $\Gamma_i \vdash_\omega \neg K_k\psi \vee \bigtriangledown_k X$.
   iii On the other hand, by E4, since $\psi \notin X : \Gamma_i \vdash_\omega \neg(K_k\psi \wedge \bigtriangledown_k X)$, i.e., $\Gamma_i \vdash_\omega \neg K_k\psi \vee \neg \bigtriangledown_k X$. Combined with (ii) this means that $\Gamma_i \vdash_\omega \neg K_k\psi$, but this contradicts the assumption in the construction of Γ_{i+1}.

Thus each Γ_i is consistent.

To show that Γ' is consistent, we first show that

$$\Gamma'' \vdash_\omega \phi \Rightarrow (\Gamma'' \subseteq \Gamma' \Rightarrow \phi \in \Gamma') \tag{1}$$

holds for all derivations $\Gamma'' \vdash_\omega \phi$, by induction over the derivation. The base cases are **Ax**, **MP** and **R***, and the inductive steps are **W** and **Cut**. Let i be the index of the formula ϕ, i.e. $\phi = \phi_i$.

Ax: If $\vdash_\omega \phi$, then $\phi \in \Gamma_i$ by the first case in the definition of Γ_i.

MP: $\Gamma'' = \{\phi', \phi' \to \phi\}$. If $\Gamma'' \subseteq \Gamma'$, there exists k, l such that $\phi' \in \Gamma_k$ and $\phi' \to \phi \in \Gamma_l$. If $\phi \notin \Gamma'$, $\neg\phi \in \Gamma'$ by maximality, i.e. there exists a m such that $\neg\phi \in \Gamma_m$. But then $\neg\phi, \phi', \phi' \to \phi \in \Gamma_{\max(k,l,m)}$, contradicting consistency of $\Gamma_{\max(k,l,m)}$.

R*: $\Gamma'' = \cup_{j\in J}\{\alpha_j \to \neg K_k\psi : \psi \notin X_j\}$ and $\phi = \bigwedge_j \alpha_j \to \bigtriangledown_k X$, where $X = \bigcap_j X_j$, and $\Gamma'' \subseteq \Gamma'$. If $\phi \notin \Gamma'$ then, by maximality, $\neg\phi \in \Gamma'$, and thus $\neg\phi \in \Gamma_i$. Then, by construction of Γ_i, $\Gamma_{i-1} \nvdash_\omega \phi$ (otherwise $\phi \in \Gamma'$) and $K_k\psi \in \Gamma_i$ for some $\psi \notin X$. By the same argument as in point 3.(ii) above, $\Gamma_i \vdash_\omega \bigwedge_j \alpha_j$, and hence also $\Gamma' \vdash_\omega \bigwedge_j \alpha_j$. But then, for an appropriate m

(namely, for which $\phi_m = \alpha_j \to \neg K_k\psi$): $\Gamma_{m-1} \vdash_\omega \alpha_j$ and $\Gamma_{m-1} \vdash_\omega K_k\psi$, i.e., $\neg(\alpha_j \to \neg K_k\psi) \in \Gamma_m$, and so $\alpha_j \to \neg K_k\psi \notin \Gamma'$, which contradicts the assumption that $\Gamma'' \subseteq \Gamma'$.

W: $\Gamma'' = \Gamma''' \cup \Delta$, and $\Gamma''' \vdash_\omega \phi$. If $\Gamma'' \subseteq \Gamma'$, $\Gamma''' \subseteq \Gamma$ and by the induction hypothesis $\phi \in \Gamma'$.

Cut: $\Gamma'' \vdash_\omega \Delta$ and $\Gamma'' \cup \Delta \vdash_\omega \phi$. Let $\Gamma'' \subseteq \Gamma'$. By the induction hypothesis on the first derivation (once for each of the formulae in Δ), $\Delta \subseteq \Gamma'$. Then $\Gamma'' \cup \Delta \subseteq \Gamma'$, and by the induction hypothesis on the second derivation $\phi \in \Gamma'$.

Thus (1) holds for all $\Gamma'' \vdash_\omega \phi$; particularly for $\Gamma' \vdash_\omega \phi$. Consistency of Γ' follows: if $\Gamma' \vdash_\omega \bot$, then $\bot \in \Gamma'$, i.e. $\bot \in \Gamma_l$ for some l, contradicting the fact that each Γ_l is consistent. ■

The following Lemma is needed in the proof of the thereafter following Lemma stating satisfiability of maximal consistent theories.

