Asynchronous Behavior of Double-Quiescent Elementary Cellular Automata

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Abstract. In this paper we propose a probabilistic analysis of the relaxation time of elementary finite cellular automata (i.e., $\{0,1\}$ states, radius 1 and unidimensional) for which both states are quiescent (i.e., $(0,0,0) \mapsto 0$ and $(1,1,1) \mapsto 1$), under α-asynchronous dynamics (i.e., each cell is updated at each time step independently with probability $0 < \alpha \leqslant 1$). This work generalizes previous work in [1], in the sense that we study here a continuous range of asynchronism that goes from full asynchronism to full synchronism. We characterize formally the sensitivity to asynchronism of the relaxation times for 52 of the 64 considered automata. Our work relies on the design of probabilistic tools that enable to predict the global behaviour by counting local configuration patterns. These tools may be of independent interest since they provide a convenient framework to deal *exhaustively* with the tedious case analysis inherent to this kind of study. The remaining 12 automata (only 5 after symmetries) appear to exhibit interesting complex phenomena (such as polynomial/exponential/infinite phase transitions).

1 Introduction

The aim of this article is to analyze the asynchronous behavior of unbounded finite cellular automata. Cellular automata are widely used to model systems involving a huge number of interacting elements such as agents in economy, particles in physics, proteins in biology, distributed systems, etc. In most of these applications, in particular in many real system models, agents are not synchronous. Depending on the transition rules, the behaviour of the system may vary widely when asynchronism increases in the dynamics. More generally one can ask how much does asynchronous in real system perturbs computation. In spite of this lack of synchronism, real living systems are very resilient over time. One might then expect the cellular automata used to model these systems to be robust to asynchronism and to other kind of failure as well (such as misreading the states of the neighbors). It turns out that the resilience to asynchronism widely varies from one automata to another (e.g., [2, 3]). Only few theoretical studies exist on the influence of asynchronism. Most of them usually focus on one specific cellular

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automata (e.g., [4, 5, 6]) and do not address the problem globally. Recently, Gács shows in [7] that it is undecidable to determining if in a given automota, the sequences of changes of states followed by a given cell is independent of the history of the updates. Related work on the existence of stationary distribution on infinite configurations for probabilistic automata can be found in [8].

One can see cellular automata as physical systems where cell states change according to local constraints (the transition rules). One typical example consists of a network where each cell have two states, e.g., "I have a token" and "I don't have a token", and where transitions from one state to the other depends on the states of the neighbours, e.g., "I get a token if both of my neighbors have one" or "I have a token if and only if my right neighbor has one", etc. One natural question for such systems, ask for the *relaxation time*, i.e. the time needed to reach a stable configuration (e.g., "everyone has a token" or "no one has a token"). As opposed to classic work in asynchronous distributed computing, where one tries to *design* efficient transitions rules that guarantees fast convergence to a stable configuration (e.g., [9]), we study here how asynchrony affects the global evolution of the system given an *arbitrary* set of local constraints, and in particular how does asynchronicity affects its relaxation time. In [1], the authors carried out a complete analysis of the class of one-dimensional double quiescent elementary cellular automata (DQECA), where each cell has two states `0` and `1` which are quiescent (i.e., where each cell for which every cell in its neighbourhood are in the same state, remains in the same state) and where each cell updates according to its state and the states of its two immediate neighbours. They study the behaviour of these automata under fully asynchronous dynamics, where only one random cell is updated at each time step. They show that one can classify the 64 DQECAs in six categories according to their relaxation times under full asynchronism (either constant, logarithmic, linear, quadratic, exponential or infinite) and furthermore that the relaxation time characterizes their behaviour, i.e., that all automata with equivalent relaxation times present the same kind of space-time diagrams.

The present paper extends this study to a continuous range of asynchyronism from fully asynchronous dynamics to fully synchronous dynamics: the *α-asynchronous dynamics*, with $0 < \alpha \leqslant 1$. In this setting, each cell is updated independently with probability α at each time step. When α varies from 1 down to 0, the α-asynchronous dynamics evolves from the fully synchronous regime to a more and more asynchronous regime. As α approaches 0, the probability that the updates involve at most one cell tends to 1, and the dynamics gets closer and closer to a kind of fully asynchronous dynamics up to a time rescaling by a factor $1/\alpha$. Abusing of the notation, we thus refer the fully asynchronous dynamics as the 0-asynchronous regime.

Figure 1 page 457 presents the space-time diagrams of the 24 representatives of the DQECAs as α increases (by steps of 0.25) starting from the same random configuration of length $n = 100$. The last column plots the density of black cells at time step $t = 1000/\alpha$ on one single random configuration. This class exhibits a rich variety of behaviours. Thirteen representatives of the DQECAs (ECAs 204 to 128, 198, and 142 on Fig. 1) appear to be marginally sensitive to asynchronism.

