On the Quotient Structure of Computably Enumerable Degrees Modulo the Noncuppable Ideal*

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Abstract. We show that minimal pairs exist in the quotient structure of $\mathcal{R}$ modulo the ideal of noncuppable degrees.

In the study of mathematical structures it is very common to form quotient structures by identifying elements in some equivalence classes. By varying the equivalence relations, the corresponding quotient structures often reveal certain hidden features of the original structure. In this paper, we focus on the upper semi-lattice of computably enumerable degrees and the equivalence relations are induced by definable ideals.

We begin with introducing some notations and terminologies. Let $\mathcal{R}$ be the class of computably enumerable degrees or simply c.e. degrees.

Definition 1. *We say that a nonempty subset I of $\mathcal{R}$ is an* ideal *of $\mathcal{R}$ if I is downward closed and closed under join. In other words, the following conditions are satisfied.*

(a) If $\mathbf{a}$ is in I and $\mathbf{b} \leq \mathbf{a}$ then $\mathbf{b}$ is in I;
(b) If $\mathbf{a}$ and $\mathbf{b}$ are in I, then their least upper bound, denoted by $\mathbf{a} \vee \mathbf{b}$, is in I.

We say that an ideal I is definable *if there is a first-order formula $\varphi(x)$ over the partial order language $L = \{\leq\}$ such that a c.e. degree $\mathbf{a} \in I$ if and only if $\mathcal{R} \models \varphi(\mathbf{a})$.*

Each ideal I of $\mathcal{R}$ naturally induced an equivalence relation $\equiv_I$ as follows. For any two c.e. degrees $\mathbf{a}$ and $\mathbf{b}$, define

$$\mathbf{a} \leq_I \mathbf{b} \text{ if and only if } \exists \mathbf{x} \in I(\mathbf{a} \leq_T \mathbf{b} \vee \mathbf{x}),$$

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and

$$\mathbf{a} \equiv_I \mathbf{b} \text{ if and only if } \mathbf{a} \leq_I \mathbf{b} \text{ and } \mathbf{b} \leq_I \mathbf{a}.$$

It is easy to see that $\equiv_I$ is an equivalence relation. We use $[\mathbf{a}]$ to denote the equivalence class containing the c.e. degree $\mathbf{a}$. The *quotient structure* $\mathcal{R}/I$ then consists of all equivalence classes $[\mathbf{a}]$. Clearly, the least element $[\mathbf{0}]$ is the ideal I and the greatest element is $\{\mathbf{0}'\}$. Furthermore, with respect to the induced join relation, $\mathcal{R}/I$ is also an upper-semi lattice. We now look at some quotient structures of $\mathcal{R}$ modulo some definable ideals.

The topic of definable ideals in $\mathcal{R}$ is beyond the scope of this paper. Some recent developments can be found in Nies [4], Yu and Yang [7] and Jockusch, Li and Yang [2]. All newly discovered ideals are defined by formulas involving coding techniques, hence very complicated. Until now, there are only two proper ideals which can be defined by relatively simple formulas: One consists of the cappable degrees, the other of noncuppable ones. Recall:

Definition 2. *1. A c.e. degree* $\mathbf{a}$ *is called* cappable *if it is a half of a minimal pair, that is, there exists a nonzero* $\mathbf{b} \in \mathcal{R}$ *such that the infimum of* $\mathbf{a}$ *and* $\mathbf{b}$ *exists and equal to* $\mathbf{0}$.

2. A c.e. degree $\mathbf{a}$ *is called* noncuppable *if for all incomplete degrees* $\mathbf{b} \in \mathcal{R}$*, the join of* $\mathbf{a}$ *and* $\mathbf{b}$ *remains incomplete.*

It is easy to verify from definition that the noncuppable degrees form an ideal; and the existence of nonzero noncuppable c.e. degrees was first proved by Cooper and Yates and later generalized by Harrington (see Miller [3]). However, it is highly nontrivial that the cappable degrees are closed under join, in fact, it follows from a deep result by Ambos-Spies, Jockusch, Shore and Soare [1]. We use M to denote the ideal of cappable degrees.

In the 1980's, partially motivated by Shoenfield conjecture (see Schwarz [5]), people started to investigate the quotient structure $\mathcal{R}/M$, for example, Ambos-Spies and Schwarz showed that $\mathcal{R}/M$ satisfies the splitting property (see Yi [6]).

