# Complete Axiomatizations of Finite Syntactic Epistemic States

Thomas Ågotnes and Michal Walicki

Department of Informatics, University of Bergen,
PB. 7800, N-5020 Bergen, Norway
{agotnes, walicki}@ii.uib.no

**Abstract.** An agent who bases his actions upon explicit logical formulae has at any given point in time a finite set of formulae he has computed. Closure or consistency conditions on this set cannot in general be assumed – reasoning takes time and real agents frequently have contradictory beliefs. This paper discusses a formal model of knowledge as explicitly computed sets of formulae. It is assumed that agents represent their knowledge syntactically, and that they can only know finitely many formulae at a given time. In order to express interesting properties of such finite syntactic epistemic states, we extend the standard epistemic language with an operator expressing that an agent knows *at most* a particular finite set of formulae, and investigate axiomatization of the resulting logic. This syntactic operator has also been studied elsewhere without the assumption about finite epistemic states [5]. A strongly complete logic is impossible, and the main results are non-trivial characterizations of the theories for which we can get completeness. The paper presents a part of a general abstract theory of resource bounded agents. Interesting results, e.g., complex algebraic conditions for completeness, are obtained from very simple assumptions, i.e., epistemic states as arbitrary finite sets and operators for knowing at least and at most.

## 1 Introduction

Traditional epistemic logics [11, 16], based on modal logic, are logics about knowledge closed under logical consequence – they describe agents who know all the infinitely many consequences of their knowledge. Such logics are very useful for many purposes, including modelling the information implicitly held by the agents or modelling the special case of extremely powerful reasoners. They fail, however, to model the explicit knowledge of real reasoners. Models of explicit knowledge are needed, e.g., if we want to model agents who represent their knowledge syntactically and base their actions upon the logical formulae they know. An example is when an agent is required to answer questions about whether he knows a certain formula or not. The agent must then decide whether this exact formula is true from his perspective — when he, e.g., is asked whether he knows $q \wedge p$ and he has already computed that $p \wedge q$ is true but not (yet) that $q \wedge p$ is true, then he cannot answer positively before he has performed a (trivial) act of reasoning. Real agents do not have unrestricted memory or unbounded time available for reasoning. In reality, an agent who bases his actions on explicit logical formulae has at any given time a finite set of formulae he has computed. In the general case, we

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cannot assume any closure conditions on this set: we cannot assume that the agent has had time to deduce something yet, nor can we assume consistency or other connections to reality — real agents often hold contradictory or otherwise false beliefs. The topic of this paper is formal models of knowledge as explicitly computed sets of formulae.

We represent an agent's state as a finite set of formulae, called a finite epistemic state. Modal epistemic logic can be seen not only as a description of knowledge but also as a very particular model of *reasoning* which is not valid for resource bounded agents. With a syntactic approach, we can get a theory of knowledge without any unrealistic assumptions about the reasoning abilities of the agents. The logic we present here is a logic about knowledge in a system of resource bounded agents *at a point in time*. We are not concerned with *how* the agents obtain their knowledge, but in reasoning about their static states of knowledge. Properties of reasoning can be modelled in an abstract way by considering only the set of epistemic states which a reasoning mechanism could actually produce. For example, we can choose to consider only epistemic states which do not contain both a formula and its negation. The question is, of course, whether anything interesting can be said about static properties of such general states. That depends on the available language.

Syntactic characterizations of states of knowledge are of course nothing new [7, 11, 12, 17]. The general idea is that the truth value of a formula such as $K_i\phi$, representing the fact that agent $i$ knows the formula $\phi$, need not depend on the truth value of any other formula of the form $K_i\psi$. Of course, syntactic characterization is an extremely general approach which can be used for several different models of knowledge – including also closure under logical consequence. It is, however, with the classical epistemic meta language, too general to have any interesting logical properties.

The formula $K_i\phi$ denotes that fact that $i$ knows *at least* $\phi$ – he knows $\phi$ but he may know more. We can generalize this to finite sets $X$ of formulae:

$$\triangle_i X \equiv \bigwedge\{K_i\phi : \phi \in X\}$$

representing the fact that $i$ knows at least $X$. In this paper we also use a dual operator, introduced in [3, 5], to denote the fact that $i$ knows *at most* $X$:

$$\triangledown_i X$$

denotes the fact that every formula an agent knows is included in $X$, but he may not know all the formulae in $X$. We call the language the agents represent their knowledge in *the object language* ($OL$). In the case that $OL$ is finite, the operator $\triangledown_i$ can, like $\triangle_i$, be defined in terms of $K_i$:

$$\triangledown_i X \equiv \bigwedge\{\neg K_i\phi : \phi \in OL \setminus X\}$$

But in the general case when $OL$ is infinite, e.g. if $OL$ is closed under propositional connectives, $\triangledown_i$ is not definable by $K_i$. We also use a third, derived, epistemic operator: $\Diamond_i X \equiv \triangle_i X \wedge \triangledown_i X$ meaning that the agent knows exactly $X$.

The second difference from the traditional syntactic treatments of knowledge, in addition to the new operator $\triangledown_i$, is that we restrict the set of formulae an agent can know at a given time to be finite. The problem we consider in this paper is axiomatizing the resulting logic. We present a sound axiomatization, and show that it is impossible to obtain strong completeness. The main results are proof-theoretical and semantical characterizations of the sets of premises for which the system *is* complete; these sets include the empty set so the system is weakly complete. In [5] we studied the axiomatization of a similar logic with the "knowing at most" operator, albeit without the finiteness assumption. Proving completeness (for the mentioned class of premises) turns out to be quite difficult when we assume that only finitely many formulae can be known, but this can be seen as a price paid for the treatment of the inherently difficult issue of finiteness.

In the next section, the language and semantics for the logic are presented. In Section 3 it is shown that strong completeness is impossible, and a sound axiomatization presented. The rest of the paper is concerned with finding the sets of premises for which the system *is* complete. Section 5 gives a proof-theoretic account of these premise sets, while a semantic one consisting of complex algebraic conditions on possible epistemic states is given in Section 6. The results in Sections 5 and 6 build on results from the similar logic from [5] mentioned above, presented in Section 4. In Section 7 some actual completeness results, including weak completeness, are shown, and Section 8 concludes.

## 2 Language and Semantics

The logic is parameterized by an object language $OL$. The object language is the language in which the agents reason, e.g. propositional logic or first order logic. No assumptions about the structure of $OL$ is made, and the results in this paper are valid for arbitrary object languages, but the interesting case is the usual one when $OL$ is infinite. An example of a possible property of the object language, which is often assumed in this paper, is that it is closed under the usual propositional connectives. Another possible property of an object language is that it is a subset of the meta language, allowing e.g. the expression of the *knowledge axiom* in the meta language: $\triangle_i\{\alpha\} \to \alpha$.

$$\wp^{fin}(OL)$$

denotes the set of all finite epistemic states, and a state $T \in \wp^{fin}(OL)$ is used as a term in an expression such as $\triangle_i T$. In addition, we allow set-building operators $\sqcup, \sqcap$ on terms in order to be able to express things like $(\triangle_i T \wedge \triangle_i U) \to \triangle_i(T \sqcup U)$ in the meta language. $TL$ is the language of all terms:

**Definition 1 ($TL(OL)$).** $TL(OL)$, or just $TL$, is the least set such that

- $\wp^{fin}(OL) \subseteq TL$
- If $T, U \in TL$ then $(T \sqcup U), (T \sqcap U) \in TL$

The *interpretation* $[T] \in \wp^{fin}(OL)$ of a term $T \in TL$ is defined as expected: $[X] = X$ when $X \in \wp^{fin}(OL)$, $[T \sqcup U] = [T] \cup [U]$, $[T \sqcap U] = [T] \cap [U]$. □

An expression like $\triangle_i T$ relates the current epistemic state of an agent to the state described by the term $T$. In addition, we allow reasoning about the relationship between the two states denoted by terms $T$ and $U$ in the meta language by introducing formulae of the form $T \doteq U$, meaning that $[T] = [U]$.