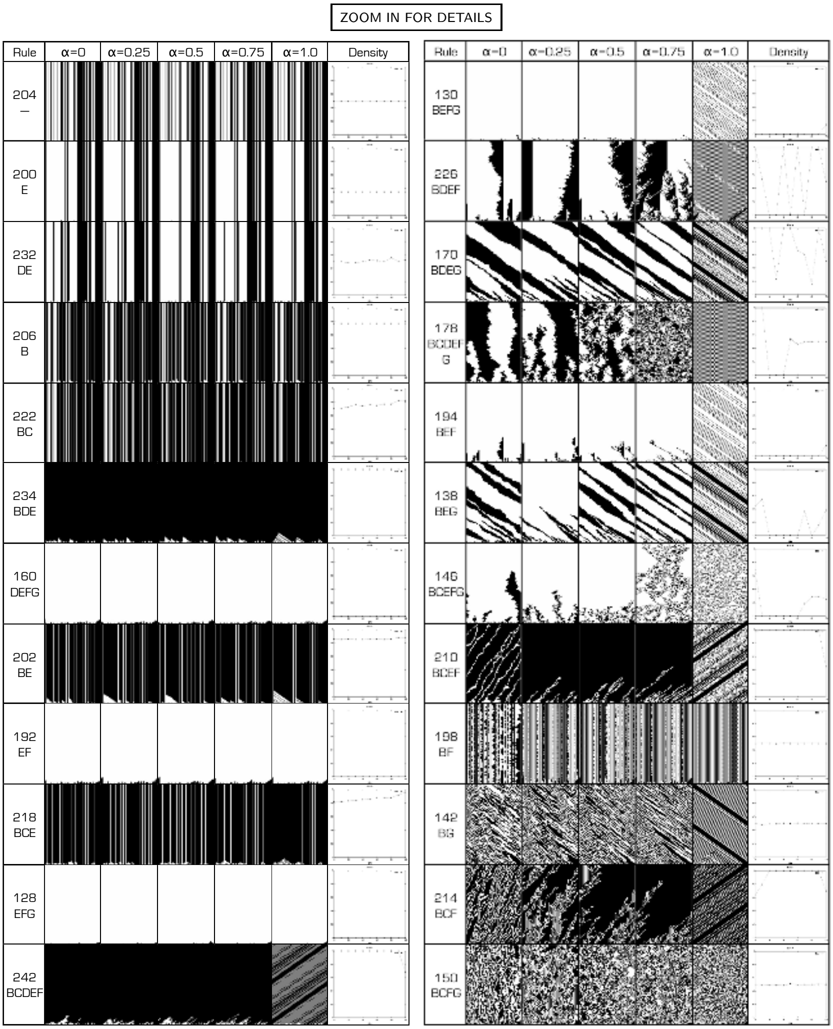


Fig. 1. Behaviour of DQECAs as a function of the synchronicity rate α

Six of them (ECAs 242 to 170, 194, and 138 on Fig. 1) present a brutal transition from the synchronous to asynchronous dynamics: they converge in polynomial time to an all-zero or all-one configuration as soon as (even a small amount of) asynchronism is introduced, while diverge under synchronous dynamics. One can observed that their space-time diagrams exhibit random walks like behaviour. The most interesting behaviour are observed on the remaining five representatives. The relaxation time of ECAs 210 and 214 are respectively exponential and infinite under fully asynchronous dynamics, and both infinite under synchronous dynamics, but appears to be polynomial under α-asynchronous dynamics. The relaxation time as well as the time-space diagrams of ECAs 178 and 146 evolve continuously as α increases, but seem to present an interesting phase transition at some α_c and α'_c, respectively, such that the relaxation time appears to be polynomial for $\alpha < \alpha'_c$, and exponential for $\alpha > \alpha'_c$. Finally, the relaxation time of ECA 150 appears to be exponential when $0 < \alpha < 1$, and is infinite otherwise.

Section 2 introduces the main definitions and presents our main result. Section 3 presents the key phenomena that differentiate the different dynamics: fully synchronous, α-asynchronous (studied here), and fully asynchronous (studied in [1]). These observations will guide the design of probabilistic tools that are presented in Section 4 and used in Section 5 to bound the relaxation times. Finally, Section 6 sums up the intuitions, hints and conjectures on the behaviours of the remaining automata that could not be treated theoretically here, leaving the determination of their relaxation times open.

2 Definitions, Notations and Main Results

In this paper, we consider the class of the two-state cellular automata on finite size configurations with periodic boundary conditions.

Definition 1. *An* Elementary Cellular Automata (ECA) *is given by its* transition function $\{\delta : \{0,1\}^3 \to \{0,1\}\}$. *We denote by* $Q = \{0,1\}$ *the set of* states. *A state* q *is* quiescent *if* $\delta(q,q,q) = q$. *An ECA is* double-quiescent (DQECA) *if both states* 0 *and* 1 *are quiescent.*

We denote by $U = \mathbb{Z}/n\mathbb{Z}$ *the set of* cells. *A* finite configuration *with periodic boundary conditions* $x \in Q^U$ *is a word indexed by* U *with letters in* Q.

