Domination Search on Graphs with Low Dominating-Target-Number

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Abstract. We settle two conjectures on domination-search, a game proposed by Fomin et.al. [1], one in affirmative and the other in negative. The two results presented here are (1) domination search number can be greater than domination-target number, (2) domination search number for asteroidal-triple-free graphs is at most 2.

1 Introduction

Domination search is a game proposed by Fomin et.al. [1] which is a variant of *node-search-game*, see [2]. It is also a graph variant of polygonal search problem, [3,4,5,6,7]. It is a problem of sweeping out a mobile fugitive out of a graph (think of a house where vertices are rooms) with k guards. A guard at a node can check its node and all nodes adjacent to it. The fugitive can move in zero time from node x to y if there is a path between the nodes which does not pass through the nodes under any guard's watch. In each step one guard can move from its current node to any other vertex. During the move this guard is absent from the graph and fugitive can take the advantage. Search is successful if after a finite number of moves entire graph is cleared of the fugitive.

We present a formal definition of domination-search game differently from the original but it is equivalent to that. Here $N[X]$ denotes the closed neighborhood of the vertex set X. The search algorithm with k *guards* on a graph $G = (V, E)$ places k guards on k vertices initially. $D(0)$ denotes these vertices. In each move one guard is moved from it current position (vertex) to a new position. $D(i)$ denotes the set of vertices where the guards are placed after i moves. Formally, the search is a sequence of k-sets: $D(0), D(1), \ldots, D(M)$, where $D(i-1) \cap D(i)$ is denoted by $S(i)$ and has cardinality $k-1$ for all $i > 0$. A vertex is said to *clear* if it was in the neighborhood of some guard in some previous move and since then no path has been established between this vertex and a contaminated (fugitive may potentially be on it) vertex without passing through the neighborhood of a guard in the current position. We define vertex sets $U_a(i)$ (set of clear vertices after i moves) for $0 \le i \le M$ and $U_d(i)$ (clear set during move-i) for $1 \le i \le M$. These sets are recursively defined by the following equations. $U_a(0)$

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is the closed neighborhood of $D(0)$, i.e., $N[D(0)]$. $U_a(i) = U_d(i) \cup N[D(i) - S(i)]$, and $U_d(i)$ is the set $\{v \in U_a(i-1)$: *every path from* v *to any vertex in* $V - U_a(i-1)$ *passes through* $N[S(i)]\}$. Finally, $U_a(M) = V$. The domination search number of a graph G is the smallest k for which such a sequence exists. It is denoted by $ds(G)$.

Domination-search number is found to be strongly related with another graph-parameter, dominating-target number denoted by $dt(G)$. A vertex subset, T, of a graph is said to be a dominating-target [8] if every connected subgraph which contains T, dominates the entire graph. The cardinality of the smallest dominating-target is called the dominating-target number, denoted by $dt(G)$. Fomin et.al. [1] have shown that for arbitrary connected graph $ds(G) \leq 2 \cdot dt(G) + 3$. But they have found that this is perhaps not a tight bound and conjectured that $ds(G) \leq dt(G)$.

In their work Fomin et.al. have also studied $ds(G)$ of graphs of small dominating-target number. These include cocomparability graphs; AT-free graphs [9,10,11,12]; and DP graphs [13,14]. These graph classes are defined as follows. An asteroidal-triple is a set of three vertices such that there is a path between any two vertices without entering the neighborhood of the third. A graph is said to be AT-free if it contains no asteroidal-triple. The family of graphs having dominating target number equal to two is denoted by DP. Their results include (i) $ds(G) \leq 4$ for the DP graphs; (ii) $ds(G)$ of cocomparability graphs is 2; (iii) $ds(G) \leq 2$ for AT-free claw-free graphs; and (iv) $ds(G) \leq 3$ for AT-free graphs. They also conjecture that $ds(G) \leq 2$ for AT-free graphs.

In this work we will show that there exists a DP graph for which domination-search number is greater than 2. This settle the conjecture "$ds(G) \leq dt(G)$" in negative. We also present a domination search algorithm for AT-free graphs with $ds(G) \leq 2$ which settles the second conjecture in affirmative. The paper is organized as follows. Section 2 presents a DP graph and shows that it cannot be searched with two guards. Section 3 describes a partial ordering on graphs which plays an important role in developing the domination-search algorithm for AT-free graphs, presented in Section 4.