The meta language $EL$, and the semantic structures, are parameterized by the number of agents $n$ and a set of primitive propositions $\Theta$, in addition to the object language. The primitive propositions $\Theta$ play a very minor role in the rest of this paper; they are only used to model an arbitrary propositional language which is then extended with epistemic (and term) formulae. Particularly, no relation between $OL$ and $\Theta$ is assumed.

**Definition 2 ($EL(n, \Theta, OL)$).** Given a number of agents $n$, a set of primitive formulae $\Theta$, and an object language $OL$, the epistemic language $EL(n, \Theta, OL)$, or just $EL$, is the least set such that:

- $\Theta \subseteq EL$
- If $T \in TL(OL)$ and $i \in [1, n]$ then $\triangle_i T, \bigtriangledown_i T \in EL$
- If $T, U \in TL(OL)$ then $(T \doteq U) \in EL$
- If $\phi, \psi \in EL$ then $\neg\phi, (\phi \wedge \psi) \in EL$ □

The usual derived propositional connectives are used, in addition to $T \preceq U$ for $T \sqcup U \doteq U$ and $\Diamond_i \phi$ for $(\triangle_i \phi \wedge \bigtriangledown_i \phi)$. The operators $\triangle_i, \bigtriangledown_i$ and $\Diamond_i$ are called epistemic operators. A boolean combination of formulae of the form $T \doteq U$ is called a *term formula*. Members of $OL$ will be denoted $\alpha, \beta, \ldots$, members of $EL$ denoted $\phi, \psi, \ldots$, and members of $TL$ denoted $T, U, \ldots$.

The semantics of $EL$ is defined as follows. Again, $\Theta$ and its interpretation does not play an important role here.

**Definition 3 (Knowledge Set Structure).** A Knowledge Set Structure (KSS) for $n$ agents, primitive propositions $\Theta$ and object language $OL$ is an $n+1$-tuple

$$M = (s_1, \ldots, s_n, \pi) \text{ where } s_i \in \wp^{fin}(OL)$$

and $\pi : \Theta \to \{\mathbf{true}, \mathbf{false}\}$ is a truth assignment. $s_i$ is the epistemic state of agent $i$, and the set of all epistemic states is $\mathcal{S}^f = \wp^{fin}(OL)$. The set of all KSSs is denoted $\mathcal{M}_{fin}$. The set of all truth assignments is denoted $\Pi$. □

Truth of an $EL$ formula $\phi$ in a KSS $M$, written $M \models_f \phi$, is defined as follows (the subscript $f$ means "finite" and the reason for it will become clear later).

**Definition 4 (Satisfaction).** Satisfaction of a $EL$-formula $\phi$ in a KSS $M = (s_1, \ldots, s_n, \pi) \in \mathcal{M}_{fin}$, written $M \models_f \phi$ ($M$ is a model of $\phi$), is defined as follows:

$$\begin{array}{lcl}
M \models_f p & \Leftrightarrow & \pi(p) = \mathbf{true} \\
M \models_f \neg\phi & \Leftrightarrow & M \not\models_f \phi \\
M \models_f (\phi \wedge \psi) & \Leftrightarrow & M \models_f \phi \text{ and } M \models_f \psi \\
M \models_f \triangle_i T & \Leftrightarrow & [T] \subseteq s_i \\
M \models_f \bigtriangledown_i T & \Leftrightarrow & s_i \subseteq [T] \\
M \models_f T \doteq U & \Leftrightarrow & [T] = [U]
\end{array}$$
□

As usual, if $\Gamma$ is a set of formulae then we write $M \models_f \Gamma$ iff $M$ is a model of all formulae in $\Gamma$ and $\Gamma \models_f \phi$ ($\phi$ is a logical consequence of $\Gamma$) iff every model of $\Gamma$ is also a model of $\phi$. If $\emptyset \models_f \phi$, written $\models_f \phi$, then $\phi$ is valid. The class of all models of $\Gamma$ is denoted $mod^f(\Gamma)$.

The logic consisting of the language $EL$, the set of structures $\mathcal{M}_{fin}$ and the relation $\models_f$ can be used to describe the current epistemic states of agents and how epistemic states are related to each other — without any restrictions on the possible epistemic states. For example, the epistemic states are neither required to be consistent – an agent can know both a formula and its negation – nor closed under any form of logical consequence – an agent can know $\alpha \wedge \beta$ without knowing $\beta \wedge \alpha$. Both consequence conditions and closure conditions can be modelled by a set of structures $\mathcal{M}' \subset \mathcal{M}_{fin}$ where only epistemic states not violating the conditions are allowed. For example, we can construct a set of structures allowing only epistemic states not including both a formula $\alpha$ and $\neg\alpha$ at the same time, or including $\beta \wedge \alpha$ whenever $\alpha \wedge \beta$ is included. If we restrict the class of models considered under logical consequence to $\mathcal{M}'$, we get a new variant of the logic. We say that "*$\Gamma \models_f \phi$ with respect to $\mathcal{M}'$*" if every model of $\Gamma$ in $\mathcal{M}'$ is a model of $\phi$.

The question of how to completely axiomatize these logics, the general logic described by $\mathcal{M}_{fin}$ and the more special logics described by removing "illegal" epistemic states, is the main problem considered in this paper and is introduced in the next section.

## 3 Axiomatizations

The usual terminology and notation for Hilbert-style proof systems are used. A proof system is *sound* with respect to $\mathcal{M}' \subseteq \mathcal{M}_{fin}$ iff $\Gamma \vdash \phi$ implies that $\Gamma \models_f \phi$ wrt. $\mathcal{M}'$, *weakly complete* wrt. $\mathcal{M}'$ iff $\models_f \phi$ wrt. $\mathcal{M}'$ implies that $\vdash \phi$, and *strongly complete* wrt. $\mathcal{M}'$ iff $\Gamma \models_f \phi$ wrt. $\mathcal{M}'$ implies that $\Gamma \vdash \phi$.

