Optimality Zone Algorithms for Hybrid Systems: Efficient Algorithms for Optimal Location and Control Computation

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Abstract. A general Hybrid Minimum Principle (HMP) for hybrid optimal control problems (HOCPs) is presented in [1, 2, 3, 4] and in [4, 5], a class of efficient, provably convergent Hybrid Minimum Principle (HMP) algorithms were obtained based upon the HMP. The notion of optimality zones (OZs) ([3, 4]) provides a theoretical framework for the computation of optimal location (i.e. discrete state) schedules for HOCPs (i.e. discrete state sequences with the associated switching times and states). This paper presents the algorithm HMPOZ which fully integrates the prior computation of the OZs into the HMP algorithms class. Summing (a) the computational investment in the construction of the OZs for a given HOCP, and (b) the complexity of (i) the computation of the optimal schedule, (ii) the optimal switching time and optimal switching state sequence, and (iii) the optimal continuous control input, yields a complexity estimate for the algorithm HMPOZ which is linear (i.e. $O(L)$) in the number of switching times L.

1 Introduction

Over the last few years the notion of a hybrid control system with continuous and discrete states and dynamics has crystallized and various classes of optimal control problems for such systems have been formalized (see for example [3, 6, 2, 1, 7, 8, 9]). In particular, generalizing the standard Minimum Principle (MP), Sussmann [10] and Riedinger et al. [11], among other authors, have given versions of the Hybrid Minimum Principle (HMP) with indications of proof methods. An explicit theory for the two stage controlled switching optimal control problem was given by Tomiyama in [12] and a complete, rigorous treatment of the HMP is given in [13, 14] for the case of *a priori* fixed location sequences. In [1, 2, 3, 4] a set of necessary conditions for hybrid optimal control problems (HOCPs) was derived which constitutes a general Hybrid Minimum Principle (HMP); based upon this, a class of efficient Hybrid Minimum Principle (HMP)

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algorithms has been constructed [5] and their convergence established. Next, in [3, 4] the notion of optimality zones (OZs) was introduced as a theoretical framework enabling the computation of optimal schedules (i.e. location sequences with the associated switching times and states) for HOCPs. A distinct approach to the computational solution of HOCPs with fixed schedules is to be found in [7], while [15, 16, 17] present progress on parallel work on the solution of HOCPs including schedule optimization using a location (i.e. discrete state) insertion method.

The contributions of this paper include: (i) the algorithm HMPOZ which fully integrates the prior computation (termed the PREP computation) of the OZs into the HMP algorithms of [4, 5]; and (ii) computed examples of the application of HMPOZ to a bilinear quadratic regulator HOCP, demonstrating the efficacy of HMPOZ.

The computational complexity of HMPOZ has two components: (a) the complexity of the construction of the optimality zones for a given HOCP, which depends upon the cardinality of the discrete state set Q and the number of grid points $|G|$ but is independent of the number of switchings, and (b) the complexity of a single run of the HMP algorithm which is linear (i.e. $O(L)$) in the number of switchings L. This gives the overall complexity of HMPOZ as $O(|G|^2 \cdot |Q|) + O(L)$; this is to be compared with the geometric (i.e. $O(|Q|^L)$) growth of a direct combinatoric search over the set of location sequences.

Efficient Dynamic Programming (DP) based computational methods exist for certain classes of standard optimal control problems (see [18, 19]); furthermore, in case the upper bound $\bar{L}$ on the number of switchings is infinite (see [8, 4]) or the switchings occur at fixed instants, numerical methods for HOCP may, in principle, be formulated within a DP framework. However, severe complexity issues arise for DP based methods when the constraint $\bar{L} < \infty$ must be taken into account at each iterative step of a DP procedure; these do not arise for local optima seeking methods such as HMPOZ.

While the computational complexity of PREP for HMPOZ and DP methods increases geometrically with the dimension of the continuous state space $\mathbb{R}^n$, the complexity of HMPOZ implementations increase proportionally to that of the TPBVP methods used by HMP. For reasons of space, and to concentrate on the dependence on $\bar{L}$, the examples in this paper concern scalar systems.

The notion of optimality zones must be distinguished from the so-called "switching regions" presented in [20, 21, 22]; switching regions partition the continuous state space of autonomous (steady state) hybrid systems whereas optimality zones partition the Cartesian product of the system's time and state space ($\mathbb{R}^1 \times \mathbb{R}^n$) with itself, that is to say, they partition $(\mathbb{R}^1 \times \mathbb{R}^n)^2$. As explained in Section 4, these partitions are defined for any given finite horizon hybrid optimal control problem (HOCP) and their specification is completely independent of the number of switchings L in the associated HOCP.

