Designing Diagrammatic Catalogues of Types of Basic Interval Equation: A Case Study*

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Abstract. The use of diagrammatic representations as *catalogues of cases* is analyzed using an example of the catalogue of types of the basic interval equation $a \cdot x = b$. The procedure of finding and describing the types is outlined and a number of different diagrammatic and tabular catalogues are presented and their drawbacks and merits discussed. Suggestions for other solutions, like different forms of the catalogue and interactive catalogue are included. Some preliminary guidelines for designing such catalogues are formulated as well.

1 Introduction

As advocated in the previous *MKM* Conference paper [6], diagrams can be used for efficient representation of complex mathematical knowledge. They offer readable general comprehension of some part of knowledge "at a glance," allowing also for representation of precise structural relationships. One of the several kinds of uses of mathematical diagrams proposed in that paper is a *catalogue of cases*. The purpose of such a catalogue is to group a number of similar objects, types of objects, or reasoning cases, with the main emphasis on comparing the listed objects and delineate differences and similarities between them. The current state of research on mathematical diagrams does not provide any ready for use guidelines for the design of such catalogues. Thus, this paper is structured as a case study—a detailed exposition of problems and experiences with some particular set of mathematical data and various approaches to catalogue them. On the basis of these experiences, an attempt is made to formulate some preliminary guidelines for the design of such catalogues.

In general graphic design practice [15, 16] a notion of *multiples* is used, meaning structures built from similar repeating components. Multiples allow representation of a number of similar objects, facilitating their comparison end enhancing the dimensionality of otherwise flat diagramming medium [16]. Some kinds of catalogues of cases can be designed as such multiples. Other forms are also possible, like region maps or graphs (networks).

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In this paper, various forms of the catalogue of structural types of the interval equation $a \cdot x = b$ are presented and discussed. The detailed analysis of solutions of this simple interval equation was first conducted in [3] and its basic types were listed there diagrammatically. Another, simplified form of this basic catalogue was included as an example in [6] as well. However, there is a much greater number of intermediate and degenerate types of this equation. That makes compiling, handling and use of the complete catalogue of types rather troublesome without more attention to proper design, as discussed in this paper. Several basic forms of the complete catalogue were listed for reference in the report [7], but the design issues were not discussed either there or in other works that included various versions of the catalogues [3, 5, 8]. This paper is devoted to the design issues of the catalogues. Further extensions and improvements of the catalogues (like an interactive version) are also proposed, and some general design guidelines are formulated. Several issues are only sketched, as work on them is still under way.

To make the paper self-contained, basic material on intervals, interval linear equations and interval space diagrams is also included, together with the diagrammatic procedure of solving the basic equation and finding its structural types. These details are needed to fully understand the structure and contents of the catalogues and relative merits and usability of their various versions. However, some of the details can be skipped by the reader not interested in the exact meaning of data items included in the catalogues.

The importance of the basic equation $a \cdot x = b$ itself and its solutions comes from the fact that, as was shown in [4, 5, 8], various characterizations of solution sets of the general many-dimensional system of interval equations $A \cdot x = b$ are provided by solution sets of that simple one-dimensional equation with coefficients a and b obtained as functions of appropriate coefficients of the general system of equations.

2 Real Intervals and Interval Equations

Interval analysis, a new approach to reliable numerical computing allowing for proper tackling of inexact data and rounding errors, is based on the notion of a *real interval.* Generally, a (proper) *real interval*, say u, is defined as a pair of real numbers $u = [\underline{u}, \overline{u}]$, so that its endpoints (*beginning* $\underline{u}$ and *end* $\overline{u}$) obey the inequality $\underline{u} \leq \overline{u}$. For most purposes, real intervals can be identified with the closed set of numbers $u = \{x \,|\, \underline{u} \leq x \leq \overline{u}\} \subset \mathbb{R}$. For real intervals, two other parameters are in use, namely *midpoint* $\check{u}$ and *radius* $\hat{u}$, so that $\check{u} = (\underline{u} + \overline{u})/2$ and $\hat{u} = (\overline{u} - \underline{u})/2$. That leads to the *centred* notation for intervals (see e.g. [9]), namely $\check{u} \pm \hat{u} = [\check{u} - \hat{u}, \check{u} + \hat{u}]$. An interval is called *thick* if $\underline{u} < \overline{u}$ (or $\hat{u} > 0$); *thin* (or *point*) interval if $\underline{u} = \overline{u}$ (or $\hat{u} = 0$). Point intervals can be for most purposes identified with the corresponding real number, i.e., $[x, x] = x \in \mathbb{R}$. An interval for which $\underline{u} = -\overline{u}$ (or $\check{u} = 0$) is called zero-symmetric or just symmetric. Intervals not containing zero can be positive or negative, according to the relations $u > 0 \Leftrightarrow \underline{u} > 0$ and $u < 0 \Leftrightarrow \overline{u} < 0$.

Occasionally, a concept of an exterval will be also used here. A *real exterval* e is a sort of interval-like object that contains infinity,i.e., the set $]\overline{e}, \underline{e}[= (-\infty, \overline{e}] \cup [\underline{e}, +\infty) \subseteq \mathbb{R}$ (with $\overline{e} < \underline{e}$). A *one-sided exterval* is the set $]\overline{e}, \infty[\, = (-\infty, \overline{e}]$ or $]-\infty, \underline{e}[\, = [\underline{e}, +\infty)$. When $\overline{e} \geq \underline{e}$, then $]\overline{e}, \underline{e}[\, = \mathbb{R}$.

An important parameter of an interval is the function rex (for *r̲elative e̲x̲tent*), first introduced in [2] and defined as $\operatorname{rex} u = \hat{u} \,/\, \check{u}$. Its variant named κ (kappa) in [9] is sometimes more convenient: $\kappa\, u = \hat{u} \,/\, |\check{u}| = |\operatorname{rex} u|$ (for proper intervals). For u containing 0 we have $\kappa\, u \geq 1$ while $0 \leq \kappa\, u < 1$ for u without 0. It is assumed to equal infinity for symmetric intervals (including 0).

