The Layered Net Surface Problems in Discrete Geometry and Medical Image Segmentation

Xiaodong Wu[1,⋆], Danny Z. Chen[2,⋆⋆], Kang Li[3], and Milan Sonka[4,⋆⋆⋆]

[1] Departments of Electrical & Computer Engineering and Radiation Oncology, University of Iowa, Iowa City, Iowa 52242, USA
xiaodong-wu@uiowa.edu
[2] Department of Computer Science and Engineering, University of Notre Dame, Notre Dame, IN 46556, USA
chen@cse.nd.edu
[3] Dept. of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA 15213, USA
kangl@cmu.edu
[4] Dept. of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242-1595, USA
sonka@engineering.uiowa.edu

Abstract. Efficient detection of multiple inter-related surfaces representing the boundaries of objects of interest in d-D images ($d \geq 3$) is important and remains challenging in many medical image analysis applications. In this paper, we study several *layered net surface (LNS)* problems captured by an interesting type of geometric graphs called *ordered multi-column graphs* in the d-D discrete space ($d \geq 3$). The LNS problems model the simultaneous detection of multiple mutually related surfaces in three or higher dimensional medical images. Although we prove that the d-D LNS problem ($d \geq 3$) on a general ordered multi-column graph is NP-hard, the (special) ordered multi-column graphs that model medical image segmentation have the self-closure structures, and admit polynomial time exact algorithms for solving the LNS problems. Our techniques also solve the related *net surface volume (NSV)* problems of computing well-shaped geometric regions of an optimal total volume in a d-D weighted voxel grid. The NSV problems find applications in medical image segmentation and data mining. Our techniques yield the first polynomial time exact algorithms for several high dimensional medical image segmentation problems. The practical efficiency and accuracy of the algorithms are showcased by experiments based on real medical data.

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1 Introduction

In this paper, we study the *layered net surface* (LNS) problems and their extensions in discrete geometry. These problems arise in d-D medical image segmentation ($d \geq 3$) and other applications.

As a central problem in image analysis, image segmentation aims to define accurate boundaries for the objects of interest captured by image data. Accurate 3-D image segmentation techniques promise to improve medical diagnosis and revolutionize the current medical imaging practice. Although intensive research has been done on image segmentation in several decades, efficient and effective high dimensional image segmentation still poses one of the major challenges in image understanding. As one common practice, to identify surface representing the boundary of the sought 3-D object, 2-D image slices are more or less analyzed independently, and the 2-D results are stacked together to form the 3-D segmentation. One most successful and widely used technique is based on 2-D dynamic programming and optimal graph searching [11]. These approaches have inherent limitations – the most fundamental one stems from the lack of contextual slice-to-slice information when analyzing a sequence of consecutive 2-D images. Performing the segmentation directly on a 3-D image can produce a more consistent segmentation result, yielding 3-D surfaces for object boundaries instead of a set of individual 2-D contours. Another active and far-reaching line of research in this area rely on variational calculus and numerical methods, e.g. level set methods and deformable models [9]. Although these approaches are theoretically powerful, the interfacing between continuous formulations and discrete solutions involve numerical approximation and stability issues.

We present a novel graph-theoretic technique for the problem of simultaneous segmentation of multiple inter-related surfaces in three or higher dimensional medical images, namely, the LNS problem. This technique is practically significant since many surfaces in medical images appear in mutual relations. A number of medical imaging problems can benefit from an efficient method for simultaneous detection of multiple inter-related 3-D surfaces [11, 15, 9].

The simultaneous detection of multiple inter-related surfaces has been studied by the medical image analysis community for a long time. For the 2-D case, there are several satisfactory results [11, 1, 14]. However, little work has been done on the three and higher dimensional cases. Previous attempts [12, 3] on extending graph-search based segmentation methods for the 2-D case to identifying even a single optimal surface in 3-D medical images either made the methods computationally intractable or traded their ability to achieve global optima for computational efficiency. Motivated by this segmentation problem, Wu and Chen [13] introduced the optimal net surface problems and presented efficient polynomial time exact algorithms for them. But, the algorithms in [13] can detect only one optimal surface in 3-D. An implementation of their algorithms and experimental validation based on real 3-D medical images were presented in [7]. More recently, Li *et al.* [8] extended the approach [13, 7] to segmenting multiple inter-related surfaces in 3-D. However, their new method does not consider the very important region information (e.g., homogeneity) for the surface detection.