2 Hybrid Optimal Control Theory

In this paper we consider hybrid systems which in each location are governed by globally controllable non-linear dynamics of the form

$$\mathbb{H} : \dot{x}_q = f_q(x_q, u), \quad q \in Q \mathrel{\underline{\Delta}} \{1, 2, \ldots, |Q|\}.$$

At a controlled location transition at an instant t, $t \in [t_0, t_f]$, the piecewise constant, right continuous, Q valued, discrete state (component) trajectory satisfies

$$\mathbb{H} : q(t-) = q_i \in Q, \quad q(t) = q_j \in Q, \quad q_i \neq q_j.$$

In this paper no constraints are imposed on the dynamics of the location transition while in [4, 5] the controlled transitions satisfy the Q-dependent dynamics of the form $q_j = \Gamma(q_i, \sigma_{ij})$, where σ_{ij} is a partially defined discrete input; however, the algorithms presented here are easily extended to the more general case.

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and an upper bound on the number of switchings $\bar{L} \leq \infty$. Let $S_L = ((t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L))$ be a hybrid switching sequence and let $I_L \mathrel{\underline{\Delta}} (S_L, u)$, $u \in \mathcal{U}^{\mathrm{cpt}}$, $\bar{L} \leq \infty$, be a hybrid input trajectory which (subject to the assumptions of [4, 5]) results in a (necessarily unique) hybrid execution and is such that $L \leq \bar{L}$ switchings occur on the time interval $[t_0, t_f]$. Here the set of *admissible input control functions* is $\mathcal{U}^{\mathrm{cpt}} \mathrel{\underline{\Delta}} \mathcal{U}(U^{\mathrm{cpt}}, L_\infty([0, t_f]))$, the set of all bounded measurable functions on some interval $[0, t_f]$, taking values in the compact set U^{cpt}. Further let the collection of such inputs be denoted $\{I_L\}$. We define the hybrid cost function as:

$$J : J(t_0, t_f, h_0; I_L, \bar{L}, \mathcal{U}^{\mathrm{cpt}}) \mathrel{\underline{\Delta}} \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s))\, ds + g(x_{q_L}(t_f)), \quad (1)$$

where for $i = 0, 1, \ldots, L$,

$$\begin{aligned}
&\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad a.e.\ t \in [t_i, t_{i+1}),\\
&u(t) \in U^{\mathrm{cpt}} \subset \mathbb{R}^n,\\
&u(\cdot) \in L_\infty(U^{\mathrm{cpt}}),\\
&h_0 = (q_0, x_{q_0}(t_0)) = (q_0, x_0),\\
&x_{q_{i+1}}(t_{i+1}) = \lim_{t \uparrow t_{i+1}} x_{q_i}(t), \text{ and}\\
&t_{L+1} = t_f < \infty, \qquad L \leq \bar{L} \leq \infty.
\end{aligned}$$

Definition 1. *([4, 5], Hybrid Optimal Control Problem (HOCP)) Given a hybrid system $\mathbb{H}$, loss functions $\{l_q,\ q \in Q\}$, initial and final times, t_0, t_f, the initial hybrid state $h_0 = (q_0, x_0)$, and an upper bound on the number of switchings $\bar{L} \leq \infty$, the* hybrid optimal control problem *(HOCP($t_0, t_f, x_0, \bar{L}, \mathcal{U}^{\mathrm{cpt}}$)), is to find the infimum $J^0(t_0, t_f, h_0, \bar{L}, \mathcal{U}^{\mathrm{cpt}})$ of the hybrid cost function $J(t_0, t_f, h_0; I_L, \bar{L}, \mathcal{U}^{\mathrm{cpt}})$ over the family of input trajectories $\{I_L\}$.*

If a hybrid input trajectory I_{L^0} exists which realizes $J^0(t_0, t_f, h_0, \bar{L}, \mathcal{U}^{\mathrm{cpt}})$ it is called a hybrid optimal control *for the HOCP($t_0, t_f, x_0, \bar{L}, \mathcal{U}^{\mathrm{cpt}}$).* □

In Theorem 1 we state the necessary conditions, for the controlled switchings case, upon which the algorithms of this paper are based; the theorem is stated for the cases where the control takes values in the compact set U^{cpt}. The reader is referred to [4] and the associated paper [5] for a complete exposition of the HMP necessary conditions covering compact and open bounded control value sets and both the autonomous and controlled switchings cases.

Theorem 1. *([4, 5]) Consider a hybrid system* $\mathbb{H}$ *and the HOCP(*$t_0, t_f, x_0, \bar{L}, \mathcal{U}^{\text{cpt}}$*), and define*

$$H_q(x, \lambda, u) = \lambda^T f_q(x, u) + l(x, u), \quad x, \lambda \in \mathbb{R}^n, \quad u \in U^{\text{cpt}}, \quad q \in Q.$$

1) *Let* $J^0(t_0, t_f, h_0, \mathcal{U}^{\text{cpt}}) = \inf_{\{I_L\}} J^0(t_0, t_f, h_0, I_L, \bar{L}, \mathcal{U}^{\text{cpt}})$ *be realized at* $I^0_{L^0}$, (x^0, q^0).
2) *Let* $t_1, t_2, \ldots, t_{L^0}$, *denote the switching times along the optimal trajectory* (x^0, q^0).
3) *Assume that either (a)* $\bar{L} < \infty$ *and* $L^0 + 2 \le \bar{L}$, *or (b)* $\bar{L} = \infty$ *and* $L^0 < \infty$.