When coefficients of the matrices A and b in the system of linear equations $A \cdot x = b$ are allowed to be intervals, the formula is usually called a system of interval *linear equations* [10, 12]. Precisely speaking, however, it is not *linear* (as the space of intervals is not a linear space), and usually is not treated as a system of *equations* either. The name "equation" is justified in the situation when one considers the *interval solution* (called also *algebraic solution*, or *formal solution* [14]) to the system. This solution is defined as an interval x_{I} which fulfills the equation $A \cdot x_{\mathrm{I}} = b$ in the sense of interval arithmetic. In most cases, other definitions of a solution are considered, usually as sets of *real* vectors (not necessarily intervals), defined as follows (see e.g. [14]):

United Solution Set: $\quad \Xi(A, b) = \{x \in \mathbb{R}^n \mid A \cdot x \cap b \neq \emptyset\} =$
$$= \Xi_{\exists\exists}(A, b) = \{x \in \mathbb{R}^n \mid (\exists \tilde{A} \in A)(\exists \tilde{b} \in b) \tilde{A} \cdot x = \tilde{b}\},$$

Control Solution Set: $\quad \Xi_{\supseteq}(A, b) = \{x \in \mathbb{R}^n \mid A \cdot x \supseteq b\} =$
$$= \Xi_{\exists\forall}(A, b) = \{x \in \mathbb{R}^n \mid (\forall \tilde{b} \in b)(\exists \tilde{A} \in A) \tilde{A} \cdot x = \tilde{b}\},$$

Tolerance Solution Set: $\quad \Xi_{\subseteq}(A, b) = \{x \in \mathbb{R}^n \mid A \cdot x \subseteq b\} =$
$$= \Xi_{\forall\exists}(A, b) = \{x \in \mathbb{R}^n \mid (\forall \tilde{A} \in A)(\exists \tilde{b} \in b) \tilde{A} \cdot x = \tilde{b}\}.$$

None of the above is actually a solution to the original equation. They are sets of real solutions to a system of interval *relational expressions*, with different relations put in the place of the equal sign, namely :

$A \cdot x \between b$ for the set $\Xi(A, b)$,
$A \cdot x \supseteq b$ for the set $\Xi_{\supseteq}(A, b)$,
$A \cdot x \subseteq b$ for the set $\Xi_{\subseteq}(A, b)$, respectively.

The relation symbol "$\between$," meaning $S \between T \iff S \cap T \neq \emptyset$ was introduced here for convenience. With this meaning of the interval relational expressions, the equation $A \cdot x = b$ will define the solution set $\Xi_{=}$ equal to $\Xi_{\supseteq} \cap \Xi_{\subseteq}$, different than the interval solution. From the definitions it follows also that $\Xi_{\subseteq} \subseteq \Xi$ and $\Xi_{\supseteq} \subseteq \Xi$.

In the one-dimensional case, the matrix A shrinks to a single interval a, as does the vector b. The relational expression becomes thus one of $a \cdot x \Diamond b$, where $\Diamond \in \{\between, \supseteq, \subseteq, =\}$. Diagrammatic analysis of solution sets for this case proves to be indispensable for diagrammatic analysis of the general multidimensional case. That analysis is based on the one-dimensional *radial* and *parallel* cuts through the solution space. As demonstrated in [4, 5, 8], the arrangement of solution sets along these cuts is provided by solutions of some one-dimensional equation whose

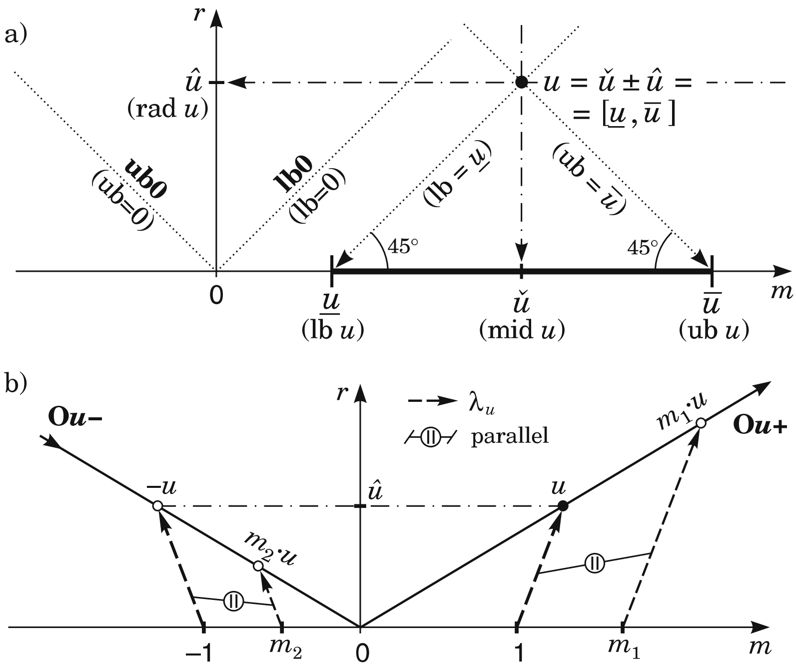

Fig. 1. The MR-diagram representation of the space of real intervals (a); interval axis, negation of intervals, and multiplication by real numbers (b)

coefficients are determined by the general equation coefficients and the direction of the cut.

3 Interval Space Diagram

The basis for the diagrammatic approach to interval analysis is the two-dimensional representation of the space of real intervals $\mathbb{IR}$ called the *MR-diagram* [1], see Fig. 1(a). In this diagram, an interval is represented by a point with its centred coordinates: midpoint $\check{u}$ and radius $\hat{u}$. Besides midpoint and radius, one can also easily obtain the endpoints $\underline{u}$ and $\overline{u}$ of the interval using the diagonal lines. In this way, the MR-diagram combines conveniently all three common representations of intervals—midpoint-radius, endpoint, and the one-dimensional representation as a segment of the real number line (here on the **Om** axis).