Lemma 13. Let $\Gamma' \subseteq \mathcal{L}_\nabla$ be an $\mathcal{L}_\nabla$-maximal and EC^ω-consistent theory. If there exists an X' such that $\Gamma' \vdash_\omega \nabla_i X'$, then there exists an X such that $\Gamma' \vdash_\omega \Diamond_i X$. □

Proof. Let Γ' be maximal consistent, and let $\Gamma' \vdash_\omega \nabla_i X'$. Let

$$X = \bigcap_{Y \subseteq X' \text{ and } \Gamma' \vdash_\omega \nabla_i Y} Y$$

Since every Y is included in the finite set X', X is finite, and $\Gamma' \vdash_\omega \nabla_i X$ can be obtained by a finite number of applications of E3. Let

$$Z = \bigcup_{\Gamma' \vdash_\omega \triangle_i Y} Y$$

If $\Gamma' \vdash_\omega \triangle_i Y$, then $Y \subseteq X$, since otherwise Γ' would be inconsistent by E4. Thus Z is finite. By a finite number of applications of E2, $\Gamma' \vdash_\omega \triangle_i Z$. If $Z \not\subseteq X$, then Γ' would be inconsistent by E4, so $Z \subseteq X$. We now show that $X \subseteq Z$. Assume the opposite: $\phi \in X$ but $\phi \notin Z$ for some ϕ. Let $X^- = X \setminus \{\phi\}$. $\Gamma' \not\vdash_\omega K_i\phi$, since otherwise $\phi \in Z$ by definition of Z. By maximality, $\Gamma' \vdash_\omega \neg K_i\phi$. By E5, $\Gamma' \vdash_\omega \nabla_i X^-$ – but by construction of X it follows that $X \subseteq X^-$ which is a contradiction. Thus, $X = Z$, and $\Gamma' \vdash_\omega \Diamond_i X$. ■

Lemma 14. Every maximal EC^ω-consistent $\mathcal{L}_\nabla$ theory is satisfiable. □

Proof. Let Γ be maximal and consistent. We construct the following syntactic structure, which is intended to satisfy Γ:

$$M^\Gamma = (\{s\}, \sigma^\Gamma)$$
$$\sigma^\Gamma(s)(p) = \textbf{true} \Leftrightarrow \Gamma \vdash_\omega p \text{ when } p \in \Theta$$
$$\sigma^\Gamma(s)(K_i\phi) = \textbf{true} \Leftrightarrow \phi \in X_i^\Gamma$$

where:

$$X_i^\Gamma = \begin{cases} Z \text{ where } \Gamma \vdash_\omega \Diamond_i Z \text{ if there is an } X' \text{ such that } \Gamma \vdash_\omega \triangledown_i X' \\ \{\gamma : \Gamma \vdash_\omega K_i\gamma\} \qquad \text{otherwise} \end{cases}$$

In the definition of X_i^Γ, the existence of a Z such that that $\Gamma \vdash_\omega \Diamond_i Z$ in the case that there exists an X' such that $\Gamma \vdash_\omega \triangledown_i X'$ is guaranteed by Lemma 13. We show, by structural induction over ϕ, that

$$(M^\Gamma, s) \models \phi \Longleftrightarrow \Gamma \vdash_\omega \phi \tag{2}$$

This is a stronger statement than the lemma; the lemma is given by the direction to the left. We use three base cases: when ϕ is in Θ, $\phi = K_i\psi$ and $\phi = \triangledown_i X$. The first base case and the two inductive steps negation and conjunction are trivial, so we show only the two interesting base cases. For each base case we consider the situations when X_i^Γ is given by a) the first and b) the second case in its definition.