Modeling the simultaneous detection of multiple inter-related surfaces in high dimensional medical images, we introduce the *layered net surface* (LNS) problems on an interesting type of geometric graphs, called *ordered multi-column graphs*, embedded in the d-D discrete space for $d \geq 3$ (to be defined in Section 2). We further extend the LNS problems to a more general ordered multi-column graph (Section 5). Motivated by segmenting anatomical structures with a relatively regular geometric shape, such as the left ventricles, kidneys, livers, and lungs, we also study several *net surface volume* (NSV) problems, which aim to find well-shaped regions of an optimal "volume" in a d-D weighted voxel grid. These well-shaped geometric regions are closely related to monotonicity and convexity in d-D discrete spaces (Section 4). Our main results in this paper are summarized as follows.

– We develop an efficient algorithm for solving the LNS problem on an interesting type of ordered multi-column graphs in polynomial time, by formulating it as computing a minimum closed set in a vertex-weighted directed graph.
– We extend our LNS technique to solving the NSV problems of computing several classes of optimal well-shaped geometric regions in a d-D weighted voxel grid. These NSV problems arise in data mining [2], image segmentation [1], and data visualization. The classes of regions that we study can be viewed as generalizations of some of the pyramid structures in [2].
– We prove that the LNS problem on a general ordered multi-column graph is NP-hard. However, the (special) ordered multi-column graphs that model medical image segmentation applications have additional properties, and the LNS problem on such graphs is polynomially solvable.
– We apply our polynomial time LNS algorithms to segmenting multiple inter-related object boundaries in 3-D medical images.

We omit the proofs of the lemmas and theorems due to the page limit.

2 The Layered Net Surface (LNS) Problems

A *multi-column graph* $G = (V, E)$ embedded in the d-D discrete space is defined as follows. For a given undirected graph $B = (V_B, E_B)$ embedded in $(d-1)$-D (called the *net model*) and an integer $\kappa > 0$, G is an undirected graph in d-D *generated* by B and κ. For each vertex $v = (x_0, x_1, \ldots, x_{d-2}) \in V_B$, there is a sequence $Col(v)$ of κ vertices in G corresponding to v; $Col(v) = \{(x_0, x_1, \ldots, x_{d-2}, k) : k = 0, 1, \ldots, \kappa - 1\}$, called the v-*column* of G. We denote the vertex $(x_0, x_1, \ldots, x_{d-2}, k)$ of $Col(v)$ by v_k. If an edge $(v, u) \in E_B$, then we say that the v-column and u-column in G are *adjacent* to each other. For each vertex $v_k \in Col(v)$, v_k has edges in G to a non-empty list of consecutive vertices in every adjacent u-column $Col(u)$ of $Col(v)$, say $u_{k'}, u_{k'+1}, \ldots, u_{k'+s}$ ($s \geq 0$); we call $(u_{k'}, u_{k'+1}, \ldots, u_{k'+s})$, in this order, the *edge interval* of v_k on $Col(u)$, denoted by $I(v_k, u)$. For an edge interval I, we denote by $Bottom(I)$ (resp., $Top(I)$) the d-th coordinate of the first (resp., last) vertex in I (e.g., $Bottom(I(v_k, u)) = k'$ and $Top(I(v_k, u)) = k' + s$ in the above example).