When it comes to completeness, it is easy to see that it is impossible to achieve *full* completeness with respect to $\mathcal{M}_{fin}$ with a sound axiomatization without rules with infinitely many antecedents, because the logic is not compact. Let $\Gamma_1$ be the following theory, assuming that the object language is closed under a $\triangle_1$ operator:

$$\Gamma_1 = \{\triangle_1\{p\}, \triangle_1\{\triangle_1\{p\}\}, \triangle_1\{\triangle_1\{\triangle_1\{p\}\}\}, \ldots\}$$

Clearly, $\Gamma_1$ is not satisfiable, intuitively since it describes an agent with an infinite epistemic state, but any finite subset of $\Gamma_1$ *is* satisfiable. However, a proof of its inconsistency would necessarily include infinitely many formulae from the theory and be of infinite length (if the proof used only a finite subset $\Gamma' \subset \Gamma$, the logical system would not be sound since $\Gamma'$ is satisfiable). Another illustrating example is the following theory, assuming that the object language has conjunction:

$$\Gamma_2 = \{\triangle_1\{\alpha, \beta\} \rightarrow \triangle_1\{\alpha \wedge \beta\} : \alpha, \beta \in OL\}$$

Unlike $\Gamma_1$, $\Gamma_2$ is satisfiable, but only in a structure in which agent 1's epistemic state is the empty set. Thus, $\Gamma_2 \models_f \triangledown_1\emptyset$. But again, a proof of $\triangledown_1\emptyset$ from $\Gamma_2$ would be infinitely long (because if would necessarily use infinitely many instances of the schema $\Gamma_2$), and an axiomatization without an infinite deduction rule would thus be (strongly) incomplete since then $\Gamma_2 \nvdash \triangledown_1\emptyset$.

### 3.1 The Basic System

Since we cannot get strong completeness, the natural question is whether we can construct a weakly complete system for the logic described by $\mathcal{M}_{\mathit{fin}}$. The answer is positive. The following system $EC$ is sound and weakly complete with respect to $\mathcal{M}_{\mathit{fin}}$. Although it is not too hard to prove weak completeness directly, we will prove a more general completeness result from which weak completeness follows as a special case – as discussed in Section 3.2 below.

**Definition 5 ($EC$).** The epistemic calculus $EC$ is the logical system for the epistemic language $EL$ consisting of the following axiom schemata:

| | |
|---|---|
| All substitution instances of tautologies of propositional calculus | **Prop** |
| A sound and complete axiomatization of term formulae | **TC** |
| $\triangle_i \emptyset$ | E1 |
| $(\triangle_i T \wedge \triangle_i U) \rightarrow \triangle_i (T \sqcup U)$ | E2 |
| $(\triangle_i T \wedge \triangledown_i U) \rightarrow T \preceq U$ | E3 |
| $(\triangledown_i (U \sqcup \{\alpha\}) \wedge \neg \triangle_i \{\alpha\}) \rightarrow \triangledown_i U$ | E4 |
| $\triangle_i T \wedge U \preceq T \rightarrow \triangle_i U$ | **KS** |
| $\triangledown_i T \wedge T \preceq U \rightarrow \triangledown_i U$ | **KG** |

and the following transformation rule

$$\frac{\phi, \phi \rightarrow \psi}{\psi} \qquad \textbf{MP}$$

□

A sound and complete term calculus is given in the appendix. $\Gamma \vdash \phi$ means that there exists a sequence $\phi_1 \cdots \phi_l$ with $\phi_l = \phi$ such that each $\phi_i$ is either an axiom, an element in $\Gamma$ or the result of applying the rule **MP** to some $\phi_j$ and $\phi_k$ with $j < i$ and $k < i$. The main axioms of $EC$ are self-explaining. **KS** and **KG** stand for "knowledge specialization" and "knowledge generalization", respectively. It is easy to see that the deduction theorem (DT) holds for $EC$.

**Theorem 6 (Soundness).** If $\Gamma \vdash \phi$ then $\Gamma \models_f \phi$ □

### 3.2 Extensions

In Section 2 we mentioned that a logic with closure conditions or consistency conditions on the epistemic states can be modelled by a class $\mathcal{M}' \subset \mathcal{M}_{\mathit{fin}}$ by restricting the set of possible epistemic states. Such subclasses can often be described by axioms. For example, the axiom

$$\triangle_i \{\alpha\} \rightarrow \neg \triangle_i \{\neg\alpha\} \qquad \textbf{D}$$

describes agents who never will believe both a formula and its negation.

The next question is whether if we add an axiom to $EC$ the resulting system will be complete with respect to the class of models of the axiom; e.g., if $EC$ extended with

$\mathbf{D}$ will be complete with respect to the class of all models with epistemic states not containing both a formula and its negation.

Weak completeness of $EC$ does, of course, entail (weak) completeness of $EC$ extended with a *finite* set of axioms (DT). An axiom schema such as $\mathbf{D}$, however, represents an *infinite* set of axioms, so completeness of $EC$ extended with such an axiom schema (with respect to the models of the schema) does not necessarily follow. The completeness proof, which is constituted by most of the remainder of the paper, is actually more than a proof of weak completeness of $EC$: it is a characterization of those sets of premises for which $EC$ is complete, called *finitary theories*, and gives a method for deciding whether a given theory is finitary. Thus, if we extend $EC$ with a finitary theory, the resulting logic is weakly complete with respect to the corresponding models.

**Examples.** If we assume that $OL$ is closed under the usual propositional connectives and the $\triangle_i$ operators, some common axioms can be written in $EL$ as follows:

| | | |
|---|---|---|
| $\triangle_i \{(\alpha \to \beta)\} \to (\triangle_i\{\alpha\} \to \triangle_i\{\beta\})$ | Distribution | $\mathbf{K}$ |
| $\triangle_i \{\alpha\} \to \neg \triangle_i \{\neg\alpha\}$ | Consistency | $\mathbf{D}$ |
| $\triangle_i \{\alpha\} \to \triangle_i\{\triangle_i\{\alpha\}\}$ | Positive Introspection | $\mathbf{4}$ |
| $\neg \triangle_i \{\alpha\} \to \triangle_i\{\neg \triangle_i \{\alpha\}\}$ | Negative Introspection | $\mathbf{5}$ |

The system $EC$ extended with axiom $\Phi$ will be denoted $EC\Phi$; e.g., the axioms above give the systems $EC\mathbf{K}$, $EC\mathbf{D}$, $EC\mathbf{4}$, $EC\mathbf{5}$.

## 4 More General Epistemic States

A semantical structure for the language $EL$, a KSS, has a finite epistemic state for each agent. In this section we introduce a *generalised* semantic structure and some corresponding results. The generalised semantics and corresponding results are taken almost directly from a similar logic presented in [5]. Their interest in this paper is as an intermediate step towards results for KSSs; the results in the two following sections build on the results for the generalised semantics given below.

Recall that the object language $OL$ is a parameter of the logic. For the rest of the paper, let $*$ be an arbitrary but fixed formula such that

$$* \notin OL$$

In other words, let $OL'$ be some arbitrary language properly extending $OL$ and let $*$ be some arbitrary element in $OL' \setminus OL$. It does not matter how $*$ is selected, as long as it is not a formula of the object language, but it is important that it is selected and fixed from now on.

The generalised semantic structures are defined as follows.