The main diagonals **lb0** and **ub0** constitute a dividing line between intervals *containing zero* (they all lie on or above the diagonals) and those *without zero* (below the diagonals). The *interval axis* $\mathbf{O}\boldsymbol{u}$ of the interval u consists of a positive half $\mathbf{O}\boldsymbol{u}+$ going through the interval u, and the negative half $\mathbf{O}\boldsymbol{u}-$ through the interval $-u$, see Fig. 1(b). Note how negation (change of sign) of an interval is obtained by reflection in the **Or** axis. All intervals v lying on the interval axis $\mathbf{O}\boldsymbol{u}$ of the interval u have the same value of the κ function: $\kappa\, v = \kappa\, u$, i.e., have the same relative extent. When $\kappa\, v \neq \kappa\, u$, u and v have different axes. An interval v is called *more extended* than u if it lies above the interval axis $\mathbf{O}\boldsymbol{u}$. Symmetric intervals (including 0) are considered more extended than all other intervals. Their axis coincides with the $\mathbf{Or}+$ coordinate axis.

Multiplication of an interval u by a scalar (real number) $m \in \mathbb{R}$ is defined by $m \cdot u = \{m\tilde{u} \,|\, \tilde{u} \in u\} = m\check{u} \pm |m|\,\hat{u}$. The interval axis $\mathbf{O}\boldsymbol{u}$ groups all products of the interval u and all real numbers, symbolically: $\mathbf{O}\boldsymbol{u} = \mathbb{R} \cdot u$. To find the product of an interval u and a real number m, it suffices to map appropriately the point on the $\mathbf{Om}$ axis with the coordinate m onto the interval axis $\mathbf{O}\boldsymbol{u}$. The diagrammatic construction for that is shown in Fig. 1(b). The mapping lines are parallel to the lines from the points of value $+1$ and -1 on the $\mathbf{Om}$ axis to u and $-u$, respectively (Fig. 1(b)). It is convenient to define the mapping as a function called *lambda mapping*: $\lambda_u(m) = m \cdot u$. Its inverse allows to find the real number (a point on the $\mathbf{Om}$ axis) by which the interval u has been multiplied to obtain the given point on the axis $\mathbf{O}\boldsymbol{u}$.

4 The Basic Equation and Its Structural Types

The basic one-dimensional equation can be solved diagrammatically. The expression $a \cdot x \lozenge b$ tells us that first we need a representation of all points that are in relation $\lozenge$ to the right-hand side interval b. Thus, we will need a diagrammatic representation of coimages of the coefficient b under the relations defining the solution sets. They are defined in Fig. 2(a), see [3, 5, 8] for more details. Borders of the coimages represent the *border relations* ⌐, ⊐, ⊏, and ⌙ that group intervals one of whose endpoints coincides with one of the endpoints of the coefficient b, as indicated in the figure.

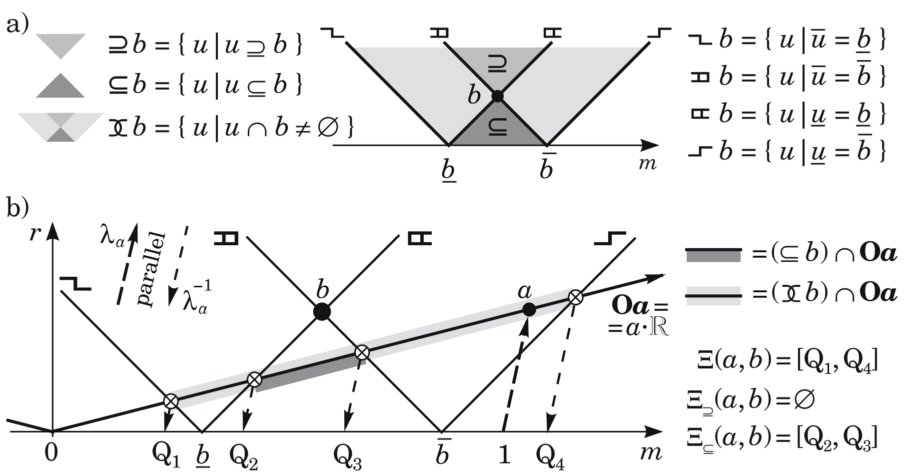

Fig. 2. Coimages of an interval b under interval relations $\between$, $\supseteq$ and $\subseteq$, and the definitions of border relations ⌐, ⊐, ⊏, and ⌙ (a), and diagrammatic solution example of the $a \cdot x = b$ equation (b)

For the given interval a and all possible real numbers x, the set of products $a \cdot x$ coincides with the axis $\mathbf{O}\boldsymbol{a}$. Thus, to find all values of x that fulfill the expression $a \cdot x \Diamond b$ for the given $\Diamond \in \{⌶, \supseteq, \subseteq, =\}$, we must first find the subset of $\mathbf{O}\boldsymbol{a}$ whose member intervals are related to b by the relation $\Diamond$. It is obviously the intersection $\mathbf{O}\boldsymbol{a} \cap (\Diamond\, b)$ of $\mathbf{O}\boldsymbol{a}$ with the coimage $\Diamond\, b$. Since $a \cdot x = \lambda_a(x)$, then $x = \lambda_a^{-1}(a \cdot x)$ and the solution set $\Xi_\Diamond$ is the result of the inverse lambda mapping of the said intersection onto the $\mathbf{Om}$ axis, that is, $\Xi_\Diamond = \lambda_a^{-1}(\mathbf{O}\boldsymbol{a} \cap (\Diamond\, b))$. An example diagrammatic construction for one of the cases is shown in Fig. 2(b), together with the resulting definitions of the three solution sets for this case.