- $\phi = K_i\psi$: $(M^\Gamma, s) \models K_i\psi$ iff $\psi \in X_i^\Gamma$.
  - $\Rightarrow$) Let $\psi \in X_i^\Gamma$. In case a), $X_i^\Gamma = Z$ where $\Gamma \vdash_\omega \Diamond_i Z$ and by **KS**, $\Gamma \vdash_\omega K_i\psi$. In case b), $\Gamma \vdash_\omega K_i\psi$ by construction of X_i^Γ.
  - $\Leftarrow$) Let $\Gamma \vdash_\omega K_i\psi$. In case a), $\Gamma \vdash_\omega \triangledown_i Z$ and thus $\psi \in Z = X_i^\Gamma$ by E4 and consistency of Γ. In case b), $\psi \in X_i^\Gamma$ by construction.
- $\phi = \triangledown_i X$: $(M^\Gamma, s) \models \triangledown_i X$ iff $X_i^\Gamma \subseteq X$.
  - $\Rightarrow$) Let $X_i^\Gamma \subseteq X$. In case a), $\Gamma \vdash_\omega \Diamond_i Z$ where $Z = X_i^\Gamma \subseteq X$, so $\Gamma \vdash_\omega \triangledown_i X$ by **KG**. In case b), X_i^Γ must be finite, since X is finite. For any $\psi \notin X_i^\Gamma$, $\Gamma \not\vdash_\omega K_i\psi$ by construction of X_i^Γ, and $\Gamma \vdash_\omega \neg K_i\psi$ by maximality. Thus, by **R*** (with $J = \{1\}$, $\alpha_1 = \top$ and $X_1 = X_i^\Gamma$), $\Gamma \vdash_\omega \triangledown_i X_i^\Gamma$, contradicting the assumption in case b). Thus, case b) is impossible.
  - $\Leftarrow$) Let $\Gamma \vdash_\omega \triangledown_i X$. In case a), $\Gamma \vdash_\omega \triangle_i Z$ and by E4 and consistency of Γ $X_i^\Gamma = Z \subseteq X$. Case b) is impossible by definition. ■

Theorem 15. EC^ω is a sound and strongly complete axiomatization of syntactic structures, in the language $\mathcal{L}_\triangledown$. □

Proof. Soundness follows from Lemma 6, and the easily seen facts that $\Gamma \models \phi$ for every instance $\Gamma \vdash_\omega \phi$ of both **MP** and of **R***, and that **W** and **Cut** preserve logical consequence, by induction over the definition of the derivation relation. Strong completeness follows from Lemmas 12 and 14.

4.2 A System for a Weaker Language

In the previous section we proved strong completeness of EC^ω by using **R***. It turns out that strong completeness can be proved without **R***, if we restrict the logical language slightly. The restriction is that for some arbitrary primitive proposition $\hat{p} \in \Theta$, $K_i\hat{p}$ and $\triangledown_i X$ are not well-formed formulae for any i and any X with $\hat{p} \in X$. The semantics is not changed; we are still interpreting the language in syntactic structures over $\mathcal{L}(\Theta, n)$ as described in Sections 2 and 3. Thus, in the restricted logic agents can know something which is not expressible in the logical language.

$\mathcal{L}_{\triangledown}^{\hat{p}} \subset \mathcal{L}_{\triangledown}$ is the restricted language for a given primitive proposition $\hat{p}$.

Definition 16 ($\mathcal{L}_{\triangledown}^{\hat{p}}$). Given a set of primitive propositions Θ, a proposition $\hat{p} \in \Theta$ and a number of agents n, $\mathcal{L}_{\triangledown}^{\hat{p}}(\Theta, n)$ (or just $\mathcal{L}_{\triangledown}^{\hat{p}}$) is the least set such that:

- $\Theta \subseteq \mathcal{L}_{\triangledown}^{\hat{p}}$
- If $\phi, \psi \in \mathcal{L}_{\triangledown}^{\hat{p}}$ then $\neg\phi, (\phi \wedge \psi) \in \mathcal{L}_{\triangledown}^{\hat{p}}$
- If $\phi \in (\mathcal{L} \setminus \{\hat{p}\})$ and $i \in \Sigma$ then $K_i\phi \in \mathcal{L}_{\triangledown}^{\hat{p}}$
- If $X \in \wp^{fin}(\mathcal{L} \setminus \hat{p})$ and $i \in \Sigma$ then $\triangledown_i X \in \mathcal{L}_{\triangledown}^{\hat{p}}$ □

The finitary logical system $EC^{\hat{p}}$ is defined by the same axiom schemata as EC^{ω}. The two systems do not, however, have the same axioms since they are defined for different languages – the extensions of the schemata are different. The derivation relation for $EC^{\hat{p}}$ is defined by the axioms and the derivation rule modus ponens. Particularly, the infinitary derivation clause **R*** from EC^{ω} is not included.