2 Lower Bound for DP Graphs

In this section we will establish that domination search cannot be performed on all weak dominating pair graphs (family of graphs with dominating target number being 2) with 2 guards. This will settle the conjecture 23 of [1], "$ds(G) \leq dt(G)$", in negative.

The open neighborhood of a vertex x, $N(x)$, in a graph is the set of vertices adjacent to x. The closed neighborhood, $N[x]$ is $N(x) \cup \{x\}$. If $N[x]$ is not a graph separator then x is called an *extreme* vertex. The set of all extreme vertices of a graph is denoted by L.

Consider the graph $G_0 = (V, E)$ in figure 1. Observe that $L = \{v_0, v_1, v_2, v_3, v_4, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. Therefore only non-extreme vertices in G_0 are v_5, v_6 and v_7. There are two connected components in the induced subgraph on $V - N[v_5]$,

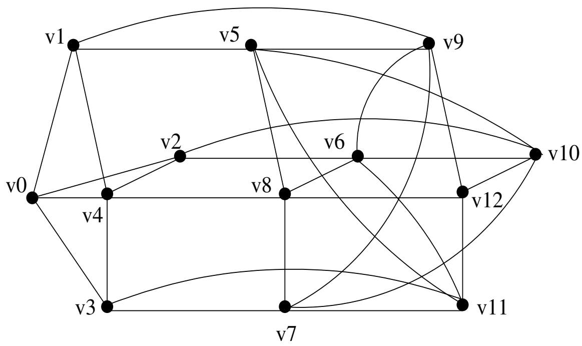

Fig. 1. A dominating pair graph with $ds(G) > 2$

denote them $C_1^{v_5}$ and $C_2^{v_5}$. Due to the symmetry in v_5, v_6 and v_7, $V - N[v_6]$ and $V - N[v_7]$ also have two components: $C_1^{v_6}$, $C_2^{v_6}$ and $C_1^{v_7}$, $C_2^{v_7}$ respectively. $C_1^{v_5} = C_1^{v_6} = C_1^{v_7}$ is the single vertex v_{12}. The second components of each case is given in figure 2 which are indeed isomorphic.

Let us assume that a domination search algorithm for G_0 exists which requires two guards. Let it be expressed by the sequence $A : D(0) = (p_1(0), p_2(0))$, $D(1) = (p_1(1), p_2(1)), \ldots, D(M) = (p_1(M), p_2(M))$. Pair $p_1(i), p_2(i)$ denote the vertices where guards are placed after i moves. $\{p_1(i-1), p_2(i-1)\} \cap \{p_1(i), p_2(i)\}$ is a singleton denoted by $S(i)$. For notational convenience we will denote the element in $S(i)$ by $S(i)$ as well without ambiguity. By $U_d(i)$ we denote the set of vertices which are clear (uncontaminated) in the graph during the i-th move when there is only one guard on the graph (at $S(i)$). After the move when there are two guards on the graph the set of clear vertices is denoted by $U_a(i)$. Without loss of generality we assume that this sequence is minimal in the sense that no step of the algorithm is redundant, i.e., no proper subsequence of A is a valid domination search. The graph does not have a dominating set of size two therefore M must be greater than zero.

Proposition 1. *$S(i) \notin L$ for all $1 \leq i \leq M$.*

Proof. Assume $S(i) \in L$. If $U_d(i)$ was equal to V then there would have been no need for the i-th move. So there is at least one contaminated vertex just before this move. During this move there is only one guard on the graph, at $S(i)$. Since the induced graph on $V - N[S(i)]$ is connected, entire set $V - N[S(i)]$ will get contaminated. So the set of clear vertices after this move will be $N[p_1(i)] \cup N[p_2(i)]$. This state can be achieved at the start of the search by placing the guards at $p_1(i)$ and $p_2(i)$. Therefore we can replace A by $A' : (p_1(i), p_2(i)), (p_1(i+1), p_2(i+1)), \ldots, (p_1(M), p_2(M))$ which will also perform the domination search. This violates the minimality condition of A.