**Definition 7 (General Epistemic States and General Knowledge Set Structures).** The set of *general epistemic states* is

$$\mathcal{S} = \wp(OL) \cup \wp^{fin}(OL \cup \{*\})$$

A *General Knowledge Set Structure (GKSS)* for $n$ agents, primitive propositions $\Theta$ and object language $OL$ is an $n+1$-tuple

$$M = (s_1, \ldots, s_n, \pi) \text{ where } s_i \in \mathcal{S}$$

and $\pi : \Theta \to \{\mathbf{true}, \mathbf{false}\}$ is a truth assignment. $s_i$ is the general epistemic state of agent $i$. The set of all GKSSs is denoted $\mathcal{M}$. □

In addition to the finite epistemic states $\mathcal{S}^f$, general epistemic states include states $s$ where:

1. $s$ is an infinite subset of $OL$: the agent knows infinitely many formulae, or
2. $s = s' \cup \{*\}$, where $s' \in \wp^{fin}(OL)$: the agent knows finitely many formulae but one of them is the special formula $*$

Observe that $\mathcal{M}_{fin} \subset \mathcal{M}$. Interpretation of the language $EL$ in the more general structures $\mathcal{M}$ is defined in exactly the same way as for the structures $\mathcal{M}_{fin}$ (Definition 4). To discern between the two logics we use the symbol $\models$ to denote the satisfiability relation between GKSSs and $EL$ formulae, and the corresponding validity and logical consequence relations, and use $\models_f$ for KSSs as before. The class of all general models (GKSSs) of $\Gamma \subseteq EL$ is denoted $mod(\Gamma)$.

While $* \notin OL$, since $EL$ is defined over $OL$, e.g., $\triangle_i\{*\}$ is not a well formed formula. So, informally speaking, the generalised semantics allows an agent to know infinitely many formulae at the same time, or to know something (the special formula $*$) which we cannot reference directly in the meta language. It turns out that our logical system $EC$ (Def. 5) is strongly complete with respect to this semantics. A variant of the following theorem was proved in [5] (for a slightly different logic; the proof is essentially the same).

**Theorem 8 (Soundness and Completeness wrt. GKSSs [5]).** For every $\Gamma \subseteq EL$, $\phi \in EL$:

$$\Gamma \models \phi \Leftrightarrow \Gamma \vdash \phi$$ □

## 5 Finitary Theories and Completeness

Since $EC$ is not strongly complete with respect to $\mathcal{M}_{fin}$, it is of interest to characterize exactly the theories for which $EC$ *is* complete, i.e., those $\Gamma$ where $\Gamma \models_f \phi \Rightarrow \Gamma \vdash \phi$ for every $\phi \in EL$. In this section we provide such a characterization. We define the concept of a *finitary theory*, and show that the set of finitary theories is exactly the set of theories for which $EC$ is complete. The proof builds upon the completeness result for the more general logic described in the previous section.

**Definition 9 (Finitary Theory).** A theory $\Gamma$ is *finitary* iff it is consistent and for all $\phi$,

$$\Gamma \vdash (\triangledown_1 X_1 \wedge \cdots \wedge \triangledown_n X_n) \to \phi \text{ for all sets } X_1, \ldots, X_n \in \wp^{fin}(OL)$$
$$\Downarrow$$
$$\Gamma \vdash \phi$$

where $n$ is the number of agents. □

Informally speaking, a theory is finitary if provability of a formula under arbitrary upper bounds on epistemic states implies provability of the formula itself.

We use the intermediate definition of a finitarily open theory, and its relation to that of a finitary theory, in order to prove completeness.

**Definition 10 (Finitarily Open Theory).** A theory $\Gamma$ is *finitarily open* iff there exist terms $T_1, \ldots, T_n$ such that

$$\Gamma \nvdash \neg(\triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n) \qquad \Box$$

Informally speaking, a theory is finitarily open if it can be consistently extended with some upper bound on the epistemic state of each agent.

**Lemma 11.**

1. A finitary theory is finitarily open.
2. If $\Gamma$ is a finitary theory and $\Gamma \nvdash \phi$, then $\Gamma \cup \{\neg\phi\}$ is finitarily open. $\Box$

PROOF.

1. Let $\Gamma$ be a finitary theory. If $\Gamma$ is not finitarily open, $\Gamma \vdash \neg(\triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n)$ for all terms $T_1, \ldots, T_n$. Then, for an arbitrary $\phi$, $\Gamma \vdash (\triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n) \rightarrow \phi$ for all $T_1, \ldots, T_n$ and thus $\Gamma \vdash \phi$ since $\Gamma$ is finitary. By the same argument $\Gamma \vdash \neg\phi$, contradicting the fact that $\Gamma$ is consistent.
2. Let $\Gamma$ be a finitary theory, and let $\Gamma \nvdash \phi$. Then there must exist terms $T_1^\phi, \ldots, T_n^\phi$ such that $\Gamma \nvdash (\triangledown_1 T_1^\phi \wedge \cdots \wedge \triangledown_n T_n^\phi) \rightarrow \phi$. By **Prop** we must have that $\Gamma \nvdash \neg\phi \rightarrow \neg(\triangledown_1 T_1^\phi \wedge \cdots \wedge \triangledown_n T_n^\phi)$ and thus that $\Gamma \cup \{\neg\phi\} \nvdash \neg(\triangledown_1 T_1^\phi \wedge \cdots \wedge \triangledown_n T_n^\phi)$, which shows that $\Gamma \cup \{\neg\phi\}$ is finitarily open. ■

It is difficult in practice to show whether a given theory satisfies a proof theoretic condition such as those for finitary or finitarily open theories, but we have a tool to convert the problem to a semantic one: the completeness result for GKSSs in the previous section (Theorem 8). For example, to show that $\Gamma \vdash \phi$, it suffices to show that $\Gamma \models \phi$ (with respect to GKSSs). This result can be used to see that the claims of *non-finitaryness* in the following example hold.

**Example 12.** The following are examples of non-finitary theories (let $n = 2$, $p \in \Theta$, $p \in OL$, and let $OL$ be closed under the $\triangle_i$ operators):

1. $\Gamma_1 = \{\triangle_1\{p\}, \triangle_1\{\triangle_1\{p\}\}, \triangle_1\{\triangle_1\{\triangle_1\{p\}\}\}, \ldots\}$. $\Gamma_1$ is not finitarily open, and describes an agent with an infinite epistemic state.
2. $\Gamma_2 = \{\neg \triangledown_1 T : T \in TL\}$. $\Gamma_2$ is not finitarily open, and describes an agent which cannot be at any finite point.
3. $\Gamma_3 = \{\triangledown_1 T \rightarrow \neg \triangledown_2 T' : T, T' \in TL\}$. $\Gamma_3$ is not finitarily open, and describes a situation where agents 1 and 2 cannot *simultaneously* be at finite points.
4. $\Gamma_4 = \{\triangledown_1 T \rightarrow p : T \in TL\}$. $\Gamma_4$ is finitarily open, but not finitary. To see the former, observe that if $\Gamma_4 \vdash \neg(\triangledown_1 T_1 \wedge \triangledown_2 T_2)$ for *arbitrary* $T_1, T_2$ then $\Gamma_4 \models_f \neg(\triangledown_1 T_1 \wedge \triangledown_2 T_2)$ by soundness (Theorem 6) – but it is easy to see that $\Gamma_4$ has

models which are not models of $\neg(\triangledown_1 T_1 \wedge \triangledown_2 T_2)$ (take e.g. $s_1 = [T_1]$, $s_2 = [T_2]$ and $\pi(p) = \mathbf{true}$). To see the latter, observe that $\Gamma_4 \nvdash p$ (if $\Gamma_4 \vdash p$, $\Delta \vdash p$ for some finite $\Delta \subset \Gamma_4$, which again contradicts soundness) but $\Gamma_4 \vdash (\triangledown_1 T_1 \wedge \triangledown_2 T_2) \rightarrow p$ for all $T_1, T_2$. □

