On the Normed Linear Space of Hausdorff Continuous Functions

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Abstract. In the present work we show that the linear operations in the space of Hausdorff continuous functions are generated by an extension property of these functions. We show that the supremum norm can be defined for Hausdorff continuous functions in a similar manner as for real functions, and that the space of all bounded Hausdorff continuous functions on an open set is a normed linear space. Some issues related to approximations in the space of Hausdorff continuous functions by subspaces are also discussed.

1 Introduction

The concept of Hausdorff continuity generalizes the concept of continuity of real functions to interval-valued functions [4, 8]. Due to a minimality condition with respect to inclusion of graphs, Hausdorff continuous (H-continuous) functions retain some important properties of continuous functions, e.g. they are completely determined by their values on a dense subset of their domain. It is well-known that the operations (addition and multiplication by scalars) associated with interval structures typically do not infer a linear space [5]. In this regard the set of H-continuous functions is a notable exception. It is shown in [9] that one can define addition and multiplication by scalars on the set $\mathbb{H}(\Omega)$ of all H-continuous functions on an open subset Ω of $\mathbb{R}^n$ in such a way that $\mathbb{H}(\Omega)$ is a linear space. Naturally, these operations are not defined in a point-wise manner. In Sections 3 and 4 of the present work we show that the linear space operations on $\mathbb{H}(\Omega)$ are a direct consequence of an extension property of H-continuous functions. In Section 5 we show that the supremum norm can be defined for H-continuous functions in a similar way as for real functions, and that the space $\mathbb{H}_b(\Omega)$ of all bounded H-continuous functions on the open set Ω is a normed linear space. However, due to the involvement of discontinuous functions, a natural metric to be associated with the space $\mathbb{H}_b(\Omega)$ is the Hausdorff metric considered in Section 6. Issues related to approximations in $\mathbb{H}(\Omega)$ by subspaces are also discussed.

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2 General Setting

The real line is denoted by $\mathbb{R}$ and the set of all finite real intervals by $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R},\ \underline{a} \le \overline{a}\}$. Given an interval $a = [\underline{a}, \overline{a}] = \{x : \underline{a} \le x \le \overline{a}\} \in \mathbb{IR}$, $w(a) = \overline{a} - \underline{a}$ is the width of a, while $|a| = \max\{|\underline{a}|, |\overline{a}|\}$ is the modulus of a. An interval a is called proper interval, if $w(a) > 0$ and point interval, if $w(a) = 0$. Identifying $a \in \mathbb{R}$ with the point interval $[a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. We denote by $\mathbb{A}(\Omega)$ the set of all locally bounded interval-valued functions defined on an arbitrary set $\Omega \subseteq \mathbb{R}^n$. The set $\mathbb{A}(\Omega)$ contains the set $\mathcal{A}(\Omega)$ of all locally bounded real functions defined on Ω. Recall that a real function or an interval-valued function f defined on Ω is called locally bounded if for every $x \in \Omega$ there exist $\delta > 0$ and $M \in \mathbb{R}$ such that $|f(y)| < M,\ y \in B_\delta(x)$, where $B_\delta(x) = \{y \in \Omega : ||x - y|| < \delta\}$ denotes the open δ-neighborhood of x in Ω.

Let D be a dense subset of Ω. The mappings $I(D, \Omega, \cdot)$, $S(D, \Omega, \cdot) : \mathbb{A}(D) \longrightarrow \mathcal{A}(\Omega)$ defined for $f \in \mathbb{A}(D)$ and $x \in \Omega$ by

$$I(D, \Omega, f)(x) = \sup_{\delta > 0} \inf\{f(y) : y \in B_\delta(x) \cap D\},$$
$$S(D, \Omega, f)(x) = \inf_{\delta > 0} \sup\{f(y) : y \in B_\delta(x) \cap D\},$$

are called lower and upper Baire operators, respectively. The mapping $F : \mathbb{A}(D) \longrightarrow \mathbb{A}(\Omega)$, called *graph completion operator*, is defined by

$$F(D, \Omega, f)(x) = [I(D, \Omega, f)(x), S(D, \Omega, f)(x)],\ x \in \Omega,\ f \in \mathbb{A}(D).$$

In the case when $D = \Omega$ the sets D and Ω will be omitted, thus we write $I(f) = I(\Omega, \Omega, f),\ S(f) = S(\Omega, \Omega, f),\ F(f) = F(\Omega, \Omega, f)$.