Then

(i) There exists a (continuous to the right), piecewise absolutely continuous adjoint process λ^0 *satisfying*

$$\dot{\lambda}^0 = -\frac{\partial H_{q^0(t)}}{\partial x}(x^0, \lambda^0, u^0), \quad t \in (t_j, t_{j+1}), \quad j \in \{0, 1, 2, \ldots, L^0\}, \tag{2}$$

where $t_{L^0+1} = t_f$ *and where the following boundary value conditions hold with* $\lambda^0(t_0)$ *free:*
(a) $\lambda^0(t_f) = \nabla_x g(x^0(t_f))$.
(b) If t_j *is a switching time, then*

$$\lambda^0(t_j-) \equiv \lambda^0(t_j) = \lambda^0(t_j+), \quad j \in \{0, 1, 2, \ldots, L^0\}. \tag{3}$$

(ii) The Hamiltonian minimization conditions are satisfied, i.e.
(a)

$$\begin{aligned} &H_{q^0(t)}(x^0(t), \lambda^0(t), u^0(t)) \le H_{q^0(t)}(x^0(t), \lambda^0(t), v), \\ &\text{a.e. } t \in [t_j, t_{j+1}),\ \forall v \in U^{\text{cpt}},\ j \in \{0, 1, 2, \ldots, L^0\}. \end{aligned} \tag{4}$$

(b)

$$\begin{aligned} &H_{q^0(t)}(x^0(t), \lambda^0(t), u^0(t)) \le H_q(x^0(t), \lambda^0(t), u^0(t)), \\ &\text{a.e. } t \in [t_j, t_{j+1}),\ j \in \{0, 1, 2, \ldots, L^0\},\ \forall q \in Q. \end{aligned} \tag{5}$$

(iii) The following Hamiltonian continuity *condition holds at a controlled switching time* $t = t_j$

$$\begin{aligned} H(t_j-) \equiv H_{q^0(t_j-)}(t_j-) &= H_{q^0(t_j-)}(t_j) = H_{q^0(t_j)}(t_j) \\ &= H_{q^0(t_j+)}(t_j+) \equiv H(t_j+), \quad j \in \{1, 2, \ldots, L^0\}. \end{aligned}$$

□

3 HMP Conceptual Algorithm

Based on the necessary conditions for hybrid optimality in Theorem 1 we proposed the HMP algorithm in [4, 5] and established its convergence properties. This algorithm is presented below for the single switching time case but can be generalized to multiple switching times case in an obvious manner. This algorithm forms the basis of the algorithm HMPOZ which is given in Section 5.

We reproduce the HMP algorithm below; in Steps 3 and 4 $\{r_k\}$ is either a constant sequence of strictly positive numbers or is an unbounded monotonically increasing sequence.

0. Algorithm Initialization: Fix $0 < \epsilon_f \ll 1$. Let (t_s, x_s) be a nominal switching time-state pair such that $t_0 < t_s < t_f$. Set the iteration counter $k = 0$. Set $t_s^k = t_s$ and $x_s^k = x_s$. Compute the optimal control functions $u_1^k(t)$, $t_0 \le t < t_s$ and $u_2^k(t)$, $t_s \le t \le t_f$. Compute the associated state and costate trajectories and Hamiltonians over the two intervals $[t_0, t_s^k]$ and $[t_s^k, t_f]$, with the terminal state pairs (x_0, x_s^k) and (x_s^k, x_f) respectively. Also compute the new total cost $J^k(t_s^k, x_s^k)$.
1. Increment k by 1.
2. Let $z_s^k \,\underline{\Delta}\, (t_s^k, x_s^k)$ and set
$$z_s^k = z_s^{k-1} - r_k \begin{pmatrix} H_1^k(t_s^{k-1}) - H_2^k(t_s^{k-1}) \\ \lambda_2^k(t_s^{k-1}) - \lambda_1^k(t_s^{k-1}) \end{pmatrix}.$$
3. Compute the optimal control functions $u_1^k(t)$, $t_0 \le t < t_s$ and $u_2^k(t)$, $t_s \le t \le t_f$. Compute the associated state and costate trajectories and Hamiltonians over the two intervals $[t_0, t_s^k]$ and $[t_s^k, t_f]$ with the terminal state pairs (x_0, x_s^k) and (x_s^k, x_f) respectively. Next, compute the $J^k \,\underline{\Delta}\, J^k(t_s^k, x_s^k)$.
4. If $|J^k - J^{k-1}| < \epsilon_f$, then Stop; else go to Step 1.

The convergence of the HMP algorithm is established in [4, 5, 23] for the case of unbounded increasing $\{r_k\}$ by use of penalty function methods and Ekeland's variational principle. The efficiency of the HMP algorithm in comparison with other fixed discrete state sequence hybrid optimal control algorithms is discussed with illustrative examples in [4, 5].