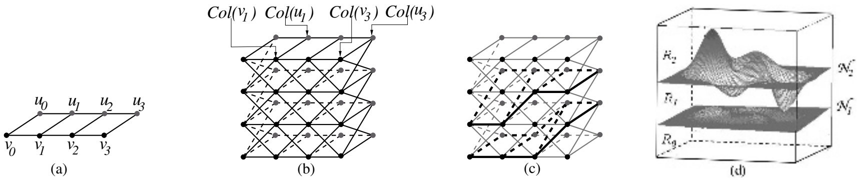

Fig. 1. (a) A 2-D net model B. (b) A 3-D properly ordered multi-column graph G generated by B and $\kappa = 4$ (the edges between $Col(u_i)$ and $Col(u_{i+1})$, $i = 0, 1, 2$, are symmetric to those between $Col(v_i)$ and $Col(v_{i+1})$, and the edges between $Col(v_j)$ and $Col(u_j)$, $j = 1, 2$, are symmetric to those between $Col(v_3)$ and $Col(u_3)$; all these edges are omitted for a better readability). (c) Two $(1, 2)$-separate net surfaces in G marked by heavy edges. (d) Two net surfaces divide the vertex set of G into three disjoint vertex subsets.

Two adjacent columns $Col(v)$ and $Col(u)$ in G are said to be in *proper order* if for any two vertices v_k and v_{k+1} in $Col(v)$,$Bottom(I(v_k, u)) \leq Bottom(I(v_{k+1}, u))$ and $Top(I(v_k, u)) \leq Top(I(v_{k+1}, u))$, and if the same holds for any two vertices u_k and u_{k+1} of $Col(u)$ on $Col(v)$. The corresponding edge $(v, u) \in E_B$ is called a *proper edge*. If all pairs of adjacent columns in G are in proper order, then we call G a *properly ordered* multi-column graph (briefly, a *properly ordered graph*). Figures 1(a)–1(b) show a net model and a properly ordered graph.

Note that in medical image segmentation, the boundaries of the target objects (e.g., organs) are often sufficiently “smooth”. The smoothness constraint on the sought surfaces is modeled by the proper ordering of the edges in a multi-column graph G, that is, the edges connecting each vertex v_k in G to every adjacent column $Col(u)$ of $Col(v)$ form an edge interval on $Col(u)$, and such edge intervals for any two adjacent columns of G are in proper order.

A *net surface* in G (also called a *net*) is a subgraph of G defined by a function $\mathcal{N}$: $V_B \rightarrow \{0, 1, \ldots, \kappa - 1\}$, such that for every edge $(v, u) \in E_B$, $(v_{k'}, u_{k''})$, with $k' = \mathcal{N}(v)$ and $k'' = \mathcal{N}(u)$, is also an edge in G. For simplicity, we denote a net by its function $\mathcal{N}$. Intuitively, a net $\mathcal{N}$ in G is a special mapping of the $(d-1)$-D net model B to the d-D space, such that $\mathcal{N}$ “intersects” each v-column of G at exactly one vertex and $\mathcal{N}$ preserves all topologies of B. $\mathcal{N}$ can be viewed as a functional “surface” of B in d-D defined on the $(d - 1)$-D space in which B is embedded.

Given two integers L and U, $0 < L < U$, two nets $\mathcal{N}_1$ and $\mathcal{N}_2$ of a properly ordered graph G are said to be (L, U)-*separate* if $L \leq \mathcal{N}_2(v) - \mathcal{N}_1(v) \leq U$ for every vertex $v \in V_B$. Roughly speaking, $\mathcal{N}_1$ and $\mathcal{N}_2$ do not cross each other and are within a specified range of distance (see Figure 1(c)). For a given set of $l-1$ integer parameter pairs $\{(L_i, U_i) : 0 < L_i < U_i, 1 \leq i < l\}$, $l \geq 2$, we consider l net surfaces $\mathcal{NS} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l\}$ in G such that $\mathcal{N}_{i+1}$ is “on top” of $\mathcal{N}_i$ (i.e., $\forall v \in V_B, \mathcal{N}_{i+1}(v) > \mathcal{N}_i(v)$), and $\mathcal{N}_i$ and $\mathcal{N}_{i+1}$ are (L_i, U_i)-separate. Then, these l net surfaces partition the vertex set V of G into $l+1$ disjoint subsets R_i, with $R_0 = \{v_k : v \in V_B, 0 \leq k \leq \mathcal{N}_1(v)\}$, $R_i = \{v_k : v \in V_B, \mathcal{N}_i(v) < k \leq \mathcal{N}_{i+1}(v)\}$ for $i = 1, 2, \ldots, l-1$, and $R_l = \{v_k : v \in V_B, \mathcal{N}_l(v) < k < \kappa\}$ (see Figure 1(d)).