Definition 1. *A function $f \in \mathbb{A}(\Omega)$ is S-continuous, if $F(f) = f$.*

Definition 2. *A function $f \in \mathbb{A}(\Omega)$ is Hausdorff continuous (H-continuous), if $g \in \mathbb{A}(\Omega)$ with $g(x) \subseteq f(x)$, $x \in \Omega$, implies $F(g)(x) = f(x)$, $x \in \Omega$.*

Theorem 1. *[1, 8] For every $f \in \mathbb{A}(\Omega)$ the functions $F(I(S(f)))$ and $F(S(I(f)))$ are H-continuous.*

H-continuous functions are similar to usual continuous real functions in that they assume point values everywhere on Ω except for a set of first Baire category. More precisely, it is shown in [1] that for every $f \in \mathbb{H}(\Omega)$ the set

$$W_f = \{x \in \Omega : w(f(x)) > 0\} \tag{1}$$

is of first Baire category and f is continuous on $\Omega \backslash W_f$. Since a finite or countable union of sets of first Baire category is also a set of first Baire category we have:

Theorem 2. *Let the set Ω be open and let $\mathcal{F}$ be a finite or countable set of H-continuous functions. Then the set $D_{\mathcal{F}} = \{x \in \Omega : w(f(x)) = 0,\ f \in \mathcal{F}\} = \Omega \setminus \bigcup_{f \in \mathcal{F}} W_f$ is dense in Ω and all functions $f \in \mathcal{F}$ are continuous on $D_{\mathcal{F}}$.*

The graph completion operator is inclusion isotone i) w. r. t. the functional argument, that is, if $f, g \in \mathbb{A}(D)$, where D is dense in Ω, then

$$f(x) \subseteq g(x),\ x \in D \implies F(D, \Omega, f)(x) \subseteq F(D, \Omega, g)(x),\ x \in \Omega, \tag{2}$$

and, ii) w. r. t. the set D in the sense that if D_1 and D_2 are dense subsets of Ω and $f \in \mathbb{A}(D_1 \cup D_2)$ then

$$D_1 \subseteq D_2 \implies F(D_1, \Omega, f)(x) \subseteq F(D_2, \Omega, f)(x),\ x \in \Omega. \tag{3}$$

In particular, (3) implies that for any dense subset D of Ω and $f \in \mathbb{A}(\Omega)$ we have $F(D, \Omega, f)(x) \subseteq F(f)(x),\ x \in \Omega$. The graph completion operator is idempotent. Moreover [2], if the sets D_1 and D_2 are both dense in Ω and $D_1 \subseteq D_2$ then

$$F(D_2, \Omega, \cdot) \circ F(D_1, \Omega, \cdot) = F(D_1, \Omega, \cdot). \tag{4}$$

Let $f \in \mathbb{A}(\Omega)$. For every $x \in \Omega$ the value of f is an interval $[\underline{f}(x), \overline{f}(x)] \in \mathbb{IR}$. Hence, f can be written in the form $f = [\underline{f}, \overline{f}]$ where $\underline{f}, \overline{f} \in \mathcal{A}(\Omega)$ and $\underline{f}(x) \leq \overline{f}(x),\ x \in \Omega$. The lower and upper Baire operators as well as the graph completion operator of an interval-valued function f can be represented in terms of $\underline{f}$ and $\overline{f}$, namely, for every dense subset D of Ω: $I(D, \Omega, f) = I(D, \Omega, \underline{f})$, $S(D, \Omega, f) = S(D, \Omega, \overline{f})$, $F(D, \Omega, f) = [I(D, \Omega, \underline{f}), S(D, \Omega, \overline{f})]$.