The endpoints of the solution sets are thus given by the points $\mathrm{Q}_i = \lambda_a^{-1}(w_i)$, where w_i denotes one of the points of intersection (marked by $\otimes$ in Fig. 2) of $\mathbf{O}\boldsymbol{a}$ with one of the border relations. As it was derived in [3, 5], the points Q_i, called *quotients* of the expression $a \cdot x \Diamond b$, are obtained according to the rule shown in Fig. 3(a), depending on the border relation whose intersection with the $\mathbf{O}\boldsymbol{a}$ axis generates the quotient and its sign (the position of the quotient with respect to the $\mathbf{Or}$ axis in the diagram). The shorthands L, S, Z, and T were chosen on a mnemonic principle, as they mimic the graphical structure formed by dashes and a division operator in the quotient expressions.

When for the given coefficients a and b we sort the quotients Q_i in an increasing order of their numerical values and then list their names $\mathrm{N}_i \in \{\text{“L”}, \text{“S”}, \text{“Z”}, \text{“T”}\}$ in the same order $\mathrm{N}_{i_1}\mathrm{N}_{i_2}\mathrm{N}_{i_3}\mathrm{N}_{i_4}$, ($\mathrm{N}_{i_j} \neq \mathrm{N}_{i_k}$ for $j \neq k$) we obtain the *characteristic quotient sequence* for these coefficients (and hence for the type of the equation with these coefficients). The sequence will be denoted by $\mathbb{Q}(a, b)$. Characteristic quotient sequences are usually augmented by the indications of the position of zero, equality, and special values of some quotients (like infinity or undefined values), see the examples further on.

After arranging quotients in a two-dimensional array as in Fig. 3(b), the sequence can be represented as a *quotient sequence diagram*. Solution sets determined by the given sequence will be indicated with the graphical annotation explained by the two examples in Fig. 3(b).

Diagrammatic analysis sketched above revealed that there are only 16 different basic quotient sequences, grouped into 6 structural types corresponding to different possible configurations of the interval axis $\mathbf{O}\boldsymbol{b}$ and the coefficient a,

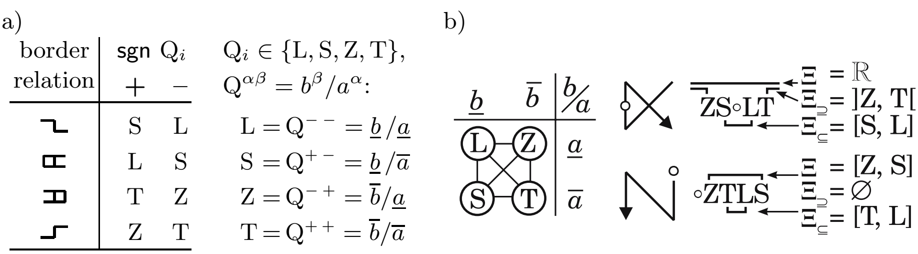

Fig. 3. Notation for quotients of the $a \cdot x \Diamond b$ relation and correspondence between them and intersections with border relations (a); quotient sequence diagram and graphical notation for solution sets (b). It is assumed that $\alpha, \beta \in \{-, +\}$ and $u^- = \underline{u}$, $u^+ = \overline{u}$. Small circles “∘” denote the position of zero.

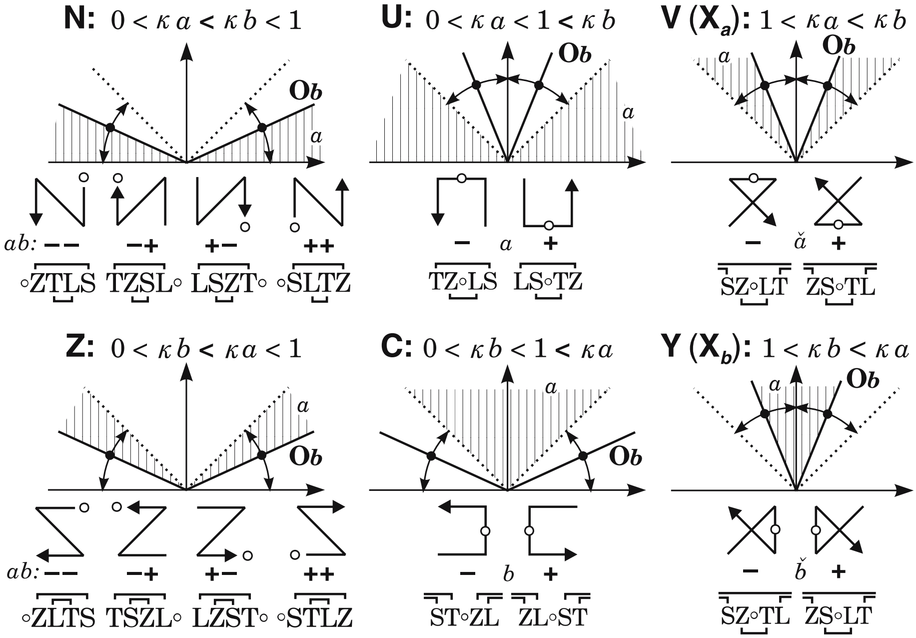

Fig. 4. The catalogue of all basic subtypes of interval equation $a \cdot x = b$. Defining conditions for the (sub)types explicitly exclude intermediate cases.

as shown in the catalogue of basic types in Fig. 4. Letter names for the types were chosen to mimic the shape of the quotient sequence diagram for the type. Concerning the names **V** and **Y**, see the remark in Section 5.

An important property of quotient sequences [3] says that they are invariant with respect to radial moves of the coefficients a and b along their respective positive interval semi-axes (excluding zero), i.e., $\mathbb{Q}(a, b) = \mathbb{Q}(sa, tb)$ for any $s, t \in \mathbb{R}^+$. As a consequence, the solution types (hence their qualitative configurations) depend only on the values of rex a and rex b. Therefore, regions with the same types have the shape of angular wedges in the MR-diagram and are independent of the scale of the diagram. That property allows for convenient diagrammatic representation of conditions for coefficients a and b defining the type of the equation as shown in Fig. 4.