4 Optimality Zones, Location Sequences and the HMPOZ Algorithm

Henceforth in this paper the HMP algorithm shall be treated as a modular unit in more general algorithmic procedures. In this section, the properties of optimal hybrid controlled trajectories are shown to permit the exploitation of the HMP algorithm in computational methods which converge to discrete and continuous control functions with certain local and global optimality properties.

4.1 Fundamental Implications of the DP Principle for Optimal Location Sequences

DP Principle. Along an optimal hybrid execution $(I^0_{L^0}, x^0)$ the Dynamic Programming Principle implies that the part of the hybrid input $I^0_{L^0}$ (and correspondingly the hybrid trajectory (q^0, x^0)) from the j-th switching time and state pair to the $j+1$-st switching time and state pair, $(t^0_j, x^0_j) \to (t^0_{j+1}, x^0_{j+1})$, $0 \leq j \leq L^0$, is optimal. Hence, in particular, $q^0(t)$, $t \in [t^0_j, t^0_{j+1})$, must be an optimal location for the trajectory from (t^0_j, x^0_j) to (t^0_{j+1}, x^0_{j+1}).

Non-hybrid Optimal Control Problem. It is to be noted that for each $q((t^0_j, x^0_j), (t^0_{j+1}, x^0_{j+1})) \in Q$ the optimization above is a standard (non-hybrid) optimal control problem which is not linked to an analogous optimization over any other interval.

$|Q|$ Complexity Search. We further note that for each time and state pair $\{(t_j, x_j), (t_{j+1}, x_{j+1})\}$ the set-up cost of a search over a set Q to find the optimal $q^0((t_j, x_j), (t_{j+1}, x_{j+1}))$ is proportional to $|Q|$ and is not linked to an analogous search over any other interval.

4.2 Variations in Switching Time and State and Local Optimality with respect to Discrete Location

Local Optimality for fixed $q \in Q$. In [4, 23] we show that under weak assumptions the value function $v(t, x, q)$ of HOCP is bounded and continuous in (t, x) for each $q \in Q$. For simplicity, consider the case where we have two locations, $Q = \{q_1, q_2\}$, and two controlled switchings at (t_1, x_1) and (t_2, x_2) with $t_0 \leq t_1 < t_2 \leq t_f$. Further assume that over the interval $[t_1, t_2]$ the optimal cost $J^0_{q_1}((t_1, x_1), (t_2, x_2))$ of a trajectory from x_1 to x_2 in location q_1 is strictly smaller than the corresponding cost $J^0_{q_2}((t_1, x_1), (t_2, x_2))$ in location q_2. Hence by the continuity of each $J^0_{q_i}$, $i = 1, 2$, in $((t_1, x_1), (t_2, x_2))$, there is a neighbourhood $N_{((t_1,x_1),(t_2,x_2))}$ of $((t_1, x_1), (t_2, x_2))$ such that for any $((t'_1, x'_1), (t'_2, x'_2)) \in N_{((t_1,x_1),(t_2,x_2))}$ the optimality of location q_1 is preserved.

Specification of OZs. The preservation of the optimality of location q_1 with respect to the perturbations of $((t_1, x_1), (t_2, x_2))$ gives rise to the notion of (the set of) optimality zones (OZs).

Under the assumptions generating the class of hybrid systems $\mathbb{H}$ (and the associated HOCP) the value function $J^0((t_1, x_1), (t_2, x_2), q)$ of HOCP is bounded and continuous in $((t_1, x_1), (t_2, x_2))$ for each $q \in Q$ (see [4]). So it is possible to define a region OZ_q of points $((t_1, x_1), (t_2, x_2))$ in the space $(\mathbb{R} \times \mathbb{R}^n)^2$ for which a specific location $q \in Q$ corresponds to the optimal hybrid system trajectory starting at (t_1, x_1) and terminating at (t_2, x_2).

We adopt the convention that if (t_2, x_2) is not accessible from (t_1, x_1) and similarly if (t_1, x_1) is not co-accessible to $(t_2, x_2))$ when the system $\mathbb{H}$ is in the location $q \in Q$ then $J_q((t_1, x_1), (t_2, x_2)) = \infty$.

Definition 2. *For $t_0 \le t_1 < t_2 \le t_f$, the optimality zone OZ_q, corresponding to the location $q \in Q$, is given by*

$$OZ_q \triangleq \{((t_1, x_1), (t_2, x_2)) \in ((t_0, t_f) \times \mathbb{R}^n)^2 : \\ J_q^0((t_1, x_1), (t_2, x_2)) \le J_{q'}^0((t_1, x_1), (t_2, x_2)),\ t_1 < t_2,\ \forall q' \in Q\}. \qquad \square$$

Under reasonable conditions [5, 24] optimality zones are closed sets with disjoint interiors.