Theorem 1. *For any nonzero* $[\mathbf{a}]$ *in* $\mathcal{R}/M$*, there are* $[\mathbf{a_1}], [\mathbf{a_2}] < [\mathbf{a}]$ *such that* $[\mathbf{a_1}] \vee [\mathbf{a_2}] = [\mathbf{a}]$.

Later, Yi proved that $\mathcal{R}/M$ does not satisfy Shoenfield conjecture by showing the following theorem:

Theorem 2 (Yi [6]). *The following property holds in* $\mathcal{R}/M$*: there are c.e. degrees* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *such that* $\mathbf{c} \leq \mathbf{a} \leq \mathbf{b}$, $[\mathbf{c}] < [\mathbf{a}]$ *and for all c.e. degrees* $\mathbf{w} \geq \mathbf{c}$*, either* $\mathbf{b} \leq \mathbf{w}$ *or* $\mathbf{b} \not\leq \mathbf{a} \vee \mathbf{w}$.

Until now, little is known about the quotient structure $\mathcal{R}$ modulo the noncuppable ideal. For notational simplicity let us use I to denote the ideal of noncuppable c.e. degrees. The main result of this paper is to show that there is a minimal pair in the quotient structure $\mathcal{R}/I$. Thus we are able to separate $\mathcal{R}/I$ from $\mathcal{R}/M$ by an elementary property, since there is no minimal pair in the quotient structure $\mathcal{R}/M$. Clearly $\mathcal{R}$ is not elementarily equivalent to $\mathcal{R}/I$, as the former has nonzero noncuppable degrees and the latter not.

We will not present the full proof in this paper, instead, we will outline the plan of the proof. We hope that we provide enough intuition so that the interested readers are able to complete the proof themselves.

In the structure $\mathcal{R}/I$, two elements $[\mathbf{a}]$ and $[\mathbf{b}]$ form a minimal pair if and only if $[\mathbf{a}] \neq 0$, $[\mathbf{b}] \neq 0$ and if $[\mathbf{e}] \leq_I [\mathbf{a}], [\mathbf{b}]$ then $[\mathbf{e}] = 0$. Thus, to build a minimal pair in $\mathcal{R}/I$, it suffices to build c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{a} \notin I \text{ and } \mathbf{b} \notin I \text{ and } \forall \mathbf{e}(\mathbf{e} \leq_I \mathbf{a} \text{ and } \mathbf{e} \leq_I \mathbf{b} \Rightarrow \mathbf{e} \in I).$$

In terms of sets, it suffices to build c.e. sets A and B, whose corresponding degrees satisfy the above conditions. Fix a complete c.e. set K. The first two conjuncts say that A and B are cuppable, which are equivalent to

$$\exists C[C \not\equiv_T K \text{ and } A \oplus C \equiv_T K]$$

and

$$\exists D[D \not\equiv_T K \text{ and } B \oplus D \equiv_T K].$$

Proposition 1. *The statement*

$$\forall \mathbf{w}[(\mathbf{w} \vee \mathbf{a} = \mathbf{0}' \textit{ and } \mathbf{w} \vee \mathbf{b} = \mathbf{0}') \Rightarrow \mathbf{w} = \mathbf{0}']$$

implies the last conjunct in the minimal pair definition.

Proof. Suppose $\mathbf{e} \leq_I \mathbf{a}$ and $\mathbf{e} \leq_I \mathbf{b}$. Then there is an $\mathbf{x} \in I$ such that $\mathbf{e} \leq \mathbf{a} \vee \mathbf{x}$ and $\mathbf{e} \leq \mathbf{b} \vee \mathbf{x}$. We show that $\mathbf{e} \in I$. If $\mathbf{w} \vee \mathbf{e} = \mathbf{0}'$, then $\mathbf{a} \vee \mathbf{w} \vee \mathbf{x} \geq \mathbf{e} \vee \mathbf{w} = \mathbf{0}'$. As $\mathbf{x}$ is noncuppable, $\mathbf{a} \vee \mathbf{w} = \mathbf{0}'$. Similarly $\mathbf{b} \vee \mathbf{w} = \mathbf{0}'$. By assumption, $\mathbf{w} = \mathbf{0}'$, which shows that $\mathbf{e} \in I$.

Theorem 3. *There exist c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $[\mathbf{a}]$ *and* $[\mathbf{b}]$ *form a minimal pair in* $\mathcal{R}/I$.

We construct c.e. sets A and B together with their companion c.e. sets C and D respectively such that they form two splitting pairs of K, i.e., C and D are incomplete and

$$A \oplus C \equiv_T K \text{ and } B \oplus D \equiv_T K,$$

and A and B share no incomplete cupping witnesses.