Definition 2. *For a given pattern* $w \in Q^*$, *we denote by* $|x|_w = \#\{i \in U : x_{i+1} \dots x_{i+|w|} = w\}$ *the number of occurrences of* w *in configuration* x.

We will use the following labels introduced in [1] which will simplify the analysis of the probabilistic evolution of the ECAs.

Notation 1. We say that a transition is *active* if it changes the state of the cell where it is applied. Each ECA is fully determined by its active transitions. We label each active transition by a letter as follow:

label	A	B	C	D	E	F	G	H
$x\ y\ z$	000	001	100	101	010	011	110	111
$\delta(x,y,z)$	1	1	1	1	0	0	0	0

We label each ECA by the set of its active transitions. Note that with these notations, the DQECAs are exactly the ECAs having a label containing neither A nor H.

We consider three kinds of dynamics for ECAs: the *synchronous dynamics*, the α-*asynchronous dynamics* and the *fully asynchronous dynamics*. The synchronous dynamics is the classic dynamics of cellular automata, where the transition function is applied at each (discrete) time step on each cell simultaneously.

Definition 3 (Synchronous Dynamics). *The* synchronous dynamics $S_\delta : Q^U \to Q^U$ *of an ECA* δ, *associates deterministically to each configuration* x *the configuration* y, *such that for all* $i \in U$, $y_i = \delta(x_{i-1}, x_i, x_{i+1})$.

Definition 4 (Asynchronous Dynamics). *An* asynchronous dynamics AS_δ *of an ECA δ associates to each configuration x a random configuration y, such that $y_i = x_i$ for $i \notin S$, and $y_i = \delta(x_{i-1}, x_i, x_{i+1})$ for $i \in S$, where S is a random subset of U chosen by a daemon. We consider two types of asynchronous dynamics:*

- *in the α-*asynchronous dynamics*, the daemon selects at each time step each cell i in S independently with probability α where $0 < \alpha \leqslant 1$. The random function which associates the random configuration y to x according to this dynamics is denoted AS_δ^α.*
- *in the* fully asynchronous dynamics*, the daemon chooses a cell i uniformly at random and sets $S = \{i\}$. The random function which associates the random configuration y to x according to this dynamics is denoted AS_δ^F.*

For a given ECA δ, we denote by x^t the random variable for the configuration obtained after t applications of the asynchronous dynamics function AS_δ on configuration x, i.e., $x^t = (AS_\delta)^t(x)$. Note that $(x^t)_{t\in\mathbb{N}}$ is an homogeneous Markov chain on Q^n. Remark that AS_δ could equivalently be seen as a function with two arguments, the configuration x and the random subset $S \subseteq U$ chosen according to the processes listed above.

Definition 5 (Fixed point). *We say that a configuration x is a* fixed point *for δ under asynchronous dynamics if $AS_\delta(x) = x$ whatever the choice of S is (the cells to be updated). $\mathfrak{F}_\delta$ denotes the set of fixed points for δ.*

The set of fixed points for the considered asynchronous dynamics is clearly identical to $\{x : S_\delta(x) = x\}$ the set of fixed points of the synchronous dynamics. The set of fixed points of an automaton can be easily deduce from its labeling as shown in [1]. Every DQECA admits two *trivial fixed points*, $\mathtt{0}^n$ and $\mathtt{1}^n$.

Definition 6 (Relaxation Time). *Given an ECA δ and a configuration x, we denote by $T_\delta(x)$ the random variable for the time elapsed until a fixed point is reached from configuration x under an asynchronous dynamics, i.e., $T_\delta(x) = \min\{t : x^t \in \mathfrak{F}_\delta\}$. The* relaxation time *of ECA δ is $\max_{x\in Q^U} \mathbb{E}[T_\delta(x)]$.*

If $\alpha < 1$ the process $(x^t)_{t\in\mathbb{N}}$ converges to a stationary distribution, but we will abusively say that an ECA *diverges from an initial configuration x* if the probability to reach a fixed point from x is 0. We can now state our main theorem.

Theorem 1 (Main result). *Under α-asynchronous dynamics, among the sixty-four DQECAs, we can determine the behaviour of 52 of them:*

- *forty-eight converge almost surely to a random fixed point from any initial configuration, and the relaxation times of these forty-eight convergent DQECAs are 0, $\Theta(\frac{\ln n}{\ln(1-\alpha)})$, $\Theta(\frac{n}{\alpha})$, $\Theta(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$, $O(\frac{n}{\alpha(1-\alpha)})$, $O(\frac{n}{\alpha^2(1-\alpha)})$, $\Theta(\frac{n^2}{\alpha(1-\alpha)})$.*
- *two diverge from any initial configuration that is neither $\mathtt{0}^n$, nor $\mathtt{1}^n$, nor $(\mathtt{01})^{n/2}$ when n is even.*