4.3 Formulation of the HMPOZ Algorithm

Discretization of Space-Time. For simplicity and for the purpose of estimation of computational complexity assume that Γ is a rectangular region in $\mathbb{R}^{n+1}$:

$$\Gamma \triangleq [t_0, t_f] \times [x_i^1, x_f^1] \times \cdots \times [x_i^n, x_f^n]$$

Let a grid G on Γ be defined as follows. The time interval $[t_0, t_f] \in \mathbb{R}$ is divided into N_0 uniform subintervals and let $\delta_0 \triangleq (t_f - t_0)/N_0$. For each point $t_0 + \delta_0 k$, $k = 0, 1, \ldots, N_0$, let each edge of Γ be divided into N_i uniform subintervals and let $\delta_i \triangleq (x_f^i - x_0^i)/N_i$, $i = 1, 2, \ldots, n$. Then

$$G \triangleq \{t_0, \ldots, t_f\} \times \left(\times_{k=1}^n \{x_i^k, \ldots, x_f^k\}\right).$$

Set-Up Computation. We shall adopt the name $PREP(G)$ for an algorithm performing the following calculation: find the optimal location $q^0 = q^0((t_1, x_1), (t_2, x_2)) \in Q$, $((t_1, x_1), (t_2, x_2)) \in G$, $t_0 \le t_1 < t_2 \le t_f$, for all such strictly ordered t_r, t_s on the lattice points of the grid G with $|G|$ elements, where the envelope of G is assumed to contain the optimal trajectory $(x^0(t); t_0 \le t \le t_f)$.

HMP with OZ Data; Conceptual Algorithm. Let the execution of the basic HOCP algorithm HMP (see [4, 5]) be modified so that, after an iterative shift of the vector of switching time and state pairs $(t_j, x_j)^{[k]}$ to $(t_j, x_j)^{[k+1]}$ in $\mathbb{R}^{L(n+1)}$, the location $q_j^{[k+1]}$ on the interval $[t_j^{[k+1]}, t_{j+1}^{[k+1]})$ is chosen so as to be optimal among all trajectories from $x_j^{[k+1]}$ to $x_{j+1}^{[k+1]}$ (such a location is generated by $PREP(G)$). Upon incrementing k to $k+1$ the HMPOZ algorithm repeats its basic HMP operation if the halting rule of HMP has not been satisfied.

4.4 Optimality and Complexity of HMPOZ

Based upon the conceptual specification of Algorithm HMPOZ above, and invoking the DP Principle 4.1 together with the global convergence analysis (subject to the associated conditions) of the Algorithm HMP in [4, 5], it is shown in Theorem 2 below that if HMPOZ halts at some $(x^H, u^H, q^H) \equiv (u^H, q^H)$ then neither (i) a change in q^H with the given $\{t_s^H, x_s^H\}$, nor (ii) a change in u^H for the given q^H can strictly decrease the cost J.

The algorithm $PREP(G)$ solves one standard (i.e. non-hybrid) optimal control problem for each pair of points in the grid G, for each location $q \in Q$. Hence the computational cost of the determination of the optimality zones for HOCP by use of $PREP(G)$ in $\mathbb{R}^{2(n+1)}$ is $O(|G|^2 \cdot |Q|)$ which is independent of the number of switchings L. The HMPOZ algorithm resulting from the enhancement of HMP with $PREP$ computes (i) the optimal continuous variables and controls, and (ii) the optimal discrete location sequence with an overall complexity cost of $O(|G|^2 \cdot |Q|) + O(L)$, where $O(L)$ corresponds to the complexity of a single run of the HMP algorithm. Hence, over k HOCP problems with possibly differing initial and terminal data, the complexity comparison between the repeated application of HMPOZ and of a full combinatorial search method employing HMP is given by:

$$\alpha|Q||G|^{2|x|+1} + \beta k|\text{Pont}(|x|)|(L+1) \; < \; \gamma k|\text{Pont}(|x|)|(L+1)|Q|^L,$$

where α, β, γ are constants, k = number of problems, $|G|$ = space-time sample point density, L = number of switchings, $|Q|$ = cardinality of Q, $|x|$ = dimension of x, and $|\text{Pont}(|x|)|$ denotes the complexity of solving one classical optimal control problem by application of a TPBVP algorithm (which constitutes the basic module of HMP).

Figure 1 shows the projections $\mathbf{P_1}(OZ_q)$ and $\mathbf{P_2}(OZ_q)$ of the optimality zone OZ_q on (t_1, x_1) and (t_2, x_2) spaces respectively.