Due to symmetry between v_5, v_6, and v_7 we may assume that $S(1) = v_5$ without loss of generality. Suppose $S(i) = v_5$ for $1 \leq i \leq i_0$. During these moves only one guard is moving to clear the parts of $V - N[v_5]$. No single vertex in

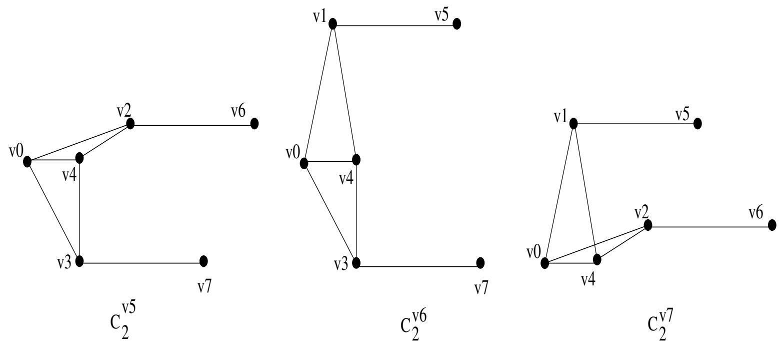

Fig. 2. Components $C_2^{v_5}$, $C_2^{v_6}$,$C_2^{v_7}$

the graph dominates entire $C_2^{v_5}$ so a single guard cannot clear it completely. Therefore during each move upto i_0 entire $C_2^{v_5}$ will be contaminated. In other words $U_d(i) \cap C_2^{v_5} = \emptyset$ for $1 \leq i \leq i_0$. As a consequence $U_a(i)$ for $1 \leq i \leq i_0$ cannot be equal to V. This indicates that the search cannot terminate if $S(i)$ remains unchanged at v_5.

Let us suppose $S(i_0 + 1) \neq v_5$. Then $S(i_0 + 1)$ must be either v_6 or v_7. Due to symmetry, we can assume $S(i_0 + 1) = v_6$ with no loss in generality. In the move i_0 one guard stays fixed at v_5 and the the other guard moves to v_6. As discussed above $U_d(i_0)$ does not contain any vertex of $C_2^{v_5}$. $U_a(i_0) = U_d(i_0) \cup N[v_6]$ which does not contain v_0. So $U_a(i_0)$ does not contain v_0. Consequently $C_2^{v_6}$ will be entirely contaminated during move $i_0 + 1$ when guard at v_6 remains fixed. Suppose $S(i) = v_6$ for $i_0 + 1 \leq i \leq i_1$. As argued above during all these moves $U(i) \cap C_2^{v_6} = \emptyset$ for $i_0 + 1 \leq i \leq i_1$. So the search cannot terminate with i_1-th move. Once again we may replace v_6 by v_5 or v_7 as the value of $S(i)$ but repeating the argument we conclude that the search will never end. We have following result.

Theorem 1. *Domination search on graph of figure 1 requires at least 3 guards.*

This graph has dominating target number 2 because $\{v_4, v_{12}\}$ is a dominating pair in it. So we establish that $dt(G_0) < ds(G_0)$.

Corollary 1. *The conjecture $ds(G) \leq dt(G)$, proposed in [1], for all connected graphs G is incorrect.*

3 A Partial Ordering on Graphs

The domination search algorithm for asteroidal-triple-free graphs proposed in the following section uses two guards. The selection of the successive positions to station the guards is determined based on a partial ordering on the vertices which is described in this section.

Let $G = (V, E)$ be an arbitrary graph and x be any vertex in it. Define relation $\succ_x$ on V as follows. $u \succ_x v$ if (i) u and v are not adjacent and (ii) every path from u to x is intercepted by v, i.e., at least one vertex on each path from u to x belongs to $N[v]$. Observe that condition (ii) can be equivalently stated as: every induced path from u to x is intercepted by v. This relation is reflexive. Define an equivalence relation $\sim_x$ on V as follows. $u \sim_x v$ if there exist $u_1, u_2, \ldots u_k$ such that $u \succ_x u_1 \succ_x u_2 \succ_x \ldots u_k \succ_x v$ and $v \succ_x u_k \succ_x \ldots u_1 \succ_x u$. The equivalence classes, x-classes, induced by $\sim_x$ will be denoted by $[u]_x$ representing the class containing u. The x-class containing x is obviously a singleton.