The six basic types (without subtypes) are also explained in [7, 8] in the form of a set of solution diagrams like that for the type **N** in Fig. 2(b).

5 Catalogues of Types

5.1 A Multiple for Basic Types

The catalogue in Fig. 4 is a two-level multiple whose six high-level components (cells) describe individual basic types. Every cell is a hybrid (partially diagrammatic, partially propositional) representation of basic data about the type:

- The name of the type (a bold-face capital letter).
- The formula providing condition on the coefficients a and b for the type.
- A diagram illustrating the condition diagrammatically. It depicts the MR-diagram showing regions in which the coefficients a (the dashed area) and b (with its axis, as indicated by arcs with arrows) should lie.
- The next level of the multiple indicates subtypes of the given type. They are defined by signs of the coefficients, or, for types **V** and **Y**, signs of their midpoints. Every small cell gives the following information:
  - A quotient sequence diagram indicating diagrammatically the sequence of quotients for the subtype.
  - The condition for the subtype, indicated by a pair of signs of the coefficients, or a single sign for one of the coefficients indicated.
  - The quotient sequence in textual form, with graphical annotations defining solution sets for the subtype, according to the rules given in Fig. 3(b).

The catalogue in this form gives an excellent general view of the whole space of basic types, but it has a number of drawbacks that indicate a need for other representations of the space of types. It is type-oriented, that is, the cells on both levels correspond to individual basic (sub)types. However, the arrangement of cells conveys little information about relations between different types, that is, the structure of the space of types is not well represented.

The main problem is that the catalogue contains only basic types (6 main types with 16 subtypes). A considerable number of intermediate and degenerate types is not shown here. These types in principle can be generated from the data provided, but the process is rather tedious and error-prone, and it is easy to miss some of the possibilities. On the other hand, some information, namely quotient sequence diagrams, seems superfluous for the end user. These diagrams encode only part of the data included anyway in the textual quotient sequence below them. They are useful at the stage of enumeration of the basic types, providing a visual classification criterion based on shapes of the diagrams which led to the choice of letter names of the types. Even in this role, these diagrams proved to be somewhat misleading, as they suggested a single type **X** for $\kappa b > 1$. When the classification was extended to multidimensional equations (see [4, 5, 8]) it was found more convenient to split this type into two, as indicated in the diagram (with old type names provided in parentheses).

5.2 Diagrammatic Catalogue of All Types

As a result of these considerations, a new form of the catalogue was developed. It disposes of quotient sequence diagrams and is complete, listing all types, including intermediate and degenerate ones. It combines only diagrammatic representation of conditions for coefficients with textual representation of quotient sequences, augmented by diagrammatic annotations defining solution sets. An additional convention concerning quotient sequences is used here, namely quotients that are equal to zero are omitted. The full diagram occupies a single page; for brevity, in Fig. 5 only a part of it is shown. Note that the original letter labels of cells (c, d, e, f) were retained to provide easy reference to the full catalogue

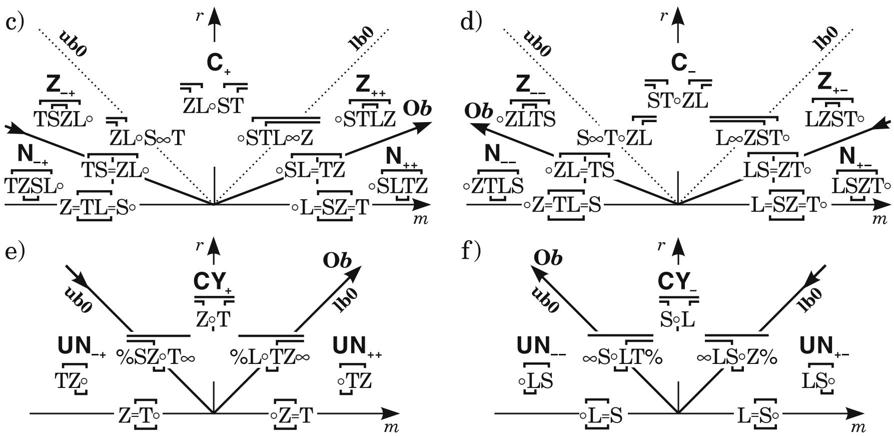

Fig. 5. Part of the diagrammatic catalogue of all subtypes of interval equation $a \cdot x = b$ for $a, b \neq 0$, and b: thick without zero (c, d), and with zero at endpoint (e, f)

of [7]. Two upper cells in this multiple provide data for all 10 basic subtypes of types **N**, **Z**, and **C**, as well as 12 intermediate types connected with them. The other two cells list 14 intermediate types obtained when $\underline{b} = 0$ or $\overline{b} = 0$.

This form of the catalogue is no longer type-oriented like that in Fig. 4. Instead, the cells here correspond to different conditions on the coefficient b. Possible positions of this coefficient in the MR-diagram are indicated by the position of the interval axis **O*b***. When the axis is placed within some region, the coefficient b is allowed to vary within the region occupied by the positive semi-axis of the axis **O*b***. When it coincides with some characteristic line of the diagram (main diagonal, **Om** or **Or** axes), the coefficient b can vary along the positive semi-axis **O*b*** only. The regions of the MR-diagram delineated by axes and main diagonals, as well as the axes and diagonals themselves, are labelled by annotated quotient sequences obtained when the coefficient a falls within the indicated region or on the indicated axis. Due to space limitations, only the regions are additionally labelled by names of subtypes. Thus, the basic information provided by this form of the catalogue is the definition of solution sets for all cases. The degenerate types (for $a = 0$ or $b = 0$) are depicted by separate cells of a slightly different design (not shown here). The complete catalogue in this form is provided in [7]; its older versions with different, less convenient layout were published in [3, 5].