3 Extension and Restriction Properties

Let $\Omega \subseteq \mathbb{R}^n$ and let D be dense in Ω. Extending a function f defined on D to Ω while preserving its properties (e.g. linearity, continuity) is an important issue in functional analysis. Recall that if f is continuous on D it does not necessarily have a continuous extension on Ω. The next theorem shows that an H-continuous function on D has a unique H-continuous extension on Ω.

Theorem 3. *Let $\varphi \in \mathbb{H}(D)$, where D is dense subset of Ω. Then there exists unique $f \in \mathbb{H}(\Omega)$, such that $f(x) = \varphi(x)$, $x \in D$. Namely, $f = F(D, \Omega, \varphi)$.*

Proof. Let $f = F(D, \Omega, \varphi)$. From the fact that φ is H-continuous on D it follows that $F(D, D, \varphi) = \varphi$. Therefore, for every $x \in D$ we have $f(x) = F(D, \Omega, \varphi)(x) = F(D, D, \varphi)(x) = \varphi(x)$. Hence f is an extension of φ over Ω. We show next that f is H-continuous on Ω. Using property (4) we obtain

$$F(f) = F(\Omega, \Omega, F(D, \Omega, \varphi)) = F(D, \Omega, \varphi) = f. \tag{5}$$

Let $g \in \mathbb{A}(\Omega)$ satisfy the inclusion

$$g(x) \subseteq f(x),\ x \in \Omega. \tag{6}$$

Then using the inclusion isotone property (2) of the operator F we have

$$F(g)(x) \subseteq F(f)(x) = f(x),\ x \in \Omega. \tag{7}$$

Relation (6) implies $g(x) \subseteq f(x) = \varphi(x),\ x \in D$. Using again the H-continuity of φ on D we obtain $F(D,D,g)(x) = \varphi(x),\ x \in D$. From this equality and properties (2) and (3) of F we obtain

$$f(x) = F(D,\Omega,F(D,D,g))(x) \subseteq F(D,\Omega,F(g))(x) \subseteq F(g)(x),\ x \in \Omega. \qquad (8)$$

Inclusions (7) and (8) give $F(g) = f$ which implies that f is H-continuous on Ω.

Finally, we prove uniqueness. Let $h \in \mathbb{H}(\Omega)$ be another extension of φ, that is, $h(x) = \varphi(x),\ x \in D$. We have $f(x) = F(D,\Omega,\varphi)(x) = F(D,\Omega,h)(x) \subseteq F(h)(x) = h(x),\ x \in \Omega$. Then the H-continuity of h implies that $F(f) = h$. Using (5) we obtain $f = h$, which completes the proof of the theorem.

Corollary 1. *Let $f,g \in \mathbb{H}(\Omega)$ and let D be a dense subset of Ω. Then*
a) $f(x) \le g(x),\ x \in D \implies f(x) \le g(x),\ x \in \Omega,$
b) $f(x) = g(x),\ x \in D \implies f(x) = g(x),\ x \in \Omega.$

Let $D \subseteq \Omega$. For $f \in \mathbb{A}(\Omega)$ denote by $f|_D$ the restriction of f on D, i. e. $f|_D \in \mathbb{A}(D)$ and $f|_D(x) = f(x),\ x \in D$. The next theorem shows that the restriction of an H-continuous function on an open subset is H-continuous.

Theorem 4. *Let D be an open subset of Ω. If $f \in \mathbb{H}(\Omega)$ then $f|_D \in \mathbb{H}(D)$.*

Proof. Since D is open for every $x \in D$ we have $B_\delta(x) \subseteq D$ for $\delta > 0$ small enough. Hence, for $x \in D$ we have $S(D,D,f|_D)(x) = S(f)(x)$, $I(D,D,f|_D)(x) = I(f)(x)$, $F(D,D,f|_D)(x) = F(f)(x)$. Then the theorem follows from Theorem 1.