Observation 2. *Let $u_1, u_2 \in [u]_x$. Then for any induced path from u_1 to x: $u_1 a_1 a_2 \ldots a_m (= x)$, u_2 is adjacent to a_1 and to no other a_i.*

We extend $\succ_x$ to the class, using the same symbol: $[u]_x \succ_x [v]_x$ if there exists $u' \in [u]_x$ and $v' \in [v]_x$ such that $u' \succ_x v'$. The above observation leads to the following result.

Observation 3. *$[u]_x \succ_x [v]_x$ iff for every $u' \in [u]_x$ and $v' \in [v]_x$ either (u', v') is an edge or $u' \succ_x v'$.*

Consider two distinct classes $[u]_x$ and $[v]_x$ such that $[u]_x \succ_x [v]_x$. Let $u' \in [u]_x$ and $v' \in [v]_x$ such that $u' \succ_x v'$. Let $u'' \in [u]_x$ such that $u'' \succ_x u'$. Assume that $u'' \succ_x v'$ is not true. From the previous result u'' must be adjacent to v'. Since $[v]_x$ is distinct from $[u]_x$ there is a path, P, from v' to x which misses u'. Now we have a path $P' = u''v'.P$. We have $u'' \succ_x u'$ and $u' \succ_x v'$ so u' is not adjacent to u'' and v'. Thus entire P' misses u'. This violates $u'' \succ_x u'$. So $u'' \succ_x v'$ must be true. If we have a chain $u^{(k)} \succ_x u^{(k-1)} \ldots u'' \succ_x u'$ then iterative application of the above argument will imply that $u^{(k)} \succ_x v'$. From the definition of the x-classes we have the following observation.

Observation 4. *Let $[u]_x$ and $[v]_x$ be distinct classes such that $[u]_x \succ_x [v]_x$. Each vertex of $[v]_x$ is either adjacent to all vertices of $[u]_x$ or to none.*

Proposition 2. *The relation $\succ_x$ on the equivalence classes is a partial ordering.*

Proof. The reflexivity and anti-symmetry are due to the definitions of $\succ_x$ and $\sim_x$. Next we show the transitivity.

Let $[u]_x \succ_x [v]_x$ and $[v]_x \succ_x [w]_x$. Our goal is to show that $[u]_x \succ_x [w]_x$. If the classes $[u]_x, [v]_x$, and $[w]_x$ are not all distinct, then the claim is true from reflexivity and anti-symmetry. So assume that all three are distinct.

There are u' in $[u]_x$, v' and v'' in $[v]_x$ and $w' \in [w]_x$ such that $u' \succ_x v'$ and $v'' \succ_x w'$. From Observation 4 we also have $v' \succ w'$. Let P be an arbitrary path from u' to x. Since it is intercepted by v', there is a path P' from v' to x in which all vertices, except perhaps v', are from P. w' intercepts P' but it is not adjacent to v' so w' intercepts P. Since P was randomly chosen, w' intercepts all paths from u' to x.

Finally we prove that w' is not adjacent to u'. From our assumption that $[v]_x$ is distinct from $[w]_x$ it follows from anti-symmetry that there is a path P'' from w' to x missing v'. If u' is adjacent to w', then we have a path from u' to w' then follow the path P'' to x. As v' is not adjacent to either u' or w' so this path is not intercepted by v'. This contradicts the fact that $u' \succ_x v'$.

Let x be an arbitrary vertex of G and G' be the induced subgraph on vertex subset V'. Any vertex y of V' will be called *x-minimal* in G' if $[y]_x$ is minimal among all the x-classes which have non-empty intersection with V'.

4 Domination Search on Asteroidal-Triple-Free Graphs

In this section we present a domination-search algorithm for AT-free graphs. We begin with some useful properties of this family.

Lemma 1. *Let $G = (V, E)$ be a connected AT-free graph and x a vertex in it. C is a connected component of the induced graph on $V - N[x]$. Then any x-minimal vertex in C intercepts all paths from any vertex in C to any vertex outside C.*

Proof. Let $y \in C$ is x-minimal. It is sufficient to show that any path from any vertex in C to any vertex in $N[x]$ passes through $N[y]$.

We will first show that any path from any vertex in C to x is intercepted by y. Let z be an arbitrary vertex in C. If $[y]_x$ and $[z]_x$ are related, then $[z]_x \succ_x [y]_x$ because $[y]_x$ is minimal by choice. Then by the definition, y is either adjacent to z or all paths from z to x pass through $N[y]$.