Definition 17 ($EC^{\hat{p}}$). $EC^{\hat{p}}$ is the logical system for the language $\mathcal{L}_{\triangledown}^{\hat{p}}$ consisting of the following axiom schemata:

All substitution instances of tautologies of propositional calculus		**Prop**
$\neg(\triangle_i X \wedge \triangledown_i Y)$	when $X \not\subseteq Y$	E4
$(\triangledown_i(Y \cup \{\gamma\}) \wedge \neg K_i\gamma) \rightarrow \triangledown_i Y$		E5
$\triangledown_i X \rightarrow \triangledown_i Y$	when $X \subseteq Y$	**KG**

The derivation relation $\vdash_{EC^{\hat{p}}}$ – written $\vdash_{\hat{p}}$ for simplicity – between sets of $\mathcal{L}_{\triangledown}^{\hat{p}}$ formulae and single $\mathcal{L}_{\triangledown}^{\hat{p}}$ formulae is the smallest relation closed under the following conditions:

$\Gamma \vdash_{\hat{p}} \phi$	when $\phi \in \Gamma$	**Prem**
$\Gamma \vdash_{\hat{p}} \phi$	when ϕ is an axiom	**Ax**
$\dfrac{\Gamma \vdash_{\hat{p}} \phi, \Gamma \vdash_{\hat{p}} \phi \rightarrow \psi}{\Gamma \vdash_{\hat{p}} \psi}$		**MP**

□

It is easy to see that E1, E2, E3, **KS** and **DT** are derivable in EC^{ω}.

The restriction $\mathcal{L}_{\triangledown}^{\hat{p}} \subset \mathcal{L}_{\triangledown}$ is sufficient to prove strong completeness without **R*** in a manner very similar to the proof in Section 4.1. The first step, existence of maximal consistent extensions, can now be proved by the standard proof since the system is finitary.

Lemma 18 (Lindenbaum lemma for $EC^{\hat{p}}$). If Γ is $EC^{\hat{p}}$-consistent, then there exists an $\mathcal{L}_{\triangledown}^{\hat{p}}$-maximal and $EC^{\hat{p}}$-consistent Γ' such that $\Gamma \subseteq \Gamma'$. □

Second, we establish the result corresponding to Lemma 13 for $\mathcal{L}_{\nabla}^{\hat{p}}$ and $EC^{\hat{p}}$.

Lemma 19. Let $\Gamma' \subseteq \mathcal{L}_{\nabla}^{\hat{p}}$ be a $\mathcal{L}_{\nabla}^{\hat{p}}$-maximal and $EC^{\hat{p}}$-consistent theory. If there exists a X' such that $\Gamma' \vdash_{\hat{p}} \nabla_i X'$, then there exists a X such that $\Gamma' \vdash_{\hat{p}} \Diamond_i X$.□

Proof. The proof is essentially the same as for Lemma 13, for the language $\mathcal{L}_{\nabla}^{\hat{p}}$ instead of $\mathcal{L}_{\nabla}$ (note that in that proof we did not rely on **R***, and that $\hat{p} \notin X$ since $X \subseteq X'$). ■

Third, we show satisfiability.

Lemma 20. Every maximal $EC^{\hat{p}}$-consistent $\mathcal{L}_{\nabla}^{\hat{p}}$ theory is satisfiable. □

Proof. Let Γ be maximal and consistent. The proof is very similar to that of the corresponding result for EC^{ω} (Lemma 14). We construct the following syntactic structure, which is intended to satisfy Γ:

$$M^{\Gamma} = (\{s\}, \sigma^{\Gamma})$$
$$\sigma^{\Gamma}(s)(p) = \mathbf{true} \Leftrightarrow \Gamma \vdash_{\hat{p}} p \text{ when } p \in \Theta$$
$$\sigma^{\Gamma}(s)(K_i\phi) = \mathbf{true} \Leftrightarrow \phi \in X_i^{\Gamma}$$

where:

$$X_i^{\Gamma} = \begin{cases} Z \text{ where } \Gamma \vdash_{\hat{p}} \Diamond_i Z & \text{if there is an } X' \text{ such that } \Gamma \vdash_{\hat{p}} \nabla_i X' \\ \{\gamma : \Gamma \vdash_{\hat{p}} K_i\gamma\} \cup \{\hat{p}\} & \text{if } \forall_{X'} \Gamma \not\vdash_{\hat{p}} \nabla_i X' \text{ and } \bigcup_{\Gamma \vdash_{\hat{p}} \triangle_i Y} Y \text{ is finite} \\ \{\gamma : \Gamma \vdash_{\hat{p}} K_i\gamma\} & \text{if } \forall_{X'} \Gamma \not\vdash_{\hat{p}} \nabla_i X' \text{ and } \bigcup_{\Gamma \vdash_{\hat{p}} \triangle_i Y} Y \text{ is infinite} \end{cases}$$

The existence of Z is guaranteed by Lemma 19, and, again, we show, by structural induction over ϕ, that

$$(M^{\Gamma}, s) \models \phi \Longleftrightarrow \Gamma \vdash_{\hat{p}} \phi \quad (3)$$

for all $\phi \in \mathcal{L}_{\nabla}^{\hat{p}}$. As in the proof of Lemma 14 we only show the epistemic base cases. For each base case we consider the situations when

a) there is an X' such that $\Gamma \vdash_{\hat{p}} \nabla_i X'$ or
b) $\Gamma \not\vdash_{\hat{p}} \nabla_i X'$ for every X'

corresponding to the first and to the second and third cases in the definition of X_i^{Γ}, respectively.