Motivated by medical image segmentation [11,16,9], we assign costs to every vertex of G as follows. Each vertex $v_k \in V$ has an *on-surface cost* $b(v_k)$, which is an arbitrary real value. For each region R_i ($i = 0, 1, \ldots, l$), every vertex $v_k \subseteq V$ is assigned a real-valued *in-region cost* $c_i(v_k)$. The on-surface cost of each vertex is inversely related to the likelihood that it may appear on a desired net surface, while the in-region costs $c_i(\cdot)$ ($i = 0, 1, \ldots, l$) measure the inverse likelihood of a given vertex preserving the expected regional properties of the partition $\{R_0, R_1, \ldots, R_l\}$. Both the on-surface and in-region costs for image segmentation can be determined using low-level image features [9,11,16].

The **layered net surface (LNS)** problem seeks l net surfaces $\mathcal{NS} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l\}$ in G such that the total cost $\alpha(\mathcal{NS})$ induced by the l net surfaces in $\mathcal{NS}$, with

$$\alpha(\mathcal{NS}) = \sum_{i=1}^{l} b(\mathcal{N}_i) + \sum_{i=0}^{l} c_i(R_i) = \sum_{i=1}^{l} \sum_{u \in V(\mathcal{N}_i)} b(u) + \sum_{i=0}^{l} \sum_{u \in R_i} c_i(u),$$

is minimized, where $V(H)$ denotes the vertex set of a graph H.

In fact, our algorithmic framework is general enough for the cases in which each vertex has *only* an on-surface cost, *only* in-region costs, or both. We will present our approach for the case where each vertex has both the on-surface and in-region costs.

3 Algorithm for the Layered Net Surface (LNS) Problem

This section gives our polynomial time algorithm for the layered net surface problem on a d-D properly ordered graph $G = (V, E)$. We first exploit the self-closure structure of the LNS problem, and then model it as a minimum-cost closed set problem based on a nontrivial graph transformation scheme.

A *closed set* $\mathcal{C}$ in a directed graph with arbitrary vertex costs $w(\cdot)$ is a subset of vertices such that all successors of any vertex in $\mathcal{C}$ are also contained in $\mathcal{C}$ [10]. The *cost* of a closed set $\mathcal{C}$, denoted by $w(\mathcal{C})$, is the total cost of all vertices in $\mathcal{C}$. Note that a closed set can be empty (with a cost zero). The minimum-cost closed set problem seeks a closed set in the graph whose cost is minimized.

3.1 The Self-closure Property of the LNS Problem

Our algorithm for the LNS problem hinges on the following observations about the self-closure structure of any feasible LNS solution. Recall that in a set of l feasible net surfaces $\mathcal{NS} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l\}$ in G, $\mathcal{N}_{i+1}$ is "on top" of $\mathcal{N}_i$, for each $i = 1, 2, \ldots, l-1$.

For a vertex $v_k \in V$ (i.e., $v \in V_B$ and $0 \leq k < \kappa$) and each adjacent column $Col(u)$ of $Col(v)$ (i.e., $(v, u) \in E_B$), the *lower-eligible-neighbor* of v_k on $Col(u)$ is the vertex in $Col(u)$ with the smallest d-th coordinate that has an edge to v_k in G (i.e., the vertex in $Col(u)$ with the smallest d-th coordinate that can possibly appear together with v_k on a same feasible net surface in G).