Table 1. DQECAs under asynchronous and synchronous dynamics (see Section 2)

ECA (#)	**Rule**	**01**	**10**	**010**	**101**	**Shift**	**Spawn**	**Fork**	**Annihil.**	**Full As.**	**α-Asynchr.**	**Synchr.**
204 (1)	·	·	·	·	·	·	·	·	·	0	0	0
200 (2)	E	·	·	+	·	·	·	·	·	$\Theta(n \ln n)$	$\Theta(\frac{\ln n}{\ln(1-\alpha)})$	1
232 (1)	DE	·	·	+	+	·	·	·	·	$\Theta(n \ln n)$	$O(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
206 (4)	B	$\leftarrow$	·	·	·	·	·	·	·	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
222 (2)	BC	$\leftarrow$	$\rightarrow$	·	·	·	·	·	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
192 (4)	EF	$\rightarrow$	·	+	·	·	·	·	·	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
128 (2)	EFG	$\rightarrow$	$\leftarrow$	+	·	·	·	·	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
234 (4)	BDE	$\leftarrow$	·	+	+	+	·	·	·	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
202 (4)	BE	$\leftarrow$	·	+	·	+	·	·	·	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
250 (2)	BCDE	$\leftarrow$	$\rightarrow$	+	+	+	·	+	+	$\Theta(n^2)$	$O(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
218 (2)	BCE	$\leftarrow$	$\rightarrow$	+	·	+	·	+	+	$\Theta(n^2)$	$O(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
242 (4)	BCDEF	$\leftrightsquigarrow$	$\rightarrow$	+	+	+	+	+	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha(1-\alpha)})$	∞
130 (4)	BEFG	$\leftrightsquigarrow$	$\leftarrow$	+	·	+	+	·	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha(1-\alpha)})$	∞
170 (2)	BDEG	$\leftarrow$	$\leftarrow$	+	+	+	·	·	·	$\Theta(n^3)$	$\Theta(\frac{n^2}{\alpha(1-\alpha)})$	∞
138 (4)	BEG	$\leftarrow$	$\leftarrow$	+	·	+	·	·	·	$\Theta(n^3)$	$\Theta(\frac{n^2}{\alpha(1-\alpha)})$	∞
226 (2)	BDEF	$\leftrightsquigarrow$	·	+	+	+	+	·	·	$\Theta(n^3)$	$O(\frac{n^2}{\alpha(1-\alpha)})$	∞
194 (4)	BEF	$\leftrightsquigarrow$	·	+	·	+	+	·	·	$\Theta(n^3)$	$O(\frac{n}{\alpha^2(1-\alpha)})$	∞
178 (1)	BCDEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	+	+	+	+	+	$\Theta(n^3)$	phase transition ? poly. for $\alpha < \alpha_c$? exp. for $\alpha > \alpha_c$?	∞
146 (2)	BCEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	·	+	+	+	+	$\Theta(n^3)$	phase transition ? poly. for $\alpha < \alpha'_c$? exp. for $\alpha > \alpha'_c$?	∞
210 (4)	BCEF	$\leftrightsquigarrow$	$\rightarrow$	+	·	+	+	+	+	$\Theta(n2^n)$	polynomial ?	∞
214 (4)	BCF	$\leftrightsquigarrow$	$\rightarrow$	·	·	·	+	·	+	∞	polynomial ?	∞
150 (1)	BCFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	·	·	·	+	·	+	∞	exponential ?	∞
198 (2)	BF	$\leftrightsquigarrow$	·	·	·	·	+	·	·	∞	∞	∞
142 (2)	BG	$\leftarrow$	$\leftarrow$	·	·	·	·	·	·	∞	∞	∞

- *two converge with a small probability from few initial configurations when n is even and diverge otherwise.*

The twelves others (5 after symetries) have different behaviours that we cannot prove presently. Some seem to exhibit a phase transition but their mathematical analysis remains a challenging problem. All the results and the conjectures (with question marks) are summed up in table 1.

3 Key Observations

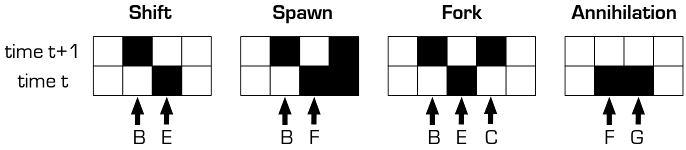
Due to `0`/`1` and reversal symmetries of configurations, we shall w.l.o.g. only consider the 24 DQECAs listed in Tab. 1 among the 64 DQECAs. For each of these 24 DQECAs, the number of the equivalent automata under symmetries is written within parentheses after their classic ECA code in the table.

From now on, we only consider the α-asynchronous dynamics; this will be implicit in all the following propositions. Our results rely on the study of the evolution of the *0-regions* and *1-regions* in the space-time diagram (i.e., of the intervals of consecutive 0s or consecutive 1s in configuration x^t). We will now enumerate the different ways the regions can be affected.