- $\phi = K_i\psi$: $(M^{\Gamma}, s) \models K_i\psi$ iff $\psi \in X_i^{\Gamma}$.
  - $\Rightarrow$) Let $\psi \in X_i^{\Gamma}$. In case a), $X_i^{\Gamma} = Z$ where $\Gamma \vdash_{\hat{p}} \Diamond_i Z$ and by **KS**, $\Gamma \vdash_{\hat{p}} K_i\psi$. In case b), $\psi \neq \hat{p}$ (since $K_i\psi \in \mathcal{L}_{\nabla}^{\hat{p}}$) and thus $\Gamma \vdash_{\hat{p}} K_i\psi$ by construction of X_i^{Γ}.
  - $\Leftarrow$) Let $\Gamma \vdash_{\hat{p}} K_i\psi$. In case a), $\Gamma \vdash_{\hat{p}} \nabla_i Z$ and thus $\psi \in Z = X_i^{\Gamma}$ by E4 and consistency of Γ. In case b), $\psi \in X_i^{\Gamma}$ by construction.
- $\phi = \nabla_i X$: $(M^{\Gamma}, s) \models \nabla_i X$ iff $X_i^{\Gamma} \subseteq X$.

⇒) Let $X_i^\Gamma \subseteq X$. In case a), $\Gamma \vdash_{\hat{p}} \Diamond_i Z$ where $Z = X_i^\Gamma \subseteq X$, so $\Gamma \vdash_{\hat{p}} \nabla_i X$ by **KG**. In case b), if $\hat{p} \in X_i^\Gamma$ then $\hat{p} \in X$ which is impossible since $\nabla_i X$ is a formula. But if $\hat{p} \notin X_i^\Gamma$ then X_i^Γ is infinite (by construction) which is also impossible since X is finite – thus case b) is impossible.

⇐) Let $\Gamma \vdash_{\hat{p}} \nabla_i X$. In case a), $\Gamma \vdash_{\hat{p}} \triangle_i Z$ and by E4 and consistency of Γ $X_i^\Gamma = Z \subseteq X$. Case b) is impossible by definition. ■

Theorem 21. $EC^{\hat{p}}$ is a sound and strongly complete axiomatization of syntactic structures, in the language $\mathcal{L}_\nabla^{\hat{p}}$. □

Proof. Soundness follows from the soundness of EC^ω and the fact that $\Gamma \vdash_{\hat{p}} \phi$ implies $\Gamma \vdash_\omega \phi$, the latter which can be seen by induction on the length of a proof in $EC^{\hat{p}}$ (every $\mathcal{L}_\nabla^{\hat{p}}$ formula is also a $\mathcal{L}_\nabla$ formula): the base case **Prem** follows by R1 (Lemma 9), the base case **Ax** follows by **Ax** and **W**, and the inductive case **MP** follows by **MP**, **W** and **Cut**. Strong completeness follows from Lemmas 20 and 18. ■

5 Only Knowing

Apart from the syntactic approaches mentioned in the introduction, the work maybe most closely related to the ideas discussed in this paper is the body of work on *only knowing* [33] which try to model concepts similar to our "knowing at most" and "knowing exactly". Here, we compare these ideas.