More precisely, we need to satisfy the following requirements:

- P: (Splitting requirement) We build Turing functionals Γ and Δ such that $\Gamma^{AC} = K$ and $\Delta^{BD} = K$.

Fix recursive enumerations of Turing functionals $\{\Theta_e\}_{e\in\omega}$, $\{\Phi_e\}_{e\in\omega}$ and $\{\Psi_e\}_{e\in\omega}$.

- N_{2e}: $\Theta_e^C \neq E$; and
- N_{2e+1}: $\Theta_e^D \neq E$, where E is an auxiliary set built by us.
- R_e: If $\Phi_{e_1}^{AW_{e_0}} = \Psi_{e_2}^{BW_{e_0}} = F$ then there is a Turing functional Ω_e such that $\Omega_e^{W_{e_0}} = K$, where $e = \langle e_0, e_1, e_2\rangle$ under standard coding and F is an auxiliary c.e. set built by us.

Note that the requirements N_e and R_e together imply that A and B are incomplete: If $A \equiv_T K$ then there exists some Φ such that $\Phi^{AD} = K$; on the other hand, $\Delta^{BD} = K$; by R-requirements, $D \equiv_T K$, contradicting to N-requirements.

First Approximation of Strategies. We give the splitting requirement P the highest priority. The construction will be divided into even and odd stages. The even stages are devoted to the definition and correction of Γ and Δ, which will be done outside the priority tree; whereas the odd stages are devoted to the satisfactions of N and R, which will be done on the priority tree.

At even stages, we satisfy the P-requirements as follows: Choose the least k such that either $\Gamma^{AC}(k) = 0 \neq K(k)$ or $\Delta^{BD}(k) = 0 \neq K(k)$ or $\Gamma^{AC}(k)$ is undefined or $\Delta^{BD}(k)$ is undefined. If it is the first case, that is, $\Gamma^{AC}(k) = 0 \neq K(k)$, then enumerate the use $\gamma(k)$ into C, redefine $\Gamma^{AC}(k) = 1$ with use -1. If it is the third case, that is, $\Gamma^{AC}(k)$ is undefined, then define $\Gamma^{AC}(k) = K(k)$ with fresh use $\gamma(k)$. We do it symmetrically for functional Δ. These off-tree activities have conflicts with the N-requirements, which we will solve in a moment.

We now look at the activities on the tree, which happen during odd stages.

The strategy to satisfy an N-requirement, say N_{2e}, is as follows: Pick a fresh witness x targeting E, wait until $\Theta_e^C(x)\downarrow= 0$, put x into E and preserve C up to the use $\theta(x)$. The requirement has two outcomes: 1 for waiting and 0 for success. Naturally we order 0 to the left of 1 on the priority tree. The net effect is a finitary restraint on C. Again, we delay the discussion of the conflict with P. The strategy for N_{2e+1} is done by replacing C by D and Γ by Δ. To avoid confusion, each strategy will choose its witness x from its own infinite computable set. This will be done by letting α choose its witnesses from $\omega^{[\alpha]}$.

The strategy to satisfy the R-requirement R_e is as follows: We will have a main R-strategy R_e and infinitely many substrategies $S_{e,i}$. The job for the mother node α is to measure the length of agreement function $l(\alpha, s)$ defined by

$$l(\alpha,s) = \mu y[\Phi_{e_1}^{AW_{e_0}}(y)\uparrow \text{ or } \Psi_{e_2}^{BW_{e_0}}(y)\uparrow \text{ or } \Phi_{e_1}^{AW_{e_0}}(y)\downarrow\neq \Psi_{e_2}^{BW_{e_0}}(y)\downarrow \\ \text{or } (\Phi_{e_1}^{AW_{e_0}}(y)\downarrow= \Psi_{e_2}^{BW_{e_0}}(y)\downarrow= z \text{ but } z \neq F(y))].$$

We say that the stage s is *α-expansionary*, if $s = 0$ or $l(\alpha, t) < l(\alpha, s)$ for all $t < s$. The outcome of R at node α is either ∞, indicating s is an α-expansionary stage, or 0 when s is not.