In the second case $[y]_x$ and $[z]_x$ are unrelated. So either (i) y and z are adjacent to each other or (ii) there exists a path from z to x not intercepted by y and a path from y to x not intercepted by z. Thus in case (ii) $\{x, y, z\}$ form an asteroidal triple. This is not possible in AT-free graphs so z must be adjacent to y. This establishes that all paths from z to x pass through $N[y]$.

Finally we consider arbitrary vertex w in $N[x]$. Consider arbitrary path, P, from z to w. If it passes through x, then we already have seen that it must pass through the neighborhood $N[y]$. Assume that P does not contain x. Extend the path to x: $P' = P.x$. It is a path from z to x. Thus P' passes through $N[y]$. y is outside $N[x]$ so x is not in $N[y]$. Therefore some vertex of P must be in $N[y]$.

Corollary 2. *Let $G = (V, E)$ be a connected AT-free graph and x a vertex in it. C is a connected component of the induced graph on $V - N[x]$. Let $y \in C$ is x-minimal in C. Then each connected component of the induced graph on $V - N[y]$ is entirely contained either in C or in $V - C$.*

It has been established that connected AT-free graphs have a pair of vertices, *poles*, such that every path between them dominates the entire graph, [9,10]. We shall use labels p_1 and p_2 for the poles.

Let $G = (V, E)$ be a graph and x be a vertex in it. If $y \in V - N[x]$, then the connected component containing y in the induced graph over $V - N[x]$ will be denoted by $C^x(y)$ and the open neighborhood $N(C^x(y))$ will be denoted by $S^x(y)$. $C^x(y)$ is defined if and only if y is not adjacent to x. A component $C^x(y)$ will be called *deep* if at least one vertex of the component is not adjacent to any vertex of $N[x]$. If a component is not deep, then it will be termed *shallow*. $C^x(p_1)$ and $C^x(p_2)$ will be called *principal* components of x, if defined. All other components of $V - N[x]$ will be termed *secondary*.

Lemma 2. *Let $G = (V, E)$ be a connected AT-free graph and $x \in V$. Then every deep component of the induced graph on $V - N[x]$ must contain exactly one pole.*

Proof. Suppose $C^x(y)$ contains no polar vertex. Assume that $z \in C^x(y)$ such that $N[z]$ is fully contained in $C^x(y)$. Then there exists a path between p_1 and p_2 which does not enter $C^x(y)$. This implies that the path will miss z which is impossible. Thus $C^x(y)$ must be a shallow component.

In case $C^x(y)$ contains both poles, then there exists a path between the poles which does not enter $N[x]$. This path will miss x. Again impossible for an AT-free graph.

Proposition 3. *Let $G = (V, E)$ be a connected AT-free graph with vertex x in it. Let C be a secondary component in the induced subgraph on $V - N[x]$. Then each vertex of C dominates at least one of p_1, p_2, $S^x(p_1)$, and $S^x(p_2)$.*

Proof. In a connected AT-free graph $G = (V, E)$, x is a vertex such that both its principal components are defined, i.e., neither pole is in $N[x]$. Let C be a secondary component of $V - N[x]$ and z be a vertex in C. Suppose there exists $u \in S^x(p_1)$ and $v \in S^x(p_2)$ such that z is adjacent to neither of these vertices. We can build a path from p_1 to p_2: $p_1 \ldots uxv \ldots p_2$ where u, x, and v are the only vertices of the path from $N[x]$. z cannot be adjacent to any vertex of this path other than u, x, and v. But by choice none of the three is adjacent to z so this path misses z. This is impossible.

If every vertex in a secondary component C dominates $S^x(p_2)$ (when p_2 is not adjacent to x) or dominates p_2 (when p_2 is adjacent to x) then C will be called a p_2-sided component.