4 The Linear Space of Hausdorff Continuous Functions

In the sequel we assume that the set Ω is open. For every two functions $f,g \in \mathbb{H}(\Omega)$ denote $D_{fg} = \Omega \setminus (W_f \cup W_g)$, where W_f and W_g are defined by (1). Using addition of intervals the point-wise sum of $f = [\underline{f},\overline{f}] \in \mathbb{H}(\Omega)$ and $g = [\underline{g},\overline{g}] \in \mathbb{H}(\Omega)$ is given by $(f+g)(x) = f(x)+g(x) = [\underline{f}(x)+\underline{g}(x), \overline{f}(x)+\overline{g}(x)],\ x \in \Omega$. It is easy to see that the point-wise sum of H-continuous functions is not always H-continuous [9]. However, the restrictions of f, g and $f+g$ on the set D_{fg} which is dense in Ω, see Theorem 2, are continuous real functions. This suggests the definition of a new operation addition "$\oplus$" on $\mathbb{H}(\Omega)$ as follows.

Definition 3. *Let $f,g \in \mathbb{H}(\Omega)$. Then $f \oplus g$ is the unique H-continuous extension of $(f+g)|_{D_{fg}}$ on Ω given by Theorem 3, that is, $f \oplus g = F(D_{fg},\Omega,f+g)$.*

Multiplication by scalars on $\mathbb{H}(\Omega)$ is defined point-wise; for $f \in \mathbb{H}(\Omega)$, $\alpha \in \mathbb{R}$

$$(\alpha * f)(x) = \alpha f(x) = \begin{cases} [\alpha\underline{f}(x), \alpha\overline{f}(x)] \text{ if } \alpha \ge 0, \\ [\alpha\overline{f}(x), \alpha\underline{f}(x)] \text{ if } \alpha < 0. \end{cases}$$

It can be verified that operations "$\oplus$" and "$*$" satisfy on $\mathbb{H}(\Omega)$ the axioms of a linear space [9]. In particular, the second distributive law, which is usually violated in interval structures, holds true; thus $(\mathbb{H}(\Omega),\oplus,*)$ is a linear space.

Denote by $\mathbb{H}_b(\Omega)$ the set of all bounded H-continuous functions on Ω. Clearly $\mathbb{H}_b(\Omega)$ is a linear subspace of $\mathbb{H}(\Omega)$. Note that the assumption that Ω is open, made in the beginning of the section, is not a significant restriction with regard to $\mathbb{H}_b(\Omega)$. One can easily see that the sets $\mathbb{H}_b(\Omega)$ and $\mathbb{H}(\overline{\Omega})$, where $\overline{\Omega}$ is the closure of Ω are identical. Indeed, according to Theorem 3 every function $f \in \mathbb{H}_b(\Omega)$ has a unique H-continuous extension $e(f)$ on $\overline{\Omega}$, that is, $e(f) \in \mathbb{H}(\overline{\Omega})$. Conversely, the restriction of every function $\mathbb{H}(\overline{\Omega})$ on Ω belongs to $\mathbb{H}_b(\Omega)$, see Theorem 4. Then the mapping $e : \mathbb{H}_b(\Omega) \longrightarrow \mathbb{H}(\overline{\Omega})$ is a bijection. Identifying f with $e(f)$ gives $\mathbb{H}_b(\Omega) = \mathbb{H}(\overline{\Omega})$. Hence by considering $\mathbb{H}_b(\Omega)$ we deal implicitly with the case when the domain is a closure of an open set. It is also easily seen that the supremum norm and the Hausdorff metric discussed in the sequel are preserved by e. Further we prefer to work with $\mathbb{H}_b(\Omega)$ rather then $\mathbb{H}(\overline{\Omega})$ since the linear operations are defined for Ω open and working with Ω closed or compact requires an extension of these definitions. We remark that similar approach is not possible for sets of continuous functions, since the set $C_b(\Omega)$ of all bounded continuous functions on Ω satisfies the inclusion $C(\overline{\Omega}) \subseteq C_b(\Omega)$ but the inverse inclusion is generally not true.