**Theorem 13.** A theory $\Gamma$ is finitarily open if and only if it is satisfiable in $\mathcal{M}_{fin}$. □

PROOF. $\Gamma$ is finitarily open iff there exist $T_i$ $(1 \leq i \leq n)$ such that $\Gamma \nvdash \neg(\triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n)$; iff, by Theorem 8, there exist $T_i$ such that $\Gamma \not\models \neg(\triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n)$; iff there exist $T_i$ and a GKSS $M \in \mathcal{M}$ such that $M \models \Gamma$ and $M \models \triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n$; iff there exist $T_i$ and $M = (s_1, \ldots, s_n, \pi) \in \mathcal{M}$ such that $s_i \subseteq [T_i]$ $(1 \leq i \leq n)$ and $M \models \Gamma$; iff there exist $s_i \in \wp^{fin}(OL)$ $(1 \leq i \leq n)$ such that $(s_1, \ldots, s_n, \pi) \models \Gamma$; iff $\Gamma$ is satisfiable in $\mathcal{M}_{fin}$. ■

**Theorem 14.** Let $\Gamma \subseteq EL$. $\Gamma \models_f \phi \Rightarrow \Gamma \vdash \phi$ for all $\phi$ iff $\Gamma$ is finitary. □

PROOF. Let $\Gamma$ be a finitary theory and let $\Gamma \models_f \phi$. By Lemma 11.1 $\Gamma$ is finitarily open and thus satisfiable by Theorem 13. $\Gamma \cup \{\neg\phi\}$ is unsatisfiable in $\mathcal{M}_{fin}$, and thus not finitarily open, and it follows from Lemma 11.2 that $\Gamma \vdash \phi$.

For the other direction, let $\Gamma \models_f \phi \Rightarrow \Gamma \vdash \phi$ for all $\phi$, and assume that $\Gamma \nvdash \phi$. Then, $\Gamma \not\models_f \phi$, that is, there is a $M = (s_1, \ldots, s_n, \pi) \in mod^f(\Gamma)$ such that $M \not\models_f \phi$. Let $T_i$ $(1 \leq i \leq n)$ be terms such that $[T_i] = s_i$. $M \models_f \triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n$, and thus $M \not\models_f (\triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n) \rightarrow \phi$. By soundness (Theorem 6) $\Gamma \nvdash (\triangledown_1 T_1 \wedge \cdots \wedge \triangledown_n T_n) \rightarrow \phi$, showing that $\Gamma$ is finitary. ■

**Lemma 15.** Let $\Gamma \subseteq EL$. The following statements are equivalent:

1. $\Gamma$ is finitary.
2. $\Gamma \models_f \phi \Rightarrow \Gamma \vdash \phi$, for any $\phi$
3. $\Gamma \models_f \phi \Rightarrow \Gamma \models \phi$, for any $\phi$
4. $(\exists_{M \in mod(\Gamma)} M \models \phi) \Rightarrow (\exists_{M \in mod^f(\Gamma)} M \models_f \phi)$, for any $\phi$
5. $\Gamma \nvdash \phi \Rightarrow \Gamma \cup \{\neg\phi\}$ is finitarily open, for any $\phi$. □

Lemma 15.4 is a finite model property, with respect to the models of $\Gamma$.

We have now given a proof-theoretic definition of all theories for which $EC$ is complete: the finitary theories. We have also shown some examples of non-finitary theories. We have not, however, given any examples of *finitary* theories. Although the problem of proving that $EC$ is complete for a theory $\Gamma$ has been reduced to proving that the theory is finitary according to Definition 9, the next problem is how to show that a given theory in fact is finitary. For example, is the empty theory finitary? If it is, then $EC$ is weakly complete. We have not been able to find a trivial or easy way to prove finitaryness in general. In the next section, we present results which can be used to prove finitaryness. The results are semantic conditions for finitaryness, but can only be used for theories of a certain class and we are only able to show that they are *sufficient* and not that they also are necessary.

## 6 Semantic Finitaryness Conditions

*Epistemic axioms* are axioms which describe legal epistemic states, like "an agent cannot know both a formula and its negation". In Section 4 we presented the notion of a *general* epistemic state, and epistemic axioms can be seen as describing sets of legal general epistemic states as well as sets of legal finite epistemic states. Although we are ultimately interested in the latter, in this section we will be mainly interested in the former – we will present conditions on the algebraic structure of sets of general epistemic states in $mod(\Phi)$ which are sufficient for the axioms $\Phi$ to be finitary.

First, epistemic axioms and their correspondence with sets of legal general epistemic states are defined. Then, conditions on these sets are defined, and it is shown that the GKSSs of a given set of epistemic axioms – being (essentially) the Cartesian product of the corresponding sets of legal general states – exhibit the finite model property if the sets of legal general states fulfil the conditions. The set of axioms is then finitary by Lemma 15.4.

### 6.1 Epistemic Axioms

Not all formulae in $EL$ should be considered as candidates for describing epistemic properties. One example is $p \rightarrow \triangle_i\{p\}$. This formula does not solely describe the *agent* – it describes a relationship between the agent and the world. Another example is $\diamondsuit_i\{p\} \rightarrow \diamondsuit_j\{q\}$, which describes a constraint on one agent's belief set contingent on another agent's belief set. Neither of these two formulae describe purely *epistemic* properties of an agent. In the following definition, $EF$ is the set of epistemic formulae and $Ax$ is the set of candidate epistemic axioms.

**Definition 16 ($EF$, $EF^i$, $Ax$).**

- $EF \subseteq EL$ is the least set such that for $1 \leq i \leq n$:

$$T \in TL \Rightarrow \triangle_i T, \triangledown_i T \in EF \qquad \phi, \psi \in EF \Rightarrow \neg\phi, (\phi \wedge \psi) \in EF$$

- $EF^i = \{\phi \in EF :$ Every epistemic operator in $\phi$ is $\triangle_i$ or $\triangledown_i\}$ $(1 \leq i \leq n)$
- $Ax = \bigcup_{1 \leq i \leq n} EF^i$ □

An example of an epistemic axiom schema is, if we assume that $OL$ has conjunction,

$$\triangle_i\{\alpha \wedge \beta\} \rightarrow \triangle_i\{\alpha\} \wedge \triangle_i\{\beta\} \tag{1}$$

Recall the set $\mathcal{S}$ of all general epistemic states, defined in Section 4.