Several authors have analyzed the knowledge state of an agent who knows a (set of) formula(e) [34, 35, 36, 37]. Levesque [33] introduced a logic in which *only knowing* can be expressed in the logical language. Briefly speaking, Levesque's language is of first order[2] and has two unary epistemic connectives **B** and **O**.[3] Semantically, a *world* is a truth assignment to the primitive sentences, and satisfaction of a formula is defined relative to a pair W, w where W is the set of worlds the agent considers possible and w is the "real" world[4] (the world corresponding to the correct state of affairs). A sentence $\mathbf{B}\alpha$ is true in W, w iff α is true in W, w' for every $w' \in W$; **B** is the traditional belief/knowledge operator in modal epistemic logic. A sentence $\mathbf{O}\alpha$ is true in W, w iff $\mathbf{B}\alpha$ is true in W, w and $w' \in W$ for every w' such that α is true in W, w'. $\mathbf{O}\alpha$ expresses that the agent only knows α; the set of possible worlds is as large as possible consistent with believing α. The **O** operator can be modeled by a "natural dual" to the **B** operator — an operator **N**. The intended meaning of $\mathbf{N}\alpha$ is that α at most is believed to be false, and $\mathbf{N}\alpha$ is true in W, w iff α is true in W, w' for every $w' \notin W$. Then, $\mathbf{O}\alpha$ is true iff $\mathbf{B}\alpha$ and $\mathbf{N}\neg\alpha$ is true; **B** specifies a lower bound and **N** specifies an upper bound on what is believed.

[2] The logic was only shown to be complete for the unquantified version of the language, the full version was later shown to be incomplete [38].

[3] Levesque only considers a single agent, but his approach has later been extended to the multi-agent case [39].

[4] Note that this corresponds to the semantical assumptions of the modal logic $S5$ for one agent.

Levesque's logic of only knowing and the extended syntactic epistemic logic we have discussed in this paper set out to model similar concepts, i.e. *all an agent knows* — expressed as $\mathbf{O}\alpha$ by Levesque and $\Diamond X$ by us (for simplicity, we here assume the single-agent case and write the epistemic operators without subscript). In order to compare these two notions, we take a closer look at a possible correspondence between the operators $\mathbf{N}$ and $\triangledown$.

The first question is whether given a formula α there is an X such that $\triangledown X$ corresponds to $\mathbf{N}\alpha$. The intended interpretation of $\mathbf{N}\alpha$ is that the agent "knows at most $\neg\alpha$", so "corresponds" should at least require that $\neg\alpha \in X$. However, the following is a sound inference rule in Levesque's logic:

$$\frac{\alpha \rightarrow \beta}{\mathbf{N}\alpha \rightarrow \mathbf{N}\beta}$$

and it should thus be the case that $\neg\beta \in X$ too. That does not follow automatically in our logic, and we cannot define X to include all such $\neg\beta$s since there are infinitely many and X must be finite. Thus, we cannot express $\mathbf{N}\alpha$ directly by $\triangledown_i X$.

The second question is the other direction: given a set X, is there an α such that $\mathbf{N}\alpha$ corresponds to $\triangledown X$? Again, we should at least require $\mathbf{N}\neg\bigwedge X$ to hold, since otherwise the agent might know something which is not specified by X. It follows that, to get the proper semantics for negation, we should require that $\neg\mathbf{N}\neg\bigwedge X$ holds whenever $\neg\triangledown X$ holds. But take X such that the conjunction is an inconsistency: $\bigwedge X = \bot$. Now $\mathbf{N}\neg\bot$ does hold — but it holds trivially: it is in fact valid in Levesque's logic. So if $\neg\triangledown X$, for the given X, it can never be the case that $\neg\mathbf{N}\neg\bigwedge X$ holds. Thus, for inconsistent X, these two formulae $\triangledown X$ and $\mathbf{N}\alpha$ do not have corresponding semantics since the latter can never be false while the former can. In other words, we cannot express $\triangledown X$ directly by $\mathbf{N}\alpha$, either.

As an illustration of a situation where our $\Diamond$ operator might express an agent's knowledge more realistically than the $\mathbf{O}$ operator is when we want to model an agent's explicit knowledge at a point in time when it has computed *only* the formulae $p \rightarrow q$ and p (and not yet q). From $\mathbf{O}((p \rightarrow q) \wedge p)$ it follows that $\mathbf{B}q$ – which is not true – but from $\Diamond\{p \rightarrow q, p\}$ it does not follow that Kq.

Although these observations are not a full formal analysis of the respective expressive power of the two logics, they seem to confirm the idea that the syntactic and semantic approaches are fundamentally different.

6 Conclusions

In this paper we investigated syntactic operators, similar to those used in several logical models of multi-agent systems such as the logic of general awareness [11].

We introduced a "knows at most" operator in order to increase the expressiveness of the epistemic language with respect to syntactic structures, and investigated strong axiomatization of the resulting logic. The new operator destroyed semantic compactness and thus the possibility of a strongly complete finitary axiomatization, but we presented a strongly complete infinitary axiomatization. An interesting result is that we have a strongly complete finitary axiomatization if we make the assumption that the agents can know something which is not expressible in the logical language. The results are a contribution to the logical foundation of multi-agent systems.