A dual a-oriented form of the catalogue can be constructed as well. In it the cells would correspond to appropriate conditions on the coefficient a, while the types would be selected by the position of the coefficient b in the MR-diagram. It may be more convenient for certain purposes. In fact, in the form provided in [7], the degenerate types for $a = 0$ are grouped in a separate cell and represented in that a-oriented form. That, however, violates the b-oriented structure of the whole catalogue. A possible solution, assuring uniformity of the catalogue, is used in the construction of the restricted catalogue in Fig. 6.

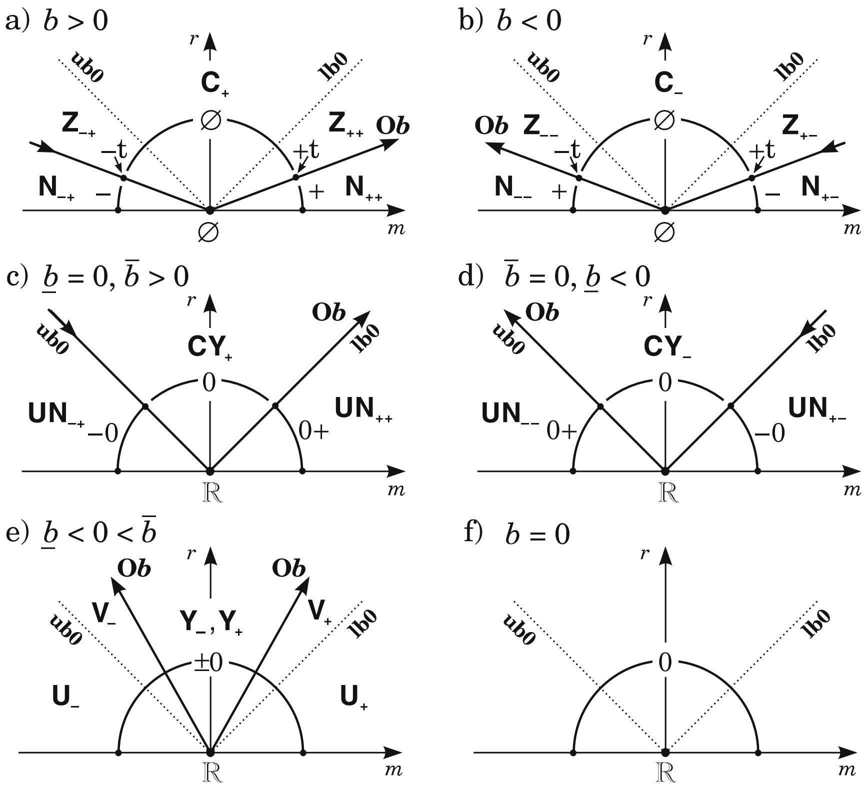

Fig. 6. Diagrammatic catalogue of tolerance solution set configurations for various conditions on the coefficient b

Despite its completeness and compactness, this form of the catalogue has its drawbacks. First, the lack of formulae defining conditions for types is inconvenient in some situations, forcing the user to translate from formulae to the situation in the diagram and back. That can be corrected for the conditions on the coefficient b by adding appropriate formulae as headers of the cells, as in the version in Fig. 6 (see [8] for such improvement). Second, although the representation is aimed at describing solution sets, it does not provide a good view of the distribution of solution set structures depending on positions of the coefficients in the MR-diagram. As information about all basic solution sets is lumped together into annotations of quotient sequences, it is easy to read out locally the particular definition of the solution sets for any single type. E.g., in Fig. 5(e) one may easily read that solution sets for the subtype intermediate between CY_+ and UN_{++} (with $\underline{a} = 0$), are $\Xi = \mathbb{R}, \Xi_{\supseteq} = [\mathrm{T}, \infty)$, $\Xi_{\subseteq} = [0, \mathrm{T}]$, and thus $\Xi_{=} = \{\mathrm{T}\}$. However, the overall picture is hard to comprehend.

5.3 Catalogues for Individual Solution Sets

For providing an overall view of possible structures of a single given solution set, useful in finding conditions for occurrence of interesting structures for that set, a specialized catalogues for individual sets may be more useful. Such a catalogue for the tolerance solution set is shown in Fig. 6. Analogous catalogues for control and united solution sets can be found in [7, 8]. The catalogue shown here (in Fig. 6) is additionally augmented with formulae describing conditions on the coefficient b for every cell, as it was discussed above. This catalogue is significantly smaller and simpler than the full catalogue whose part is shown in Fig. 5. This is due to omission of quotient sequences and replacing exact definitions of solution sets by qualitative codes (see [7] for their explanation). That, with the fact that the structure of only a single solution set is represented, allows for aggregation of the cases into a smaller number of cells. The price for that is losing exact definitions of solution sets and losing the possibility to directly compare structures of different sets.

In this version the degenerate cases for $a = 0$ are not depicted with a separate cell of different kind. Instead, descriptors of solution sets for these cases are put at the point $(0, 0)$ in the b-oriented cells. Such a solution can be also adopted in the full catalogue of Fig. 5, though with some difficulties due to a rather large size of quotient sequence descriptors for this case (see [7]).

5.4 Tabular Catalogue

For certain regular structures of information it may be convenient to represent the catalogue in a tabular form. A small part of the table of types, containing 6 intermediate types shown in top parts of Fig. 5(e) and 5(f), from among 73 subtype entries in the complete catalogue, is provided in Table 1. The data in the cells are represented only propositionally. Solution sets are described with both exact definitions in terms of quotients, and with qualitative descriptors used in Fig. 6. The fourth column links table entries to corresponding cells of the

Table 1. Part of a detailed table of descriptions of solution sets for intermediate subtypes