Proposition 4. *Let $G = (V, E)$ be a connected AT-free graph with non-adjacent poles. Let x be either p_1 or a vertex for which both principal components are defined. Let $y \in C^x(p_2)$ be an x-minimal vertex. If z is a vertex of $C^x(p_2)$ which dominates $S^y(x)$, then z is also x-minimal.*

Proof. $S^y(x)$ is a graph separator which contains at least one vertex of each edge connecting $C^x(p_2)$ with $V - C^x(p_2)$ because all paths between the two pass through $N[y]$. Therefore each component of the induced graph on $V - S^y(x)$ is either completely contained in $C^x(p_2)$ or in $V - C^x(p_2)$. Thus every path from $C^x(p_2)$ to $V - C^x(p_2)$ must touch $S^y(x)$. If $N[z]$ contains $S^y(x)$ then all such path also touch $N[z]$. Therefore z also x-minimal.

Let x and y be vertices in a graph. Then by $|C^x(y)|$ we denote the cardinality of $C^x(y)$ if y is not adjacent to x. If the two vertices are adjacent, then $|C^x(y)|$ is defined to be zero.

Lemma 3. *Let G be a connected AT-free graph and x a vertex which is not adjacent to p_2. Further x is either p_1 or not adjacent to p_1. Let y be x-minimal in $C^x(p_2)$ but different from p_2, $|C^y(p_2)| \geq |C^{y'}(p_2)|$ for all x-minimal y', and p_2 does not dominate $S^y(p_1)$. Then any secondary component of $V - N[y]$ having non-empty intersection with $C^x(p_2)$, is p_2-sided.*

Proof. We consider two cases: $p_2 \notin N[y]$ and $p_2 \in N(y)$ since $y \neq p_2$.

(a) $p_2 \notin N[y]$. Suppose C is a secondary component of induced graph on $V - N[y]$ such that $C \cap C^x(p_2)$ is non-empty. Assume that C is not p_2-sided. Therefore there is a vertex z in C such that it does not dominate $S^y(p_2)$. From Corollary 2 we know that entire C is contained in $C^x(p_2)$ so z belongs to $C^x(p_2)$. From Proposition 3 z dominates $S^y(p_1)$.

Next we show that z does not dominate $S^y(x)$. Assume the contrary. From Proposition 4 it is an x-minimal vertex of $C^x(p_2)$. Since $C^y(p_2) \cap N[z]$ is empty, $C^z(p_2)$ will contain $C^y(p_2)$. In addition, by choice, z does not dominate $S^y(p_2)$ so there is a path from y to p_2 not intercepted by z. Thus y is also contained in $C^z(p_2)$. This implies that $|C^z(p_2)| > |C^y(p_2)|$. But Due to the choice of y, $|C^z(p_2)|$ can never be larger than $|C^y(p_2)|$.

Now we will show that $\{x, z, p_2\}$ is an asteroidal triple. Since p_2 and z belong to $C^x(p_2)$ so there is a path between z and p_2 not passing through $N[x]$.

To show that there is path between z and x which misses p_2 observe that p_2 does not dominate $S^y(p_1)$ so there exists a vertex u in $S^y(p_1)$ which is not adjacent to p_2 but adjacent to z since the latter dominates $S^y(p_1)$. Consider two cases. In the first case $x \in C^y(p_1)$. Consider the path $zu.P$ where P joins u to x and is confined to $C^y(p_1)$. This path misses p_2. In case $x \notin C^y(p_1)$ $N[x]$ contains $S^y(p_1)$ since $N[x]$ separates p_1 from y. Thus x is adjacent to u and xuz is a path that misses p_2.

Finally it needs to be shown that there is a path between x and p_2 not intercepted by z. We have seen that z does not dominate $S^y(x)$ so there is a vertex v in it which is not adjacent to z. Also there is a vertex w in $S^y(p_2)$ not adjacent to z since by choice z does not dominate $S^y(p_2)$. So there is a path $xvyw.P$ where P is a path from w to p_2 contained in $C^y(p_2)$. This path is not intercepted by z. Consequently the entire path, from x to p_2 misses z. Thus $\{x, z, p_2\}$ form an asteroidal set which is not possible.

(b) $p \in N(y)$. Again C is a secondary component of y such that $C \cap C^x(p_2)$ is non-empty. Assume C is not contained in $N[p_2]$. Therefore there is a vertex z in C such that it is not adjacent to p_2. From Corollary 2 we know that entire C is contained in $C^x(p_2)$ so z belongs to $C^x(p_2)$. From Proposition 3 z dominates $S^y(p_1)$.