Related work include the classical syntactic treatment of knowledge mentioned in the introduction and modeled in a possible worlds framework by [1] as described in Section 2. The $\triangledown_i$ operator is new in the context of syntactic models. It is however, as we discussed in Section 5, similar to Levesque's **N** operator [33]. Although a full formal comparison between the relative expressive power of these two logics are outside the scope of this paper, and is left as an opportunity for future work, the discussion in Section 5 indicates that despite apparent similarities the syntactic and the semantic approaches are fundamentally different — also when it comes to "only knowing". We saw that a correspondence between the operators was obstructed by that fact that the syntactic logic has no closure condition (in the first "question" in Section 5) and the fact that it has no consistency condition (in the second "question" in Section 5). The syntactic "at most" operator is an alternative to the "only knowing" operator when these two conditions cannot be assumed.

In [31] we investigate the $\triangle_i$ and $\triangledown_i$ operators in the special case of agents who can know only *finitely* many syntactic formulae at the same time. Completeness results for such finitely restricted agents build upon the results presented in this paper. Another possibility for future work is to study other special classes of syntactic structures.

In this paper we have only studied the *static* aspect of syntactic knowledge. In [14], we discuss how syntactic knowledge can evolve as a result of reasoning and communication, i.e. a *dynamic* aspect of knowledge.

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References

1. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: Reasoning About Knowledge. The MIT Press, Cambridge, Massachusetts (1995)
2. Meyer, J.J.C., van der Hoek, W.: Epistemic Logic for AI and Computer Science. Cambridge University Press, Cambridge, England (1995)
3. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge University Press (2001)
4. Hintikka, J.: Impossible possible worlds vindicated. Journal of Philosophical Logic **4** (1975) 475–484
5. Eberle, R.A.: A logic of believing, knowing and inferring. Synthese **26** (1974) 356–382
6. Moore, R.C., Hendrix, G.: Computational models of beliefs and the semantics of belief sentences. Technical Note 187, SRI International, Menlo Park, CA (1979)

7. Halpern, J.Y., Moses, Y.: Knowledge and common knowledge in a distributed environment. Journal of the ACM **37** (1990) 549–587
8. Konolige, K.: A Deduction Model of Belief and its Logics. PhD thesis, Stanford University (1984)
9. Konolige, K.: Belief and incompleteness. In Hobbs, J.R., Moore, R.C., eds.: Formal Theories of the Commonsense World. Ablex Publishing Corporation, New Jersey (1985) 359 – 403
10. Konolige, K.: A Deduction Model of Belief. Morgan Kaufmann Publishers, Los Altos, California (1986)
11. Fagin, R., Halpern, J.Y.: Belief, awareness and limited reasoning. Artificial Intelligence **34** (1988) 39–76 A preliminary version appeared in [16].
12. Drapkin, J., Perlis, D.: Step-logics: An alternative approach to limited reasoning. In: Proceedings of the European Conference on Artificial Intelligence, Brighton, England (1986) 160–163
13. Elgot-Drapkin, J., Kraus, S., Miller, M., Nirkhe, M., Perlis, D.: Active logics: A unified formal approach to episodic reasoning. Techn. Rep. CS-TR-4072 (1999)
14. Ågotnes, T., Walicki, M.: Syntactic knowledge: A logic of reasoning, communication and cooperation. In Ghidini, C., Giorgini, P., van der Hoek, W., eds.: Proceedings of the Second European Workshop on Multi-Agent Systems (EUMAS), Barcelona, Spain (2004)
15. Alechina, N., Logan, B., Whitsey, M.: A complete and decidable logic for resource-bounded agents. In: Proc. of the Third Intern. Joint Conf. on Autonomous Agents and Multi-Agent Syst. (AAMAS 2004), ACM Press (2004) 606–613
16. Fagin, R., Halpern, J.Y.: Belief, awareness and limited reasoning. In: Proceedings of the Ninth International Joint Conference on Artificial Intelligence, Los Angeles, CA (1985) 491–501
17. Hadley, R.F.: Fagin and halpern on logical omniscience: A critique with an alternative. In: Proc. Sixth Canadian Conference on Artificial Intelligence, Montreal, University of Quebec Press (1986) 49 – 56
18. Konolige, K.: What awareness isn't: A sentential view of implicit and explicit belief. In Halpern, J.Y., ed.: Theoretical Aspects of Reasoning About Knowledge: Proceedings of the First Conference, Los Altos, California, Morgan Kaufmann Publishers, Inc. (1986) 241–250
19. Huang, Z., Kwast, K.: Awareness, negation and logical omniscience. In van Eijk, J., ed.: Logics in AI, Proceedings JELIA'90. Volume 478 of Lecture Notes in Computer Science. Springer-Verlag, Berlin (1991) 282–300
20. Thijsse, E.: On total awareness logics. In de Rijke, M., ed.: Diamonds and Defaults. Kluwer Academic Publishers, Dordrecht (1993) 309–347
21. Halpern, J.: Alternative semantics for unawareness. Games and Economic Behaviour **37** (2001) 321–339
22. Montague, R.: Pragmatics. In Klibansky, R., ed.: Contemporary Philosophy: A Survey. I. La Nuova Italia Editrice, Florence (1968) 102–122 Reprinted in [40, pp. 95 – 118].
23. Montague, R.: Universal grammar. Theoria **36** (1970) 373–398 Reprinted in [40, pp. 222 – 246].
24. Scott, D.S.: Advice on modal logic. In Lambert, K., ed.: Philosophical Problems in Logic. D. Reidel Publishing Co., Dordrecht (1970) 143–173
25. Moreno, A.: Avoiding logical omniscience and perfect reasoning: a survey. AI Communications **11** (1998) 101–122
26. Sim, K.M.: Epistemic logic and logical omniscience: A survey. International Journal of Intelligent Systems **12** (1997) 57–81