Type	Quotient sequence	Conditions $a \neq 0,\ b \neq 0$	Fig. 5	Solution sets $\Xi_{\subseteq}$	$\Xi_{\supseteq}$	$\Xi_{=}$	Ξ
		...			...		
CY		$\kappa b\ = 1 < \kappa a$					
–	S∘L	$\overset{\vee}{b} < 0$	f	0	$\pm\infty$:]S, L[	$\emptyset$	$\supseteq$
+	Z∘T	$\overset{\vee}{b} > 0$	e	0	$\pm\infty$:]Z, T[	$\emptyset$	$\supseteq$
		...			...		
CU		$\kappa a\ = \kappa b\ = 1$					
– +	%SZ∘T∞	$\overset{\vee}{a} < 0,\ \overset{\vee}{b} > 0$	e	-0: [Z, 0]	$-\infty$:]Z, ∞[	$-t$: Z	$\mathbb{R}$
+ –	∞LS∘Z%	$\overset{\vee}{a} > 0,\ \overset{\vee}{b} < 0$	f	-0: [S, 0]	$-\infty$:]S, ∞[	$-t$: S	$\mathbb{R}$
– –	∞S∘LT%	$\overset{\vee}{a} < 0,\ \overset{\vee}{b} < 0$	f	0+: [0, L]	$+\infty$:]∞, L[	$+t$: L	$\mathbb{R}$
++	%L∘TZ∞	$\overset{\vee}{a} > 0,\ \overset{\vee}{b} > 0$	e	0+: [0, T]	$+\infty$:]∞, T[	$+t$: T	$\mathbb{R}$
		...			...		

diagrammatic catalogue in Fig. 5. Although most convenient for some purposes, these tables have their drawbacks too. First, they tend to be large—the complete catalogue in [7, 8] uses four pages (for 73 subtype entries), compared to the single page of the diagrammatic catalogue in Fig. 5. Second, the overall picture of relations between different types and structures of their solution sets is almost completely lost.

5.5 RR-Diagram Maps and Graphs of Types

New forms of the catalogue can be based on the RR-diagram, introduced in [3, 5]. In this diagram, values of $\operatorname{rex} a$ and $\operatorname{rex} b$ (or their reciprocals) are put on the coordinate axes. Because types do not change when extent functions of the coefficients a and b do not change, to every point in the RR-diagram corresponds some type. It is unique except when one of the coefficients is thin, because then the value of the extent function is zero, independently of the sign of the interval. Thus, the sign of a thin interval cannot be distinguished by its position in the RR-diagram. Labelling appropriate regions in the diagram by the type of its points we obtain a sort of map, partitioning the diagram into typed regions as in Fig. 7(a) and 7(b). Intermediate and degenerate types (not shown in the figure) correspond to borders and vertices of the regions. Obtaining the complete catalogue in this way is, however, troublesome, as some different intermediate types involving thin intervals of different signs (that includes also all degenerate types) fall on the same points and segments of the $\operatorname{rex} a$ and $\operatorname{rex} b$ axes.

Representing regions as nodes and neighbourhood relations between them as edges, we can obtain various graphs (networks) of types. Two such graphs are shown in Fig. 7(c) and 7(d). The second graph is useful for enumerating types of multidimensional equations, see [4, 5, 8]. The RR-diagram and graph representations of the catalogue combine the catalogue aspect with another usage type of diagrams, namely showing the structure of the space of types, see [6–Fig. 2(a)], the feature lacking in the multiple-like catalogues.

6 Discussion

Design of a diagrammatic catalogue of types of the basic interval equation $a \cdot x = b$ presented several nontrivial problems, leading to the development of various forms of the catalogue and searching for new ways of structuring them. The nontrivial, though not overwhelming complexity of the catalogue has made it a convenient case study of the problem of designing catalogues of various pieces of mathematical knowledge. The main obstacle here is the lack of general guidelines for designing such catalogues. On the basis of this case study one may try to formulate some preliminary design guidelines.

Catalogues of cases serve as reference databases, but also as research tools for searching patterns of differences and similarities between the cases and their various constituent parameters, in this case especially the structure and definitions of solution sets for every type. As observed by Tufte [15]: "Comparisons must be enforced within the scope of the eyespan." Therefore, the catalogue

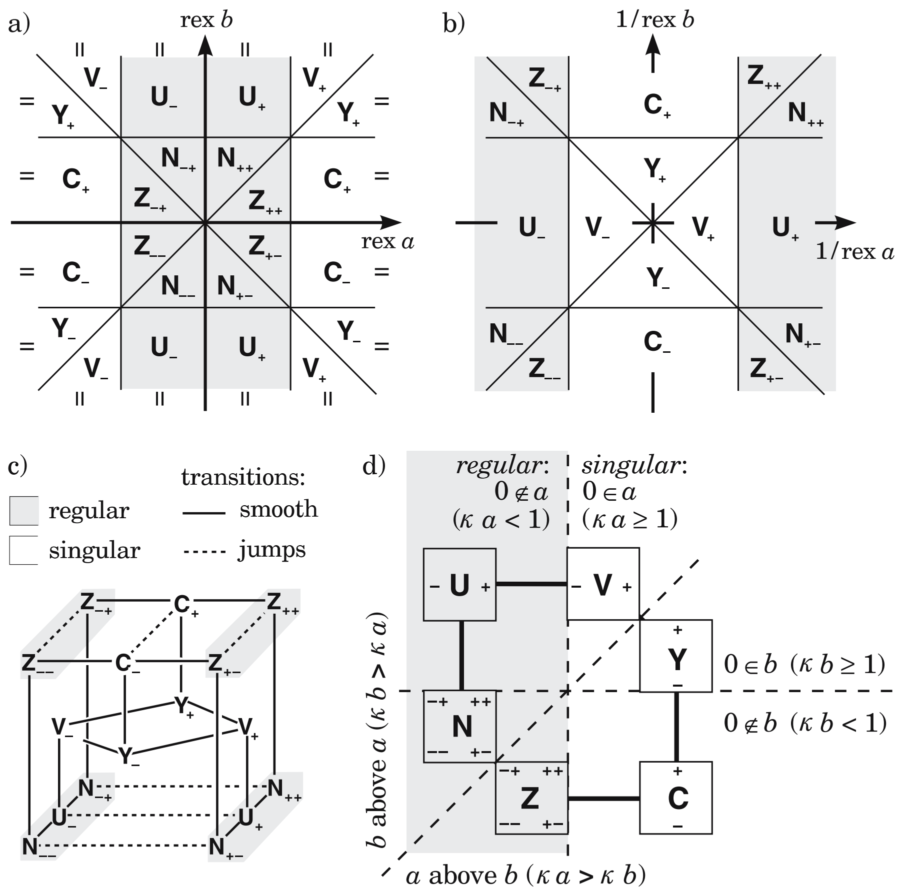

Fig. 7. Type catalogues on the RR-diagram (a), 1/RR-diagram (b); and in the form of graphs (c, d)