First we consider the cases where a cell updates and none of its two neighbours update:

- Transitions D and E are thus responsible for decreasing the number of regions in the space-time diagram: D "erases" the isolated 1s and E the isolated 0s.
- Transitions B and F act on patterns 01. Intuitively, transition B moves a pattern 01 to the left, and transition F moves it to the right. In particular, patterns 01 perform a kind of random walk for DQECA with both transitions B and F if no others phenomena occurs. The arrows in Tab. 1 represent the different behavior of the patterns: ← or →, for left or right moves of the patterns 01 or 10; ↭, for random walks of these patterns.
- Similarly, transitions C and G act on patterns 10. Transition C moves a pattern 10 to the right, and transition G moves it to the left.

One important observation made during the study of the fully asynchronous dynamics in [1] is that the number of regions can only decrease and each activation of D or E makes the number of regions decrease by one. This statement is not true anymore under the α-asynchronous dynamics, as we will see now. Here are the new phenomena when two or three neighboring cells update at the same time:



- **Shift** phenomenon occurs with the activation of rules B and E, or C and E, or F and D, or G and D together: in this case an isolated 0 or an isolated 1 is shifted. Here even if a transition D or E is activated, no regions is erased.
- **Spawn** phenomenon occurs with the activation of rules B and F, or C and G together: a pattern 0011 can create a new region. This is an important phenomenon because it increases the number of regions by one each time it occurs.
- **Fork** phenomenon occurs with the activation of rules B, C and E or F, G and D together: here three neighboring cellules update at the same time and an isolated point is duplicated. This phenomenon increases the number of regions by one each time it occurs.
- **Annihilation** phenomenon occurs with the activation of rules B and C or F and G together: the activation of these two rules erases a region of length 2. This is a very important new phenomenon because it is another way to decrease the number of region. In particular, it is the only way to decrease the number of regions in automaton where neither D nor neither E is activated.

The next section is devoted to the tools which will be used to prove our main theorem.

4 Lyapunov Functions Based on Local Neighbourhoods

Definition 7 (Mask). *A* mask $\dot{m}$ *is a word on* $\{\mathtt{0},\mathtt{1},\dot{\mathtt{0}},\dot{\mathtt{1}}\}$ *containing exactly one dotted letter in* $\{\dot{\mathtt{0}},\dot{\mathtt{1}}\}$*. We say that the cell* i *in configuration* x *matches the mask* $\dot{m} = m_{-k}\dots m_{-1}\dot{m}_0 m_1 \dots m_l$ *if* $x_{i-k}\dots x_i \dots x_{i+l} = m_{-k}\dots m_0 \dots m_l$*. We denote by* m *the undotted word* $m_{-k}\dots m_0 \dots m_l$*.*

Fact 2. *The number of cells matching a given mask* $\dot{m}$ *in a configuration* x *is exactly* $|x|_m$*, the number of occurrences of the undotted word* m*.*

Definition 8 (Masks basis). *A* masks basis $\mathcal{B}$ *is a finite set of masks such that for any configuration* x *and any cell* i*, there exists an unique* $\dot{m} \in \mathcal{B}$ *that matches cell* i*.*

A masks basis $\mathcal{B}$ can be represented by a binary tree where the children of a node are labelled by adding $\mathtt{0}$ and $\mathtt{1}$ to the node label, on the right or the left (the children of the root receive $\dot{\mathtt{0}}$ and $\dot{\mathtt{1}}$), and where the masks of $\mathcal{B}$ are the labels of the leaves. Reciprocally, any binary tree observing these properties defines a unique masks basis by taking the labels of its leaves. Figure 2b page 463 illustrates the construction of the tree for the masks basis $\mathcal{B} = \{\mathtt{1}\dot{\mathtt{1}}, \mathtt{00}\dot{\mathtt{1}}\mathtt{0}, \mathtt{00}\dot{\mathtt{1}}\mathtt{1}, \mathtt{010}\dot{\mathtt{1}}, \mathtt{110}\dot{\mathtt{1}}, \dot{\mathtt{0}}\mathtt{0}, \mathtt{0}\dot{\mathtt{0}}\mathtt{10}, \mathtt{0}\dot{\mathtt{0}}\mathtt{11}, \mathtt{01}\dot{\mathtt{0}}\mathtt{1}, \mathtt{11}\dot{\mathtt{0}}\mathtt{1}\}$.

Masks bases will be used to define Lyapunov weight functions from local patterns. It provides an efficient tool to validate exhaustive case analysis.