5 Supremum Norm and Approximations

The supremum norm on $\mathbb{H}_b(\Omega)$ can be defined as usually by

$$||f|| = \sup_{x\in\Omega} |f(x)|, \ f \in \mathbb{H}_b(\Omega). \tag{9}$$

Lemma 1. *If D is dense in Ω, then for $f \in \mathbb{H}_b(\Omega)$ we have $||f|| = \sup_{x\in D} |f(x)|$.*

Proof. The inequality $\sup_{x\in D} |f(x)| \le ||f||$ is obvious. To prove the inverse inequality denote $m = \sup_{x\in D} |f(x)|$. From $-m \le f(x) \le m, \ x \in D$, it follows $-m \le F(D, \Omega, f)(x) \le m, \ x \in \Omega$. Since f is H-continuous the inclusion $F(D, \Omega, f)(x) \subseteq F(f)(x) = f(x), \ x \in \Omega$, implies $f = F(D, \Omega, f)$. Therefore $|f(x)| \le m \ x \in \Omega$, which gives $||f|| \le m$. This completes the proof of Lemma 1.

Theorem 5. *The mapping $||\cdot|| : H_b(\Omega) \longrightarrow \mathbb{R}$ given in (9) is a norm on the linear space $\mathbb{H}_b(\Omega)$.*

Proof. Let $f, g \in \mathbb{H}_b(\Omega)$. According to Definition 3 for every $x \in D_{fg}$ we have $(f \oplus g)(x) = f(x) + g(x)$. Hence,

$$\sup_{x\in D_{fg}} |(f+g)(x)| = \sup_{x\in D_{fg}} |f(x)+g(x)| \le \sup_{x\in D_{fg}} |f(x)| + \sup_{x\in D_{fg}} |g(x)|.$$

Using Lemma 1 the above inequality implies $||f+g|| \le ||f|| + ||g||$. The remaining properties of the norm are trivially satisfied.

Theorem 5 shows that $\mathbb{H}_b(\Omega)$ considered with the operations "$\oplus$", "$*$" and the supremum norm is a normed linear space. Clearly the supremum norm on $\mathbb{H}_b(\Omega)$ is an extension of the supremum norm on the set of usual bounded continuous functions which is a subset of $\mathbb{H}_b(\Omega)$. Thus the familiar normed linear space $C_b(\Omega)$ is a subspace of $\mathbb{H}_b(\Omega)$.

It is well-known that the supremum norm has limited applications in the approximation of discontinuous functions. It is easy to construct examples of approximations in $\mathbb{H}_b(\Omega)$ by subspaces where the error of the approximation remains bounded away from zero irrespective of the dimension of the subspace. However, approximations with respect to the supremum norm work, in the case when the approximated function and/or some of its derivatives have only "jump" type of discontinuities at a finite number of points which are known. This is a situation which may arise e. g. in the solution of PDE's where discontinuities of the given boundary conditions are propagated in a predictable way within the interior of the domain of the solution [3].

6 Hausdorff Distance and Approximations by Finite Dimensional Subspaces

A natural metric to be associated with H-continuous functions is the Hausdorff metric denoted here by ρ. Let us recall that for $f, g \in \mathbb{H}_b(\Omega)$ the distance $\rho(f,g)$ is defined as the Hausdorff distance between the graphs of f and g considered as subsets of $\mathbb{R}^{n+1}$ [8]. It should be noted that the operation "$\oplus$" is not continuous with respect to the Hausdorff metric as can be shown by easy examples. Hence $\mathbb{H}_b(\Omega)$ is not a linear metric space in the sense of [6]. However, the next theorem shows that the operation "$\oplus$" satisfies a condition rather close to continuity. In the sequel "convergence" is meant in the sense of Hausdorff metric.