27. Wansing, H.: A general possible worlds framework for reasoning about knowledge and belief. Studia Logica **49** (1990) 523–539
28. Rantala, V.: Impossible worlds semantics and logical omniscience. Acta Philosophica Fennica **35** (1982) 106–115
29. Rantala, V.: Quantified modal logic: non-normal worlds and propositional attitudes. Studia Logica **41** (1982) 41–65
30. Ågotnes, T., Walicki, M.: A logic for reasoning about agents with finite explicit knowledge. In Tessem, B., Ala-Siuru, P., Doherty, P., Mayoh, B., eds.: Proc. of the 8th Scandinavian Conference on Artificial Intelligence. Frontiers in Artificial Intelligence and Applications, IOS Press (2003) 163–174
31. Ågotnes, T., Walicki, M.: Complete axiomatizations of finite syntactic epistemic states. In Baldoni, M., Endriss, U., Omicini, A., Torroni, P., eds.: The Third International Workshop on Declarative Agent Languages and Technologies (DALT 2005), Workshop Notes, Utrecht, the Netherlands (2005) To appear in Lecture Notes in Artificial Intelligence (LNAI), Springer-Verlag, 2006.
32. de Lavalette, G.R., Kooi, B., Verbrugge, R.: Strong completeness for propositional dynamic logic. In Balbiani, P., Suzuki, N.Y., Wolter, F., eds.: Preliminary Proceedings of AiML2002, Institut de Recherche en Informatique de Toulouse IRIT (2002) 377–393
33. Levesque, H.J.: All I know: a study in autoepistemic logic. Artificial Intelligence **42** (1990) 263–309
34. Konolige, K.: Circumscriptive ignorance. In Waltz, D., ed.: Proceedings of the National Conference on Artificial Intelligence, Pittsburgh, PA, AAAI Press (1982) 202–204
35. Moore, R.C.: Semantical considerations on nonmonotonic logic. In Bundy, A., ed.: Proceedings of the 8th International Joint Conference on Artificial Intelligence, Karlsruhe, FRG, William Kaufmann (1983) 272–279
36. Halpern, J.Y., Moses, Y.: Towards a theory of knowledge and ignorance. In Apt, K.R., ed.: Logics and Models of Concurrent Systems. Springer-Verlag, Berlin (1985) 459–476
37. Halpern, J.Y.: A theory of knowledge and ignorance for many agents. Journal of Logic and Computation **7** (1997) 79–108
38. Halpern, J.Y., Lakemeyer, G.: Levesque's axiomatization of only knowing is incomplete. Artificial Intelligence **74** (1995) 381–387
39. Halpern, J.Y., Lakemeyer, G.: Multi-agent only knowing. In Shoham, Y., ed.: Theoretical Aspects of Rationality and Knowledge: Proceedings of the Sixth Conference (TARK 1996). Morgan Kaufmann, San Francisco (1996) 251–265
40. Montague, R.: Formal Philosophy. Yale University Press, New Haven, CT (1974)