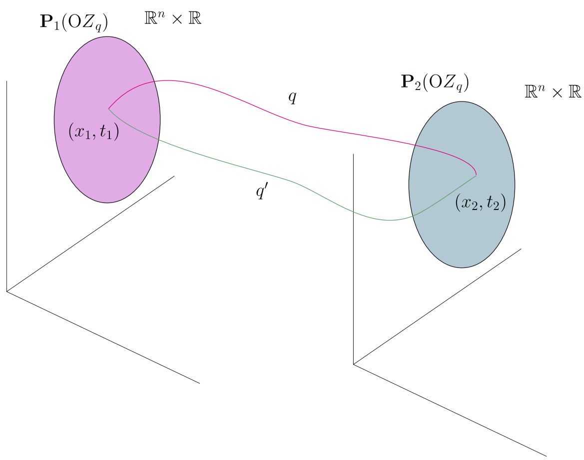

Fig. 1. Optimality zones

5 Halting and Convergence of HMPOZ Algorithms

Let $z_i \underline{\Delta} (t_i, x_i)$ and let $OZ : \mathbb{R}^{(n+1)L} \rightarrow Q^{L+1}$ be such that $OZ(\{z_i\}_{i=1}^{L}) = \{q_i\}_{i=0}^{L}$, i.e. for a given HOCP, the function OZ takes a sequence of time and state pairs and returns a sequence of locations from the precomputed Optimality Zones database computed by $PREP(G)$. Notice that the initial and final time-state are not passed to the OZ as they are part of the specification of HOCP.

Let $HMP : \mathbb{R}^{(n+1)L} \times Q^{L+1} \to \mathbb{R}^{(n+1)L}$ be such that for a given HOCP it performs the switching time and switching state update step of the Algorithm HMP of [4, 5].

Also, let $SC : \mathbb{R}^{2(n+1)L} \times Q^{2(L+1)} \times \mathbb{Z}_+ \to \mathbb{R}_+$ be a function which, for a given HOCP, computes a quantity to be compared to the stopping condition tolerance $\epsilon > 0$ of the Algorithm HMP of [4, 5].

Then the Algorithm HMPOZ may be specified as follows:

1. Initialization: Fix $0 < \epsilon \ll 1$. Set the iteration counter $k = 0$. Let $\{z_i\}_{i=1}^{L} \underline{\Delta} \{(t_i, x_i)\}_{i=1}^{L}$ be initial switching time and state pairs satisfying $t_0 < t_1 < t_2 < \cdots < t_L < t_f$. Also let $\{q_i\}_{i=0}^{L} = OZ(\{z_i\}_{i=1}^{L})$ be the initial location sequence.
2. $\{z_i\}_{i=1}^{L} \leftarrow HMP\left(\{z_i\}_{i=1}^{L}, \{q_i\}_{i=0}^{L}\right)$.
3. $\{q_i\}_{i=0}^{L} \leftarrow OZ(\{z_i\}_{i=1}^{L})$.
4. If $SC\left(\{z_i\}_{i=1}^{L}, \{q_i\}_{i=0}^{L}; k, k-1\right) \leq \epsilon$ then STOP;
   else $k \leftarrow k+1$, go to Step 2. □

Figures 2 and 3 show a typical iteration of the Algorithm HMPOZ where an OZ boundary crossing takes place.

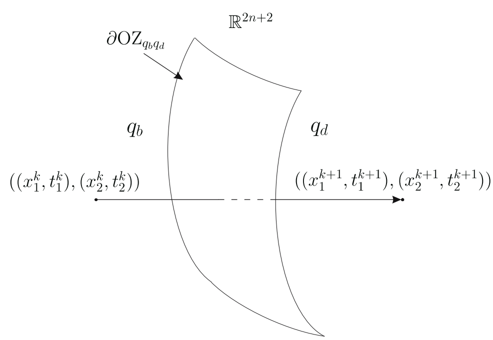

Fig. 2. An iteration of the Algorithm HMPOZ: ∂OZ crossing

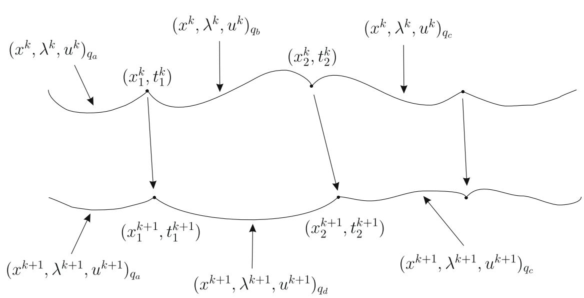

Fig. 3. An iteration of the Algorithm HMPOZ: switching time and switching state update

5.1 Convergence of HMPOZ

For brevity let us denote the sequence of switching times, switching states and locations $\left(\{(t_i, x_i)\}_{i=1}^{L}, \{q_i\}_{i=0}^{L}\right)$ as (z, q). Also let $u^0(z, q)$ denote the optimal continuous control for the sequence (z, q), i.e. for $i = 0, 1, \cdots, L$, the restriction of $u^0(z, q)$ to the interval $[t_i, t_{i+1}]$ is optimal for transferring the system from the continuous state x_i to continuous state x_{i+1} when the discrete state of the system is q_i. We define a product optimality zone OZ_q corresponding to the location sequence $\{q_i\}_{i=0}^{L}$ as

$$OZ_q \triangleq OZ_{q_0} \times OZ_{q_1} \times \cdots \times OZ_{q_L} \subset \mathbb{R}^{2(n+1)(L+1)},$$

and denote its interior as $\overset{\circ}{OZ}_q$. Then the following theorem gives the properties of the halting point of the Algorithm HMPOZ.