**Definition 17 ($\mathcal{M}^\phi$, $S_i^\phi$, $\mathcal{M}^\Phi$, $S_i^\Phi$).** For each epistemic formula $\phi \in EF^i$,

$$\mathcal{M}^\phi = S_1^\phi \times \cdots \times S_n^\phi \times \Pi$$

where $S_j^\phi = \mathcal{S}$ for $j \neq i$ and $S_i^\phi$ is constructed by structural induction over $\phi$ as follows:

$$S_i^{\triangle_i T} = \{X \in \mathcal{S} : [T] \subseteq X\} \qquad S_i^{\triangledown_i T} = \{X \in \mathcal{S} : X \subseteq [T]\}$$
$$S_i^{\neg\psi} = \mathcal{S} \setminus S_i^\psi \qquad S_i^{\psi_1 \wedge \psi_2} = S_i^{\psi_1} \cap S_i^{\psi_2}$$

When $\Phi \subseteq Ax$ then: $S_i^\Phi = (\bigcap_{\phi \in \Phi \cap EF^i} S_i^\phi) \cap \mathcal{S}$ and $\mathcal{M}^\Phi = S_1^\Phi \times \cdots \times S_n^\Phi \times \Pi$ □

In the construction of $\mathcal{M}^{\phi}$ we remove the impossible (general) epistemic states by restricting the set of epistemic states to $S_i^{\phi}$. The epistemic states which are not removed are the possible states — an agent can be placed in any of these states and will satisfy the epistemic axiom $\phi$.

Given $\Phi \subseteq Ax$, the corresponding KSS models are:

$$\mathcal{M}^{\Phi}_{\mathit{fin}} = \mathcal{M}^{\Phi} \cap \mathcal{M}_{\mathit{fin}} = (S_1^{\Phi} \cap \mathcal{S}^f) \times \cdots \times (S_n^{\Phi} \cap \mathcal{S}^f) \times \Pi$$

That $\mathcal{M}^{\Phi}$ and $\mathcal{M}^{\Phi}_{\mathit{fin}}$ indeed are the class of GKSS models and the class of KSS models of $\Phi$, respectively, can easily be shown:

**Lemma 18.** If $\Phi \subseteq Ax$, $\mathcal{M}^{\Phi} = mod(\Phi)$ and $\mathcal{M}^{\Phi}_{\mathit{fin}} = mod^f(\Phi)$ □

Thus, the model class for epistemic axioms is constructed by removing certain states from the set of legal epistemic states. For example, (1) corresponds to removing epistemic states where the agent knows a conjunction without knowing the conjuncts.

Note that $\emptyset$ is trivially a set of epistemic axioms, and that $S_i^{\emptyset} = \mathcal{S}$ and $\mathcal{M}^{\emptyset} = \mathcal{M}$.

### 6.2 Finitaryness of Epistemic Axioms

Lemmas 15.1 and 15.4 say that $\Gamma$ is finitary iff $mod(\Gamma)$ has the finite model property. We make the following intermediate definition, and the following Lemma is an immediate consequence.

**Definition 19 (Finitary set of GKSSs).** A class of GKSSs $\mathcal{M}' \subseteq \mathcal{M}$ is finitary iff, for all $\phi$:

$$\exists_{M \in \mathcal{M}'} M \models \phi \Rightarrow \exists_{M^f \in \mathcal{M}'^f} M^f \models \phi$$

where $\mathcal{M}'^f = \mathcal{M}' \cap \mathcal{M}_{\mathit{fin}}$. □

**Lemma 20.** Let $\Gamma \subseteq EL$. $\Gamma$ is finitary iff $mod(\Gamma)$ is finitary. □

In the definition of the conditions on sets of general epistemic states, the following two general algebraic conditions will be used.

**Directed Set.** A set $A$ with a reflexive and transitive relation $\leq$ is *directed* iff for every finite subset $B$ of $A$, there is an element $a \in A$ such that $b \leq a$ for every $b \in B$. In the following, directedness of a set of sets is implicitly taken to be with respect to subset inclusion.

**Cover.** A family of subsets of a set $A$ whose union includes $A$ is a *cover* of $A$.

The main result is that the following conditions on sets of general epistemic states are sufficient for the corresponding GKSSs to be finitary (Def. 19), and furthermore, if the sets are induced by epistemic axioms, that the axioms are finitary. The conditions are quite complicated, but simpler ones are given below.

**Definition 21 (Finitary Set of Epistemic States).** If $S \subseteq \mathcal{S}$ is a set of general epistemic states and $s \in \wp(OL)$, then the set of finite subsets of s included in $S$ is denoted

$$S|_s^f = S \cap \wp^{fin}(s)$$

$S$ is *finitary* iff both:

1. For every infinite $s \in S$:
   - (a) $S|_s^f$ is directed
   - (b) $S|_s^f$ is a cover of $s$
2. $\forall_{s \cup \{*\} \in S} \forall_{s' \in \wp^{fin}(OL)} \exists_{\alpha \notin s'}$:
   - (a) $\exists_{s^f \in S \cap \wp(s \cup \{\alpha\})} s' \cap s \subseteq s^f$
   - (b) $\exists_{s^f \in S \cap \wp(s \cup \{\alpha\})} s^f \not\subseteq s'$
   - (c) $S \cap \wp(s \cup \{\alpha\})$ is directed □

The definition specifies conditions for each infinite set in $S$ (condition 1) and each finite set in $S$ containing $*$ (condition 2). Condition 2 is similar to condition 1, but is complicated by the fact that, informally speaking, the existence of a proper formula $\alpha$ to "replace" $*$ is needed. In practice, the simplified (and stronger) conditions presented in Corollary 24 below can often be used.

The following Lemma is the main technical result in this section. The proof is somewhat involved, and must be left out due to space restrictions. It can be found in [1].

**Lemma 22.** If $S_1, \ldots, S_n$ are finitary sets of epistemic states (Def. 21), then

$$S_1 \times \cdots \times S_n \times \mathit{\Pi}$$

is a finitary set of GKSSs (Def. 19). □

Recall that a set $\mathit{\Phi}$ of epistemic axioms induces sets of legal epistemic states $S_i^{\mathit{\Phi}}$ (Def. 17).

**Theorem 23.** If $\mathit{\Phi}$ is a set of epistemic axioms such that $S_1^{\mathit{\Phi}}, \ldots, S_n^{\mathit{\Phi}}$ are finitary sets of epistemic states, then $\mathit{\Phi}$ is finitary. □

PROOF. Since $\mathit{\Phi}$ are epistemic axioms, $\mathcal{M}^{\mathit{\Phi}} = S_1^{\mathit{\Phi}} \times \cdots \times S_n^{\mathit{\Phi}} \times \mathit{\Pi}$. Since all $S_i^{\mathit{\Phi}}$ are finitary, by Lemma 22 $\mathcal{M}^{\mathit{\Phi}}$ is a finitary set of GKSSs. Since $\mathcal{M}^{\mathit{\Phi}} = mod(\mathit{\Phi})$ (Lemma 18), $\mathit{\Phi}$ is finitary by Lemma 20. ■

Theorem 23 shows that the conditions in Def. 21 on the set of legal epistemic states induced by epistemic axioms are sufficient to conclude that the axioms are finitary. In the following Corollary, we present several alternative sufficient conditions which are stronger. It can easily be shown that these conditions imply Def. 21.