should be, if possible, not larger than a single page of paper or a single computer screen. The considerable number of types and their parameters makes such an attempt rather hopeless in this case (and in many others, unfortunately). Either the catalogue becomes too cluttered and unreadable, or it must occupy larger area, or it must omit a substantial amount of information. All these outcomes occurred in the catalogues discussed in the paper. A possible solution is to produce different catalogues for different purposes, differing by the selection of represented data and the form of their presentation. Such catalogues can be generated (semi-)automatically from some underlying complete database, or prepared separately beforehand and then browsed through (as it is currently the case with the catalogues of equation types [7]). Such a solution, however, blocks or makes troublesome some possible comparisons of data, hence other solutions should still be searched for, like interactive catalogues.

The observation of our case shows that there seem to be three basic types of such catalogues: *multiples* (including *tables*), *maps*, and *graphs* (or *networks*).

Multiples. These are regular structures (usually rectangular) of similar cells containing chunks of data (including diagrams) pertaining to the particular case (Figs. 4, 5, and 6). Essentially, tables can be also considered as multiples. They are usually distinguished from more general multiples by an explicit use of the two dimensions to structure the multiple. Namely, cells occupying the same column contain the same type of data (described by column headers, Table 1), while rows contain attribute descriptions of individual objects or cases (the equation types in our case). The roles of columns and rows can be sometimes interchanged. Multiples (including tables) can be structured hierarchically, with cells structured as lower-level multiples, Fig. 4 and Table 1. The division of data between different levels of the hierarchy is usually dictated by the intrinsic structure of the data, but often can be varied depending on the *intended use* (see below) of the catalogue (compare catalogues in Figs. 4 and 5). The main drawback of multiples is that they do not provide adequate means for representing more complex structural relations between cases (represented by data in the cells). The hierarchical grouping and grouping by data type in columns (or rows) are practically the only possibilities that are available in pure multiples. See hierarchical grouping in Fig. 4, typed columns in Table 1, and two-column multiples (corresponding to signs of the coefficient b) in Figs. 5 and 6.

Maps. They are arrangements of regions on a plane (usually; sometimes other arrangements, e.g. three-dimensional, can be used). Regions contain data pertaining to individual cases, and their shapes and relative positions encode additional data about properties and relations between the cases. Multiples can be also considered a special case of maps. Another special case can be distinguished, let us call it *constructions*, where cases are distinguished by diagrammatic constructions placing the results belonging to different cases in different regions of the space. An example is provided by the catalogue of definitions of means in [6–Fig. 4]. The appropriate division of a plane into regions is often obtained with the help of a coordinate system whose coordinates correspond to parameters distinguishing the cases. The maps used within cells of multiples in Figs. 4, 5, and 6 use the MR-diagram midpoint-radius coordinate system, while the RR-diagram based catalogues in Figs. 7(a) and (b) use the values of the function rex for the coefficients a and b. The advantage of maps comes from a richer layout structure that can be used to represent relations between cases, especially when the intrinsic structure of the set of cases conforms well to the structure of the Euclidean plane. Otherwise, the structure must be "planarized", for the price of losing information or introducing information noise. This is the case with our catalogue of types which is essentially at least three-dimensional, see Fig. 7(c).

Graphs (networks). In this form, nodes of a graph represent cases, and edges relations between them. That allows for representation of arbitrary systems of relations between cases, but often for the price of making them hard to comprehend, especially for more complex systems. The proper layout of complex graphs of relations is a nontrivial problem—it has given rise to the whole discipline of graph drawing [11]. Sometimes the proper layout can be obtained by using an appropriate map as a guide for placing the nodes. The graph in Fig. 7(d) is superimposed over a partition of the plane generated by different conditions for the coefficients a and b. Note another hybrid element in this graph—the data within the nodes are arranged as small multiples of subtypes.

Hybrid solutions and user's goals. In practice, as was indicated above, hybrid solutions are used, with different presentation means used for different portions of a catalogue. This includes combining different types of representations, like multiples containing maps (Figs. 4, 5, and 6) or graphs containing multiples and superimposed on maps (Fig. 7(d)). To some extent this depends on the structure of data, but in most part on the intended use (user's goals) of the catalogue. The importance of user's goals for proper design of information presentation has been recognized some time ago (see e.g. [13]). Like for the presentation graphics of quantitative (statistical) data, the design of mathematical diagrams should also be based of the analysis of user's goals and selecting the way of presentation appropriate for them, possibly in a similar way as developed in [13].

Interactive catalogues. The use of many catalogues for different purposes solves some of the problems but makes relating of different pieces of information contained in different catalogues difficult. A possible solution would be to make the set of catalogues interactive. In such a system, selection of certain piece of information in some catalogue may either highlight the corresponding piece of information in another catalogue, or provide that information in a separate small window or "balloon" near the place pointed to. That may not solve all the problems, especially as proper organization of such interaction, when there are several differently structured catalogues, can be a considerable problem in itself.

A useful addition to such a catalogue is an algorithmic component, namely a program producing for any given numerical values of coefficients the type data (including location in the catalogue) and solution sets for the equation with this coefficient. Besides allowing the user of the catalogue to browse the space of possibilities also quantitatively, such a subroutine is a necessary component of any program using solutions of this equation to characterize solution sets of a general multidimensional equation (e.g., calculating radial and parallel cuts through its solution space, see [8]). Such a subroutine was developed and is available for interested users, see [7].

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