Theorem 2. *Assume A1 and A2 hold and assume HMPOZ halts at (z^H, q^H), then (z^H, q^H) has the following properties:*

(i) For all $q \in Q^{N+1}$: $J(u^0(z^H, q^H)) \leq J(u^0(z^H, q))$.

(ii) Let $z^H \in \overset{\circ}{OZ}_{q^H}$, then there exists a neighbourhood $N(z^H)$ of z^H such that for all $z \in N(z^H)$: $J(u^0(z^H, q^H)) \leq J(u^0(z, q^H))$.

Proof. (i) The optimality with respect to location sequence, for a given sequence of switching times and states, follows from the specification of $PREP(G)$ and the construction of the function OZ.
(ii) In this case $N(z^H)$ can be taken to be a subset of $\overset{\circ}{OZ}_{q^H}$ for which necessarily $N(z^H) \cap \overset{\circ}{OZ}_{q^H} = N(z^H)$. Then locally (i.e. for the iterations of HMPOZ which result in switching times and states which lie in $N(z^H)$) HMPOZ behaves as HMP and its convergence proof in [4, 5] is applicable. □

6 The Hybrid Bilinear Quadratic Regulator (BLQR) Problem

Consider the HOCP specified by a hybrid system whose discrete state set consists of the two locations corresponding to the bilinear dynamics:

$$q_1: \quad \dot{x} = x + xu, \qquad q_2: \quad \dot{x} = -x + xu,$$

with initial condition x_0 at t_0 and final condition x_f at t_f, and for which the cost function is

$$J(u) = \frac{1}{2} \int_{t_0}^{t_f} u^2(s)\, ds.$$

In the set of computational experiments applying HMPOZ to this problem, the program $PREP$ was first applied to the product time-space $(R^{1+1})^2$ and this generated the OZ region data which was stored in the main program look-up

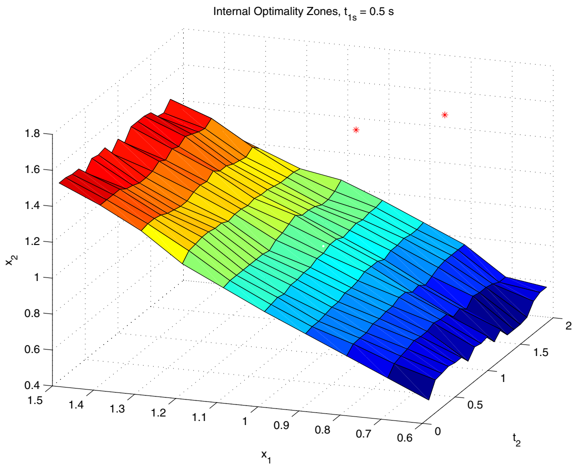

Fig. 4. OZ boundary for x_1, x_2, t_2 varying with $t_1 = 0.5$

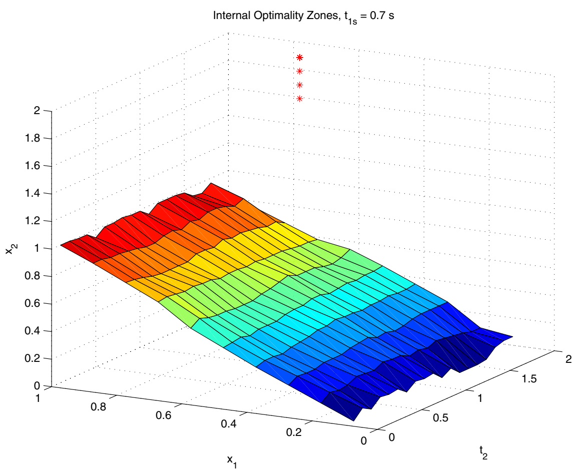

Fig. 5. OZ boundary for x_1, x_2, t_2 varying with $t_1 = 0.7$

table. The zonal boundary for the OZs corresponding to $Q = \{q_1, q_2\}$ is shown in Figures 4 and 5.

For this particular HOCP it is possible to obtain closed form expressions for the optimal cost of transferring the system from a general (t_1, x_1) to a general (t_2, x_2) under the two dynamics $(i = 1, 2)$ respectively:

$$J_i((t_1, x_1), (t_2, x_2)) = \begin{cases} \frac{1}{2}\left[1 + \frac{(-1)^i}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right)\right]^2 (t_2 - t_1), & \text{if } t_1 \neq t_2 \wedge x_1 x_2 > 0 \\ 0, & \text{if } t_1 = t_2 \wedge x_1 = x_2 \wedge x_1 x_2 > 0 \\ \infty, & \text{if } (t_1 = t_2 \wedge x_1 \neq x_2) \vee (x_1 x_2 \leq 0). \end{cases}$$

The interesting case is that of $t_1 \neq t_2$ and in this case equating the costs corresponding to the two distinct dynamics gives:

$$\frac{1}{2}\left[1 - \frac{1}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right)\right]^2 (t_2 - t_1) = \frac{1}{2}\left[1 + \frac{1}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right)\right]^2 (t_2 - t_1),$$

Hence $\frac{1}{t_2 - t_1}\log\left(\frac{x_2}{x_1}\right) = 0$, so $x_1 = x_2$, and the switching surface is given in (t_1, t_2, x_1, x_2)-space by:

$$\partial \mathrm{OZ} = \{(t_1, t_2, x_1, x_2) \in \mathbb{R}^4 : x_1 = x_2\},$$

which is illustrated by the computational experiments in Examples 1 and 2 below.