We will again show that z does not dominate $S^y(x)$. Assume the contrary. From Proposition 4 it is an x-minimal vertex of $C^x(p_2)$. $|C^y(p_2)| = 0$ but $C^z(p_2)$ contains at least p_2 so again $|C^z(p_2)| > |C^y(p_2)|$. But Due to the choice of y, $|C^z(p_2)|$ can never be larger than $|C^y(p_2)|$.

Similar to the proof of part (a) we can show that $\{x, z, p_2\}$ is an asteroidal triple.

Lemma 4. *$G = (V, E)$ is a connected AT-free graph and y is a vertex in it which is not adjacent to pole p_1. Pole p_2 dominates $S^y(p_1)$. Then $\{u, p_2\}$ dominates $V - C^y(p_1)$ where u is any vertex in $C^y(p_1)$.*

Proof. Consider a path $p_2u.P$ where P is a path to p_1 confined to $C^y(p_1)$. Each vertex of V is dominated by this path. Since vertices of $V - (C^y(p_1) \cup S^y(p_1))$ are not adjacent to any vertex beyond u, $\{u, p_2\}$ dominate them. Further vertices of $S^y(p_1)$ are in the neighborhood of p_2. So $\{u, p_2\}$ dominate $V - C^y(p_1)$.

Algorithm: *Domination search on a connected AT-free graph.*

1. If the poles are adjacent (so $\{p_1, p_2\}$ is a dominating set) then put the two guards at the poles and exit;
2. Place a guard at p_1;
3. Place the second guard at any vertex in $S^{p_1}(p_2)$;
4. Relieve the second guard;
   C1: Vertices of $V - C^{p_1}(p_2)$ are cleared
5. $x = p_1$;
6. While (p_2 is not adjacent to x) Do
7. { Let vertex u in $C^x(p_2)$ is x-minimal with maximum $|C^u(p_2)|$;
8. $y = u$;
9. If p_2 dominates $S^y(p_1)$ then
10. { Place the free guard at p_2 and relieve the guard at x;
    C2: $C^y(p_1)$ being a subset of $V - C^x(p_2)$ remains clear and $N[p_2]$ is also now cleared.
11. Place the free guard at any vertex in $S^y(p_1)$;
    C3: Entire V is clear.
12. Exit;
    }
13. Else
    C4: p_2 does not dominate $S^y(p_1)$ so $y \neq p_2$.
14. { Place the free guard at y;
15. Relieve the guard from x;
    C5: All the vertices of $V - C^x(p_2)$ remain clear. In addition $N[y]$ is also cleared.
16. if p_2 is not adjacent to y
17. { Place the free guard at any vertex of $S^y(p_2)$ and relieve it;}
18. else { Place the free guard at p_2 }
    C6: if p_2 is not adjacent to y, then $V - C^y(p_2)$ is clear else entire V is cleared.
19. $x = y$;
    }
    C7: If x is not adjacent to p_2 then vertices of $V - C^x(p_2)$ are cleared else all of V is cleared.
    }
    C8: Entire V is cleared.
20. Exit.

Theorem 5. *The domination-search number of AT-free graphs is at most 2.*

Proof. The algorithm described above performs domination search for any AT-free graph with 2 guards. We prove the correctness of the algorithm by justifying the invariants mentioned in the comments.

C1: In line-2 $N[p_1]$ is cleared. From Proposition 3, line-3 clears all secondary components of p_1. No recontamination of these components occur in line-4 since the first guard is still present at p_1.

C2: From Corollary 2 $C^y(p_1)$ is either entirely contained in $C^x(p_2)$ or in $V - C^x(p_2)$. Due to Lemma 2 p_1 cannot be in $C^x(p_2)$ so $C^y(p_1)$ must be contained in $V - C^x(p_2)$. There is a guard at p_2 and p_2 dominates $S^y(p_1)$ so $C^y(p_1)$ remains clear.

C3: Due to Lemma 4.

C4: Self explanatory.

C5: Due to Corollary 2.

C6: Due to Lemma 3.

C7: Trivial.

C8: Trivial.

The algorithm is monotonic (there is no recontamination and at least one more vertex is cleared in each pass of the loop), due to C5, as long as the condition of line-9 is not true. When the condition is true the algorithm terminates after executing lines 10, 11, and 12. Therefore the algorithm always terminates.

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