**Corollary 24.** A set of epistemic states $S \subseteq \mathcal{S}$ is finitary if either one of the following three conditions hold:

1. For every $s \subseteq OL$:
   - (a) $S|_s^f$ is directed
   - (b) $S|_s^f$ is a cover of $s$
2. (a) $S|_s^f$ is directed for every $s \subseteq OL$
   - (b) $\{\alpha\} \in S$ for every $\alpha \in OL$
3. (a) $S|_s^f$ is directed for every infinite $s \in S$
   - (b) $\{\alpha\} \in S$ for every $\alpha \in OL$
   - (c) $\forall_{s \cup \{*\} \in S} \forall_{s' \in \wp^{fin}(OL)} \exists_{\alpha \notin s'} s \cup \{\alpha\} \in S$ □

## 7 Some Completeness Results

For a given axiom schema $\Phi$, the results from Sections 4, 5 and 6 can be used to test whether the system $EC\Phi$ is weakly complete, henceforth in this section called only "complete", with respect to $mod^f(\Phi) \subseteq \mathcal{M}_{fin}$. First, check that $\Phi$ is an epistemic axiom schema (Def. 16). Second, construct the GKSS (see Sec. 4) models of $\Phi$, $\mathcal{M}^{\Phi} = S_1^{\Phi} \times \cdots S_n^{\Phi} \times \Pi = mod(\Phi)$ (Def. 17, Lemma 18). Third, check that each $S_i^{\Phi}$ is finitary (Def. 21) – it suffices that they each satisfy one of the simpler conditions in Corollary 24. If these tests are positive, $EC\Phi$ is complete with respect to $\mathcal{M}_{fin}^{\Phi} = mod^f(\Phi)$, the KSSs included in $\mathcal{M}^{\Phi}$, by Theorems 23 and 14. The converse does not hold; $\mathcal{M}_{fin}^{\Phi}$ is not necessarily *incomplete* with respect to the corresponding models if the tests are negative. Many of the properties discussed in Section 5 can, however, be used to show incompleteness.

These techniques are used in Theorem 25 below to prove the assertion from Section 3 about weak completeness of $EC$, in addition to results about completeness of the systems $EC\mathbf{K}$, $EC\mathbf{D}$, $EC\mathbf{4}$ and $EC\mathbf{5}$ from Section 3.2. For the latter results it is assumed that $OL$ is closed under the usual propositional connectives and the $\triangle_i$ operators.

**Theorem 25 (Completeness Results).**

1. $EC$ is sound and complete with respect to $\mathcal{M}_{fin}$
2. $EC\mathbf{K}$ is sound and complete with respect to $\mathcal{M}_{fin}^{\mathbf{K}}$
3. $EC\mathbf{D}$ is sound and complete with respect to $\mathcal{M}_{fin}^{\mathbf{D}}$
4. $EC\mathbf{4}$ is not complete with respect to $\mathcal{M}_{fin}^{\mathbf{4}}$
5. $EC\mathbf{5}$ is not complete with respect to $\mathcal{M}_{fin}^{\mathbf{5}}$ □

PROOF. Soundness, in the first three parts of the theorem, follows immediately from Theorem 6 and the fact that $\mathbf{K}$ and $\mathbf{D}$ are valid in $\mathcal{M}_{fin}^{\mathbf{K}}$ and $\mathcal{M}_{fin}^{\mathbf{D}}$, respectively. The strategy for the completeness proofs, for the first three parts of the theorem, is as outlined above. (Weak) completeness of $EC$ can be considered by "extending" $EC$ by the empty set, and attempting to show that the empty set is a finitary theory. The empty set is trivially a set of epistemic axioms, and the axiom schemas $\mathbf{K}$ and $\mathbf{D}$ also both represent sets of epistemic axioms, with GKSS models constructed from the following sets of general epistemic states respectively:

$$\begin{aligned} S_i^{\emptyset} &= \mathcal{S} \\ S_i^{\mathbf{K}} &= \mathcal{S} \setminus \{X \in \mathcal{S} : \exists_{\alpha,\beta \in OL} \alpha \to \beta, \alpha \in X; \beta \notin X)\} \\ S_i^{\mathbf{D}} &= \mathcal{S} \setminus \{X \in \mathcal{S} : \exists_{\alpha \in OL} \alpha, \neg\alpha \in X\} \end{aligned}$$

We show that these sets all are finitary sets of epistemic states by using Corollary 24. It follows by Theorem 23 that the theories $\emptyset$, $\mathbf{K}$ and $\mathbf{D}$ are finitary theories, and thus that $EC$, $EC\mathbf{K}$ and $EC\mathbf{D}$ are (weakly) complete by Theorem 14. For the two last parts of the theorem, we show that $\mathbf{4}$ and $\mathbf{5}$ are not finitary theories; it follows by Theorem 14 that $EC\mathbf{4}$ and $EC\mathbf{5}$ are incomplete.

1. Corollary 24.1 holds for $S_i^{\emptyset} = \mathcal{S}$: Let $s \subseteq OL$. $\mathcal{S}|_s^f = \mathcal{S} \cap \wp^{fin}(s) = \wp^{fin}(s)$. $\wp^{fin}(s)$ is directed, because for every finite subset $B \subset \wp^{fin}(s)$, $\cup_{s' \in B} s' \in \wp^{fin}(s)$. $\wp^{fin}(s)$ is a cover of $s$, because $s \subseteq \bigcup \wp^{fin}(s)$.

2. Corollary 24.3 holds for $S_i^{\mathbf{K}}$:
   **Corollary 24.3.(a):** It must be shown that $S_i^{\mathbf{K}}|_s^f$ is directed for infinite $s \in S_i^{\mathbf{K}}$. Let $s', s'' \in S_i^{\mathbf{K}} \cap \wp^{fin}(s)$, and let for $0 < j$:

   $$\begin{aligned} s_0 &= s' \cup s'' \\ s_j &= s_{j-1} \cup \{\beta : \alpha \to \beta, \alpha \in s_{j-1}\} \\ s^f &= \bigcup_k s_k \end{aligned}$$

   It is easy to show that $s^f \in S_i^{\mathbf{K}}$, each $s_j$ is a finite subset of $s$, and $s^f$ is finite.
   **Corollary 24.3.(b):** Clearly, $\{\alpha\} \in S_i^{\mathbf{K}}$ for every $\alpha \in OL$.
   **Corollary 24.3.(c):** Let $s \cup \{*\} \in S_i^{\mathbf{K}}$ and $s' \in \wp^{fin}(OL)$. Let $\alpha \in OL$ be s. t.:
   - $\alpha \to \beta \notin s$ for any $\beta \in OL$
   - $\alpha \notin s'$
   - The main connective in $\alpha$ is not implication

   It is easy to see that there exist infinitely many $\alpha$ satisfying these three conditions; there are infinitely many $\alpha \in OL$ without implication as main connective, and both $s$ and $s'$ are finite. It can easily be shown that $s \cup \{\alpha\} \in S_i^{\mathbf{K}}$.
3. Corollary 24.3 holds for $S_i^{\mathbf{D}}$:
   **Corollary 24.3.(a):** It must be shown that $S_i^{\mathbf{D}}|_s^f$ is directed for infinite $s \in S_i^{\mathbf{D}}$. Let $s', s'' \in S_i^{\mathbf{D}} \cap \wp^{fin}(s)$, and let $s^f = s' \cup s''$. It can easily be shown that $s^f \in S_i^{\mathbf{D}}$, and $s^f \in \wp^{fin}(s)$ trivially.
   **Corollary 24.3.(b):** Clearly, $\{\alpha\} \in S_i^{\mathbf{D}}$ for every $\alpha \in OL$.
   **Corollary 24.3.(c):** Let $s \cup \{*\} \in S_i^{\mathbf{D}}$ and $s' \in \wp^{fin}(OL)$. Let $\alpha \in OL$ be s. t.:
   - $\neg\alpha \notin s$
   - $\alpha \notin s'$
   - $\alpha$ does not start with negation