Extending the outcome $\alpha\hat{\ }0$, there will be no substrategies working for $S_{e,i}$. Let β be a node extending $\alpha\hat{\ }\infty$ and working for the subrequirement $S_{e,i}$. β is responsible for defining $\Omega^W(i)$ and keeping the use $\omega(i) > \max\{\varphi(z_i), \psi(z_i)\}$ for some number z_i. β acts (naively) as follows:

- β first chooses a fresh number z_i, in particular, $z_i \notin F$ at this moment.
- Wait until $l(\alpha, s) > z_i$.
- Select $\omega(i) > \max\{\varphi(z_i), \psi(z_i)\}$, define $\Omega^W(i) = K(i)$ with use $\omega(i)$ and set a restraint on A and B of amount $\omega(i)$.
- If either the uses $\varphi(z_i)[s] \neq \varphi(z_i)[s^-]$ or $\psi(z_i)[s] \neq \psi(z_i)[s^-]$, where s^- is the previous stage at which β was accessible, let ∞ be the outcome.

– When i enters K at some later stage t, we enumerate z_i into F, this z_i will be discarded forever (of course, some other fresh z_i might be chosen later). At the next α-expansionary stage, W must have changed below $\omega(i)$, since we have kept A- or B-side (in fact, we did more than enough, we have kept both), thus we are able to redefine $\Omega^W(i)$.

β will have two outcomes: ∞ for divergence of $\varphi(z_i)$ or $\psi(z_i)$ and 0 for the successful definition and correctness of $\Omega^W(i) = K(i)$.

However, this naive version of β has problems about the consistency of Ω: Once $\Omega^W(i)$ is defined on the tree by a node β, any other node must respect β's restraint. This would bring conflicts to the nodes to the left of β. To address this issue, we modify the strategy as follows: Before a node σ (not necessarily β) is visited, we must make sure that all $\Omega_e^W(i)$, which were defined by some nodes to its right, are undefined (we will refer to it as "Clearing Ω-use") . More precisely, suppose that we are at a node σ^- on the priority tree, all Ω-uses to its right have been cleared and we want to visit σ. Before visiting σ, we check whether there is $\Omega^W(i)$ which is defined by some node β extending σ^- and to the right of σ. If no, we can visit σ. If yes, we must put all $z_\beta(i)$ into F and put a restraint on either A or B side. When σ^- is visited again for the next time, W must have changed below $\Omega(i)$, hence all Ω-uses are cleared. We then can visit σ.

By making this modification, we may select a complete c.e. set K_0 which is a subset of even numbers; and use F which is a subset of odd number solely for clearing Ω-uses. The revised R_e requirement looks like:

– R_e: If $\Phi_{e_1}^{AW_{e_0}} = \Psi_{e_2}^{BW_{e_0}} = K_0 \cup F$ then there is a Turing functional Ω_e such that $\Omega_e^{W_{e_0}} = K_0$, where $e = \langle e_0, e_1, e_2\rangle$ under standard coding and F is an auxiliary c.e. set built by us.

The difference now is that we do not act to correct Ω^W, which becomes automatic. Instead, we must clear Ω^W-uses.

Revised Strategies. We now discuss the conflicts among the strategies.

The actions done off the tree have no direct conflicts with the R- and S-strategies, as the numbers are put into the "buffer" sets C and D. However these actions would make C and D complete, a direct threat to the N-strategies. For this reason N-strategies must divert some of the γ-uses into A. (We state the strategies only for N_{2e}, as it is symmetric for N_{2e+1}.) We modify the N-strategies as follows.

– Besides doing the Friedberg-Muchnik diagonalization, N-strategy picks a threshold j. (This j will be larger than all γ-uses mentions by any S-strategy $\beta \leq N$, where $\leq$ is the tree order.)
– Once the threshold is chosen, N will lift all $\gamma(v)$ for $v \geq j$ over the use $\theta(x)$. More precisely:
  • For all $v \geq j$, if $\gamma(v)$ is defined and $\gamma(v) < \theta(x)$, then put $\gamma(j)$ (need to add conventions for uses in the introduction part) into A, and declare

all $\Gamma^{AC}(v)$ for $v \geq j$ undefined. Stop the construction and initialize all nodes extending and to the right of N. We stop the construction because we want to put elements into either A or B but not both. It is worth mentioning that being able to put elements into one side makes our strategy different from the usual noncuppable strategy.

- If w enters K for some $w < j$, then we initialize the N-strategy.

A crude analysis of the impact of the revised N-strategy goes as follows: If it is on the true path, then its threshold j will be fixed and it is so big that $\gamma(j)$ will not injure any S-node $\leq N$. Furthermore, after the stage at which $K \upharpoonright j$ is fixed, say t, N will never be initialized by the off-tree activities. After stage t, N will act at most once. Thus the initialization of S-node due to the action of N happens only finitely often. Eventually, S-strategy will be successful.

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