Theorem 6. *If the sequences $(f_k)_{k\in\mathbb{N}} \subseteq \mathbb{H}_b(\Omega)$ and $(g_k)_{k\in\mathbb{N}} \subseteq \mathbb{H}_b(\Omega)$ converge respectively to $f, g \in \mathbb{H}_b(\Omega)$, then the sequence $(f_k \oplus g_k)_{k\in\mathbb{N}}$ converges to an S-continuous function h, s. t. the only H-continuous function satisfying the inclusion $\phi(x) \subseteq h(x)$, $x \in \Omega$, is $\phi = f \oplus g$. Moreover, if $h \in \mathbb{H}_b(\Omega)$ then $h = f \oplus g$.*

The proof is rather technical and will be omitted.

We next illustrate the ideas of approximation in $\mathbb{H}_b(\Omega)$ by elements of finite dimensional linear subspaces in the case of functions of one variable, that is, $\Omega = (a,b) \subseteq \mathbb{R}$. Denote by φ the Π-form function:

$$\varphi(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ [0,1] & \text{if } x \in \{0,1\}; \\ 0 & \text{if } x < 0 \text{ or } x > 1. \end{cases}$$

For every $j \in \mathbb{N}$ we consider the following set of linearly independent functions

$$\{\phi_{jk} : k \in \mathbb{Z}\}, \ \ \phi_{jk}(x) = \varphi(2^j x - k), \ j, k \in \mathbb{Z}. \tag{10}$$

It is easy to see that $\phi_{jk} \in \mathbb{H}(\mathbb{R})$, $j, k \in \mathbb{Z}$. Therefore every linear combination of functions from the set (10) is also in $\mathbb{H}(\mathbb{R})$.

We now discuss approximation of H-continuous functions by linear combinations of functions from the set (10). To simplify matters we consider approximations on the interval $(0,1)$, that is, in the set $\mathbb{H}_b(0,1)$. It follows from Theorem 4 that the restrictions of the functions (10) to the interval $(0,1)$ belong to $\mathbb{H}_b(0,1)$.

In the sequel ϕ_{jk} denotes the restriction to the interval $(0,1)$ of the function ϕ_{jk} given in (10). Clearly in $\mathbb{H}_b(0,1)$ for every $j\in\mathbb{N}$ it is enough to consider the set

$$\{\phi_{jk} : k = 0,1,...,2^j-1\}. \tag{11}$$

Denote by V_j the linear subspace of $\mathbb{H}(0,1)$ spanned by the set of functions (11). Using that $\phi_{j-1,k}(x)=\phi_{j,2k}(x)+\phi_{j,2k+1}(x)$, $j,k\in\mathbb{Z}$, one can see that the inclusions $V_0\subset V_1\subset ...\subset V_j\subset ...\subset\mathbb{H}(0,1)$ hold true. Hence we have a similar situation to the adaptive multiresolution analysis discussed in [7].

Consider the operators $I_\delta:\mathbb{A}(0,1)\longrightarrow\mathcal{A}(0,1)$ and $S_\delta:\mathbb{A}(0,1)\longrightarrow\mathcal{A}(0,1)$ where $\delta>0$ and for every $f\in\mathbb{A}(0,1)$

$$I_\delta(f)(x)=\inf\{z\in f(y): y\in(0,1), |y-x|<\delta\},\ x\in(0,1),$$
$$S_\delta(f)(x)=\sup\{z\in f(y): y\in(0,1), |y-x|<\delta\},\ x\in(0,1).$$

For a given $\delta>0$ the modulus of H-continuity $\tau(f,\delta)$ of a function $f\in\mathbb{A}(0,1)$ is the Hausdorff distance between the completed graphs of $I_{\delta/2}(f)$ and $S_{\delta/2}(f)$, that is, $\tau(f,\delta)=\rho(F(I_{\delta/2}(f)),F(S_{\delta/2}(f)))$. It is shown in [8] that a function $f\in\mathbb{A}(0,1)$ is H-continuous if and only if $\lim_{\delta\to 0}\tau(f,\delta)=0$.