   It is easy to see that there exist infinitely many $\alpha$ satisfying these three conditions; there are infinitely many $\alpha \in OL$ without negation as main connective, and both $s$ and $s'$ are finite. It can easily be shown that $s \cup \{\alpha\} \in S_i^{\mathbf{D}}$.
4. Let $1 \leq i \leq n$, and let $M = (s_1, \ldots, s_n, \pi) \in \mathcal{M}_{fin}$ such that $M \models_f \mathbf{4}$. $s_i$ must be the empty set – otherwise it would not be finite. Thus, $\mathbf{4} \models_f \nabla_i \emptyset$. **4** does, however, have *infinite* models, so $\mathbf{4} \not\models \nabla_i \emptyset$. Lemma 15 gives that **4** is not finitary.
5. It is easy to see that **5** is not satisfiable in $\mathcal{M}_{fin}$ (i.e. that a model for **5** must be infinite). By Theorem 13 and Lemma 11, **5** is not finitary. ■

Although the results in Theorem 25 are hardly surprising, they seem surprisingly hard to prove.

## 8 Discussion and Conclusions

This paper presents a general and very abstract theory of resource bounded agents. We assumed that agents' epistemic states are arbitrary finite sets of formulae. The addition of the "knowing at most" operator $\nabla_i$ gives a more expressive language for a theory of knowledge without any unrealistic assumptions about the reasoning abilities of the agents. Properties of reasoning can be modelled in an abstract way by considering only the set of epistemic states which a reasoning mechanism could actually produce. If a more detailed model of reasoning is needed, the framework can be extended with a model describing transitions between finite epistemic states. This is exactly what is done in [4]. The key property of the models considered in this paper is the assumption about finite epistemic states; the results build on previous results for a similar logic without this assumption. The main results are an axiomatization of the logic, and two characterizations of the theories for which the logic is complete. The first, the notion of finitary theories, is a proof-theoretic account of all such theories. The second, algebraic conditions on certain sets of epistemic states, is a semantic one, but is only a sufficient condition for finitaryness. The latter was used to show finitaryness of the empty theory and thus weak completeness of the system. It follows from these results that the logic $EC$ is decidable. The characterizations were also used to show (in)completeness of several extensions of $EC$. The results give a general completeness proof, of which weak completeness is a special case, and the complexity of the proof is due to this generality.

Interesting results have been obtained from very weak assumptions: finite memory and a "knowing at most" operator in the meta language give complex algebraic conditions for axiomatizability.

Related works include the many approaches to the logical omniscience problem (LOP) [13]; see e.g. [11, 18, 19] for surveys. Particularly, the work in this paper is a development of the syntactic treatment of knowledge as mentioned in Section 1. [11] presents this approach in the form of *standard syntactic assignments*. It is easy to see that KSSs are equivalent to standard syntactic assignments restricted to assigning finite knowledge to each agent. The $\bigtriangledown_i$ operator, and the derived $\Diamond_i$ operator, are new in the context of syntactic models. $\Diamond_i$ is, however, similar to Levesque's *only knowing* operator $\mathbf{O}$ [15]. $\mathbf{O}\alpha$ means that the agent does not know more than $\alpha$, but knowledge in this context means knowledge closed under logical consequence and "only knowing $\alpha$" is thus quite different from "knowing exactly" a finite set of formulae syntactically. Another well-known approach to the LOP is *the logic of general awareness* [10], combining a syntactic and a semantic model of knowledge. This logic can be seen as syntactic assignments restricted to assigning truth only to formulae which actually follow, a special case of standard syntactic assignments. In this view, the logic we have discussed in this paper is not a "competing" framework to be compared to the logic of general awareness. Rather, it is an abstraction of the syntactic fragment of the latter logic, and gives a theory of two new concepts orthogonal to those modeled by the awareness logic: finite epistemic states and the "knowing at most" operator, respectively (that the $\bigtriangledown_i$ operator is indeed not definable by the usual syntactic operator $K_i$, as mentioned in the introduction, is shown formally in [2]). Adding the finiteness assumption, i.e. restricting the set of formulae an agent can be aware of (and thus his explicit knowledge), and/or the "knowing at most" operator to the awareness logic should be straightforward, but explicit definitions must be left out here due to lack of space. The application of the results in this paper to the logic of general awareness is nevertheless interesting for future work. Models of reasoning as transition between syntactic states, as mentioned above, include *Konolige's deduction model* [14], *active logics* [8] and *timed reasoning logics (TRL)* [6]. Possibilities for future work include further development of the identification of finitary theories. For the case of epistemic axioms, the presented algebraic conditions are sufficient but not necessary and tighter conditions would be interesting. Deciding finitaryness of general, not necessarily epistemic, axioms should also be investigated[1].

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# A Term Calculus

The following axioms give a sound and complete calculus of term formulae (see Definition 5).

| | | |
|---|---|---|
| $T \doteq T$ | equivalence (reflexivity) | T1 |
| $T \doteq U \to U \doteq T$ | equivalence (symmetry) | T2 |
| $T \doteq U \wedge U \doteq V \to T \doteq V$ | equivalence (transitivity) | T3 |
| $T \doteq U \wedge S \doteq V \to S \sqcup T \doteq V \sqcup U$ | join-congruence | T4 |
| $T \doteq U \wedge S \doteq V \to S \sqcap T \doteq V \sqcap U$ | meet-congruence | T5 |
| $T \sqcup U \doteq U \sqcup T$ | join-commutativity | T6 |
| $T \sqcap U \doteq U \sqcap T$ | meet-commutativity | T7 |
| $(T \sqcup U) \sqcup V \doteq T \sqcup (U \sqcup V)$ | join-associativity | T8 |
| $(T \sqcap U) \sqcap V \doteq T \sqcap (U \sqcap V)$ | meet-associativity | T9 |
| $T \sqcup (T \sqcap U) \doteq T$ | meet-absorption | T10 |
| $T \sqcap (T \sqcup U) \doteq T$ | join-absorption | T11 |
| $T \sqcap (U \sqcup V) \doteq (T \sqcap U) \sqcup (T \sqcap V)$ | distributivity | T12 |
| $\{\alpha_1, \ldots, \alpha_n\} \doteq \{\alpha_1\} \sqcup \cdots \sqcup \{\alpha_n\}$ | atomicity | T13 |
| $\{\alpha\} \doteq \{\beta\} \to \{\alpha\} \sqcap \{\beta\} \doteq \{\alpha\}$ | | T14 |
| $\neg(\{\alpha\} \doteq \{\beta\}) \to \{\alpha\} \sqcap \{\beta\} \doteq \emptyset$ | | T15 |
| $\neg X \doteq Y$ | $X, Y \in \wp^{fin}(OL), X \neq Y$ | T16 |