Definition 9 (Local weight function). *A* local weight function f *is a function from a masks basis* $\mathcal{B}$ *to* $\mathbb{Z}$*. The local weight of the cell* i *in configuration* x *given by* f *is* $F(x,i) = f(\dot{m})$ *where* $\dot{m}$ *is the unique mask in* $\mathcal{B}$ *matching cell* i*. The* weight of a configuration x *given by* f *is defined as* $F(x) = \sum_i F(x,i)$*.*

Fact 3. *Given a local weight function* $f : \mathcal{B} \to \mathbb{Z}$*, the weight of configuration* x *is equivalently defined as:* $F(x) = \sum_{\dot{m}\in\mathcal{B}} f(\dot{m}) \cdot |x|_m$*.*

Notation 2. *For a given random sequence of configurations* $(x^t)_{t\in\mathbb{N}}$ *and a weight function* F *on the configurations, we denote by* $(\Delta F(x^t))_{t\in\mathbb{N}}$ *the random sequence* $\Delta F(x^t) = F(x^{t+1}) - F(x^t)$*.*

The next lemma provides upper bounds on stopping times for the markovian sequence of configurations $(x^t)_{t\in\mathbb{N}}$ subject to a weight function F decreasing or remaining constant on average (a *Lyapunov function*). Its proof can be found in [1].

Lemma 1. *Let* $m \in \mathbb{Z}_+$ *and* $\epsilon > 0$*. Consider* (x^t) *a random sequence of configurations, and* F *a weight function such that* $(\forall x)\ F(x) \in \{0,\dots,m\}$*. Assume that if* $F(x^t) > 0$*, then* $\mathbb{E}[\Delta F(x^t)|x^t] \leqslant -\epsilon$*. Let* $T = \min\{t : F(x^t) = 0\}$ *denote the random variable for the first time* t *where* $F(x^t) = 0$*. Then,* $\mathbb{E}[T] \leqslant \frac{m+F(x^0)}{\epsilon}$*.*

5 Relaxation Times

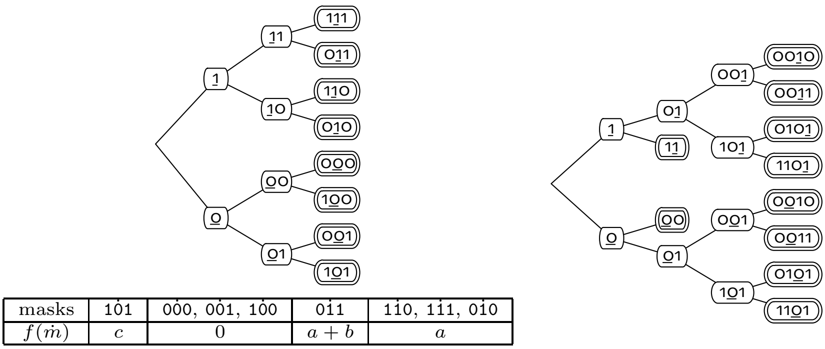
Due to space constraints, we only present the theorem for the relaxation time of the DQECA **BEF**. The results for Identity, **E**, **EF**, **EFG**, **DE**, **B**, **BC**, **BDE**, **BE**, **BCDE**, **BCE**, **BCDEF**, **BEFG**, **BDEG**, **BEG**, **BDEF**, **BF**, **BG** are given in Tab. 1 (check our websites for the full version of the paper).

5.1 Automaton BEF(194)

The fixed points of this automaton are 0^n and 1^n. Fixed point 1^n cannot be reached from any other configuration. Under fully asynchronous dynamics, the length of any 1-region follows a random walk, and thus converges in $O(n^3)$ in expectation. Here, the Spawn phenomenon (rule **B** and **F** applied together to cells $i-1$ and i) can transform the pattern $000\dot{1}11$ into the pattern 001011 with probability α^2. Even if the number of 0s and 1s are the same in these two patterns, in the pattern 001011 two 1s can become 0s at the next step (by applying rules **E** and **F**), while only one 0 can become a 1 at the next step (by applying rule **B**). So the creation of isolated 0s tends to decrease the number of 1s at the next step, leading to a speed up from a cubic relaxation time under fully asynchronous dynamics to a linear relaxation time in α-asynchronous dynamics with the respect of the size of the configuration. We consider the following variant. Let $a=-2c+2, b=-1, c=-\left\lfloor\frac{3}{\alpha}\right\rfloor-1$. We use the masks basis and local weight function f given on page 463. We have: $F(x)=a|x^t|_{1}+b|x^t|_{011}+c|x^t|_{101}$. For all configuration x, $F(x)\in\{0,\ldots,2n(\left\lfloor\frac{3}{\alpha}\right\rfloor+4)\}$ and $F(x)=0$ if and only if $x=0^n$.

Lemma 2. *For all non-fixed point configuration* x^t, $\mathbb{E}[\Delta F(x^t)]\leqslant-\alpha(1-\alpha)|x^t|_{01}$.