Let $f=[\underline{f},\overline{f}]\in\mathbb{H}(0,1)$ and let $j\in\mathbb{N}$. Using the operators $I_\delta(f)$ and $S_\delta(f)$ we can construct in V_j a lower approximation $L(f,j)$ of f and an upper approximation $U(f,j)$ of f as follows:

$$L(f,j)=\sum_{k=0}^{2^j-1} I_h(f)((2k+1)h)\phi_{jk},\quad U(f,j)=\sum_{k=0}^{2^j-1} S_h(f)((2k+1)h)\phi_{jk}, \tag{12}$$

where $h=2^{-j-1}$ and the sums are in terms of the addition "$\oplus$". The inequality

$$L(f,j)(x)\le f(x)\le U(f,j)(x),\quad x\in(0,1), \tag{13}$$

can be easily verified. Indeed, if we have $x\in(2^{-j}k,2^{-j}(k+1)$ for some $k\in\{0,1,...,2^j-1\}$ then $L(f,j)(x)=I_h(f)(2^{-j}(k+\frac{1}{2}))\phi_{jk}(x)\le f(x)$. Similarly, $U(f,j)(x)\ge\ f(x)$. Hence (13) holds on the set $\bigcup_{k=0}^{2^j-1}(2^{-j}k,2^{-j}(k+1))$ which is dense on $(0,1)$. Using that the functions involved in (13) are all H-continuous, we obtain that (13) holds for all $x\in(0,1)$, see Corollary 1.

Theorem 7. *For every* $f\in\mathbb{H}(0,1)$ *and* $j\in\mathbb{N}$ *we have*

$$\rho(L(f,j),f)\le\tau(f,2^{-j+1}),\quad \rho(U(f,j),f)\le\tau(f,2^{-j+1}).$$

Proof. Let $h=2^{-j-1}$ as in (12). From the inequalities

$$I_{2h}(f)(x)\le L(f,j)(x)\le f(x)\le U(f,j)\le S_{2h}(f)(x),\quad x\in(0,1),$$

it follows that $\rho(L(f,j),f)\le\rho(F(I_{2h}(f)),F(S_{2h}(f)))=\tau(f,4h)=\tau(f,2^{-j+1})$. Similarly, $\rho(U(f,j),f)\le\tau(f,2^{-j+1})$.

It follows from Theorem 7 that for every $f\in\mathbb{H}_b(0,1)$ both sequences $(L(f,j))_{j\in\mathbb{N}}$ and $(U(f,j))_{j\in\mathbb{N}}$ converge to f with respect to the Hausdorff distance ρ. Hence $\bigcup_{j=1}^{\infty}V_j$ is a dense subspace of $\mathbb{H}_b(0,1)$ considered as a metric space w. r. t. ρ.

7 Conclusion

H-continuous functions have a number of interesting and rather unique properties due to the fact that they share characteristics of both real-valued and interval-valued functions. The extension property discussed in Section 3 is in this category as it is typical neither for classes of real functions usually considered in Functional Analysis nor for classes of interval functions considered in Interval Analysis. We show that this extension property generates the linear space operations in $\mathbb{H}(\Omega)$ proposed in our previous work [9]. Our further discussion is devoted to issues of norm, metric and approximations of H-continuous functions. We introduce the supremum norm for H-continuous functions and prove that the set $\mathbb{H}_b(\Omega)$ of all bounded H-continuous functions is a normed linear space w. r. t. this norm. Recognizing the limitations of the supremum norm when discontinuous functions are involved we consider the Hausdorff metric on $\mathbb{H}_b(\Omega)$ and establish a strong connection between the metric and the linear space operations. The considered approximations by a subspace show that the Hausdorff metric is a natural metric to be associated with H-continuous functions.

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