Example 1. For the subsequent implementation of HMPOZ the initial and final time and initial and final state were arbitrarily chosen to be $t_0 = 0$, $t_f = 2$, $x_0 = -1.2$, and $x_0 = -1.4$ respectively.

The control objective was to transfer the continuous state from its initial value at the initial time to the final value at the final time while minimizing the cost function $J(u) = \frac{1}{2}\int_0^2 u^2(s)\,ds$. After two iterations of HMP the second and third switching time and state pairs passed through the OZ boundary; in each case this corresponded to the ratio of the subsequent switching state values passing through the value 1 as specified in the exact analysis above. These transitions resulted in the location sequence evolving from $(2, 1, 1, 2)$ to $(2, 2, 2, 2)$ as shown in the third line of Table 1. After three iterations the algorithm converged giving the optimal cost 1.21587.

The computational time for *PREP* in this experiment was 7231 seconds (about two hours). For the HMPOZ implementation the computation time was 2.5637 seconds. All computations were performed in Matlab 6.5 under Windows 2000 SP4 operating system on a P4 3.2 GHz machine with 512 MB of RAM. □

Table 1. Execution of Algorithm HMPOZ

Iter.	Loc. sequence	Cost	x_{s_1}	x_{s_2}	x_{s_3}
1	$(2, 1, 1, 2)$	1.33775	-1.3329	-1.3000	-1.2684
2	$(2, 1, 1, 2)$	1.27524	-1.3159	-1.3000	-1.2862
3	$(2, 2, 2, 2)$	1.21587	-1.2979	-1.3000	-1.3042

Example 2. To demonstrate the power of the HMPOZ algorithm we applied it to solve an HOCP involving the BLQR of Example 1 with ten switchings. *It is to be noted that the specification of the Optimality Zones for the three switchings case (Example 1) is reused for this ten switchings example without any modification. This would have been the case even if the zones had been obtained numerically.* The problem data was: $t_0 = 0$, $t_f = 2$, $x_0 = 2.4$, $x_f = 2.6$ and number of

Table 2. Execution of Algorithm HMPOZ: Ten switchings case

Iteration	Location sequence	Cost
1	$(1,1,1,1,2,1,1,2,2,1,1)$	0.75653
2	$(1,1,1,1,2,1,1,2,2,1,1)$	0.70324
3	$(1,1,1,1,2,1,1,2,2,1,1)$	0.68563
4	$(1,1,1,1,2,1,1,2,2,1,1)$	0.63887
5	$(1,1,1,1,2,1,1,2,2,1,1)$	0.61678
6	$(1,1,1,1,2,1,1,2,2,1,1)$	0.60291
7	$(1,1,1,1,2,1,1,2,2,1,1)$	0.58548
8	$(1,1,1,1,2,1,1,2,2,1,1)$	0.54783
9	$(1,1,1,1,2,1,1,2,2,1,1)$	0.49985
10	$(1,1,1,1,2,1,1,2,2,1,1)$	0.47789
11	$(1,1,1,1,2,1,1,2,2,1,1)$	0.43679
12	$(1,1,1,1,2,1,1,2,2,1,1)$	0.39453
13	$(1,1,1,1,2,1,1,2,2,1,1)$	0.35672
14	$(1,1,1,1,2,1,1,2,2,1,1)$	0.33756
15	$(1,1,1,1,2,1,1,2,2,1,1)$	0.31957
16	$(1,1,2,2,1,1,2,1,2,1,1)$	0.21986
17	$(1,1,2,2,1,1,2,1,2,1,1)$	0.21897
18	$(1,1,2,2,1,1,2,1,2,1,1)$	0.21897

switchings was set to 10. The algorithm initially computed (i) ten uniformly distributed switching times between $t_0 = 0$ and $t_f = 2$, (ii) ten randomly distributed switching states between $x_0 = 2.4$ and $x_f = 2.6$, and (iii) the initial switching sequence: $(1,1,1,1,2,1,1,2,2,1,1)$ which corresponds to the initial choice of switching times and states. The initial cost as computed by the algorithm is J = 0.75653 which drops down to J = 0.31957 by the 15th iteration. In the next (i.e. 16th) iteration the algorithm switches to the zone corresponding to the optimal switching sequence: $(1,1,2,2,1,1,2,1,2,1,1)$ giving the optimal cost J = 0.21897 at the 18th iteration. The running time was 45.596 seconds. The iterations of the program execution are shown in Table 2. □

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