Proof. By linearity of expectation: $\mathbb{E}[\Delta F(x)]=\sum_{i=0}^{n-1}\mathbb{E}[\Delta F(x,i)]$. We evaluate the variation of $F(x,i)$ using the masks basis of Figure 2b.



masks	$1\dot{0}1$	$0\dot{0}0$, $0\dot{0}1$, $1\dot{0}0$	$0\dot{1}1$	$1\dot{1}0$, $1\dot{1}1$, $0\dot{1}0$
$f(\dot{m})$	c	0	$a+b$	a

(a) Weight function (b) Analysis

Fig. 2. Mask basis for BEF

Consider that at step t, cell i matches:

- **mask $\mathtt{1\dot{1}}$:** $F(x^t, i) = a$. With probability 1 at the step $t+1$, cell i matches mask $\mathtt{\dot{1}}$. So $F(x^{t+1}, i) \in \{a, a+b\}$. Since $b < 0$, $F(x^{t+1}, i) \leqslant F(x^t, i)$. Thus, $\mathbb{E}[(\Delta F(x^t, i)] \leqslant 0$.
- **mask $\mathtt{\dot{0}0}$:** $F(x^t, i) = 0$. With probability 1 at the step $t+1$, cell i matches mask $\mathtt{\dot{0}}$. So $F(x^{t+1}, i) \in \{0, c\}$. Since $c < 0$, $F(x^{t+1}, i) \leqslant F(x^t, i)$. Thus, $\mathbb{E}[(\Delta F(x^t, i)] \leqslant 0$.
- **mask $\mathtt{00\dot{1}0}$ (and $\mathtt{0\dot{0}10}$ together):**

With probability	$\alpha(1-\alpha)$	$\alpha(1-\alpha)$	$(1-\alpha)^2$	α^2
At the step $t+1$, cell i matches mask	$\mathtt{0\dot{0}}$	$\mathtt{1\dot{1}}$	$\mathtt{0\dot{1}}$	$\mathtt{1\dot{0}}$
and $\Delta F(x^t, i-1)$	$= 0$	$= a+b$	$= 0$	$= a$
and $\Delta F(x^t, i)$	$= -a$	$= 0$	$= 0$	$= -a$

  Thus, $\mathbb{E}[\Delta F(x^t, i) + \Delta F(x^t, i-1)] = -a\alpha(1-\alpha) + (a+b)\alpha(1-\alpha) = b\alpha(1-\alpha) = -\alpha(1-\alpha)$.
- **mask $\mathtt{00\dot{1}1}$ (and $\mathtt{0\dot{0}11}$ together):**

With probability	$\alpha(1-\alpha)$	$\alpha(1-\alpha)$	$(1-\alpha)^2$	α^2
at the step $t+1$, cell i matches mask	$\mathtt{0\dot{0}}$	$\mathtt{1\dot{1}}$	$\mathtt{0\dot{1}}$	$\mathtt{1\dot{0}}$
and $\Delta F(x^t, i-1)$	$= 0$	$= a+b$	$= 0$	$= c-a-b$
and $\Delta F(x^t, i)$	$= -a-b$	$= -b$	$= 0$	$= a$

  Thus, $\mathbb{E}[\Delta F(x^t, i) + \Delta F(x^t, i-1)] = (-a-b)\alpha(1-\alpha) + a\alpha(1-\alpha) + (c-b)\alpha^2 \leqslant \alpha(1-\alpha) - 2\alpha \leqslant -\alpha(1-\alpha)$.
- **mask $\mathtt{110\dot{1}}$ (and $\mathtt{11\dot{0}1}$ together):**

With probability	α	$(1-\alpha)$
at the step $t+1$, the cell i matches mask	$\mathtt{0\dot{0}}$	$\mathtt{0\dot{1}}$
and $\Delta F(x, i-1)$	$= -c$	$= 0$
and $\Delta F(x, i)$	$= -a-b$	$= 0$

  Thus, $\mathbb{E}[\Delta F(x^t, i) + \Delta F(x^t, i-1)] = (-a-b-c)\alpha(1-\alpha) \leqslant -\alpha(1-\alpha)$.
- **mask $\mathtt{010\dot{1}}$ (and $\mathtt{01\dot{0}1}$ together):**

With probability	α	$(1-\alpha)^2$	$\alpha(1-\alpha)$
at the step $t+1$, the cell i matches mask	$\mathtt{0\dot{0}}$	$\mathtt{10\dot{1}}$	$\mathtt{00\dot{1}}$
and $\Delta F(x^t, i-1)$	$= -c$	$= 0$	$= -c$
and $\Delta F(x^t, i)$	$= -a-b$	$= 0$	$= 0$

  Thus, $\mathbb{E}[\Delta F(x^t, i) + \Delta F(x^t, i-1)] = (-a-b-c)\alpha(1-\alpha) - c\alpha(1-\alpha) \leqslant -\alpha(1-\alpha)$.

Finally $\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t, i)] \leqslant -\alpha(1-\alpha)(|x^t|_{\mathtt{0010}} + |x^t|_{\mathtt{0011}} + |x^t|_{\mathtt{1011}} + |x^t|_{\mathtt{0101}}) \leqslant -\alpha(1-\alpha)|x^t|_{\mathtt{01}}$. So, as long as x^t is not a fixed point, we have $\mathbb{E}[\Delta F(x^t)] \leqslant -\alpha(1-\alpha)|x^t|_{\mathtt{01}} \leqslant -\alpha(1-\alpha)$.

Theorem 4. *Under α-asynchronous dynamics, DQECA* ***BEF*** *converges a.s. to a fixed point from any initial configuration. The relaxation time is* $O\big(\frac{n}{\alpha^2(1-\alpha)}\big)$.

Proof. Using Lemma 1 and Lemma 2, automaton **BEF** converges a.s. from any intial configuration (except $\mathtt{1}^n$) to $\mathtt{0}^n$. The relaxation time is $O\big(\frac{n}{\alpha} \times \frac{1}{\alpha(1-\alpha)}\big) = O\big(\frac{n}{\alpha^2(1-\alpha)}\big)$.

6 Conjectures

This section presents the remaining twelve DQECAs for which the mathematical analysis is not achieved yet. However by means of simulation and by the study of special patterns, we can give some insights of the phenomena which guide their dynamics and differentiate them from the other DQECAs.

Automaton ***BCDEFG(178).*** The fixed points of this automaton are exactly $\mathtt{0}^n$ and $\mathtt{1}^n$. Simulations show a phase transition concerning the relaxation time, which can be also clearly observed on time-space diagrams and seems to occur at $\alpha = \alpha_c \approx 0,5$. If $\alpha < \alpha_c$, the overall behaviour of the automaton does not drastically change when α varies: `0`- and `1`-regions merge into larger regions reducing their number, and it seems to converge to $\mathtt{0}^n$ or $\mathtt{1}^n$ with an $O(n^2/\alpha)$ expected time. While if $\alpha > \alpha_c$, large `0`- and `1`-regions crumble quickly at their frontiers and patterns of $\mathtt{0101}\cdots\mathtt{01}$ fill the space between the regions. The closer α is to 1, the smaller is the probability of formation of large regions. In that case, we conjecture that the relaxation time is exponential in n.

Automaton ***BCEFG(146).*** The fixed points of this automaton are exactly $\mathtt{0}^n$ and $\mathtt{1}^n$. This automaton shows a phase transition which seems to appear when $\alpha = \alpha'_c \approx 0,67$. When $\alpha < \alpha'_c$, `1`-regions quickly disappear and the expected convergence time is conjectured to be polynomial in n. When α is close to 1, like the automaton **BCDEFG**, large `1`-regions do not survive because they tend to crumble very quickly. On the other hand, isolated `1`s are easily deleted and seem to multiply faster than they disappear. In that case, we conjecture that the relaxation time is exponential in n.

Automaton ***BCF(214).*** The fixed points of this automaton are $\mathtt{0}^n$, $\mathtt{1}^n$ and $(\mathtt{01})^{n/2}$ (if n is even). When starting from another configuration, it is impossible to reach one of these fixed points in the fully asynchronous dynamics, since the number of regions remains constant. With the α-asynchronous dynamics, due to the Annihilation phenomenon, any configuration converges a.s. to a fixed point within a finite time. The sizes of large `0`-regions decrease quickly. Only regions with two `0`s may disappear, but `10011` patterns may evolve into `11111` or `10101` with the same probability. This could lead to an increase of small regions, tending to slow down the convergence. However a sequence of consecutive small `0`-regions slows down the spawning phenomenon: in a `1001001` pattern, the first `00` region can not split. Thus the number of regions tends to decrease. We conjecture that the relaxation time is polynomial in n and contains an $O(\frac{1}{\alpha^2(1-\alpha)})$ term corresponding to the deletion of `00` regions.

Automaton ***BCFG(150).*** The fixed points of this automaton are $\mathtt{0}^n$, $\mathtt{1}^n$ and $(\mathtt{01})^{n/2}$ (if n is even). In the fully asynchronous dynamics, this automaton does not converge to a fixed point since it is impossible to suppress a region. However in the α-asynchronous dynamics, due to the Annihilation phenomenon, this automaton converges a.s. to a fixed point within a finite time. Simulations suggest that the relaxation time is exponential in n.

*Automaton **BCEF(210)**.* The fixed points of this automaton are exactly `0`^{n} and `1`^{n}. In the fully asynchrnous dynamics, this automaton converges to `0`^{n} with a exponential expected time. In both fully asynchrounous and α-asynchronous, dynamics, the sizes of regions of `0` tend to decrease quickly. However in the fully asynchronous dynamics, they may only disappear by merging, and the size of the last `0`-region will converge to 0 in exponential expected time. The α-asynchronous dynamics introduces the Annihilation phenomenon. On simulations, the convergence to fixed points seems to be polynomial. This case seems similar to the **BCF** automaton, but the analysis is a bit more complicated since `0`-regions may merge, and this must be taken into account in the proof of bounds for the relaxation time.

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