Given the surface separation constraints, we define below the *upstream* and *downstream* vertices of any vertex in G, to help characterize the spatial relations between feasible net surfaces in G. For every vertex $v_k \in V$ and $1 \le i < l$ (resp., $1 < i \le l$), the *i-th upstream* (resp., *downstream*) vertex of v_k is v_{k+L_i} (resp., $v_{\max\{0,k-U_{i-1}\}}$) if $k + L_i < \kappa$ (resp., $k - L_{i-1} \ge 0$). Intuitively, if $v_k \in \mathcal{N}_i$, then the i-th upstream (resp., downstream) vertex of v_k is the vertex in $Col(v)$ with the smallest d-th coordinate that can be on $\mathcal{N}_{i+1}$ (resp., $\mathcal{N}_{i-1}$).

We say that a vertex v_k is *below* (resp., *above*) a net surface $\mathcal{N}_i$ if $\mathcal{N}_i(v) > k$ (resp., $\mathcal{N}_i(v) < k$), and denote by $LO(\mathcal{N}_i)$ the subset of all vertices of G that are on or below $\mathcal{N}_i$. For every vertex $v_k \in LO(\mathcal{N}_i)$, consider its lower-eligible-neighbor $u_{k'}$ on any adjacent $Col(u)$ of $Col(v)$. Let $r = \mathcal{N}_i(v)$ and $u_{k''}$ be the lower-eligible-neighbor of v_r on $Col(u)$ ($v_r \in Col(v)$ is on $\mathcal{N}_i$). Then by the definition of net surfaces, $k'' \le \mathcal{N}_i(u)$. Since $k \le \mathcal{N}_i(v)$, we have $k' \le k''$ due to the proper ordering. Thus, $k' \le \mathcal{N}_i(u)$, and further, $u_{k'} \in LO(\mathcal{N}_i)$. Hence, we have the following observation.

Observation 1. *For any feasible net surface $\mathcal{N}_i$ in G, if a vertex v_k is in $LO(\mathcal{N}_i)$, then every lower-eligible-neighbor of v_k is also in $LO(\mathcal{N}_i)$.*

Observation 1 characterizes the self-closure property of every set $LO(\mathcal{N}_i)$. However, our task is more involved since the l net surfaces in $\mathcal{NS}$ are inter-related. We need to further examine the closure structure between the $LO(\mathcal{N}_i)$'s.

Observation 2. *Given any set $\mathcal{NS} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l\}$ of l feasible net surfaces in G, the i-th upstream (resp., downstream) vertex of each vertex in $LO(\mathcal{N}_i)$ is in $LO(\mathcal{N}_{i+1})$ (resp., $LO(\mathcal{N}_{i-1})$), for every $1 \le i < l$ (resp., $1 < i \le l$).*

Observations 1 and 2 show an important self-closure structure of the LNS problem, which is crucial to our LNS algorithm and suggests a connection between our target problem and the minimum-cost closed set problem [10]. In our LNS approach, instead of directly searching for an optimal set of l net surfaces, $\mathcal{NS}^* = \{\mathcal{N}_1^*, \mathcal{N}_2^*, \ldots, \mathcal{N}_l^*\}$, we look for l optimal subsets of vertices in G, $LO(\mathcal{N}_1^*) \subset LO(\mathcal{N}_2^*) \subset \ldots \subset LO(\mathcal{N}_l^*)$, such that each $LO(\mathcal{N}_i^*)$ uniquely defines the net surface $\mathcal{N}_i^* \in \mathcal{NS}^*$.

3.2 The Graph Transformation Scheme

Our LNS algorithm is based on a sophisticated graph transformation scheme, which enables us to simultaneously identify $l > 1$ optimal inter-related net surfaces as a whole by computing a minimum closed set in a weighted directed graph G' that we transform from G. This section presents the construction of G', which crucially relies on the self-closure structure shown in Section 3.1.

We construct the vertex-weighted directed graph $G' = (V', E')$ from the d-D properly ordered graph $G = (V, E)$, as follows. G' contains l vertex-disjoint subgraphs $\{G'_i = (V'_i, E'_i) : i = 1, 2, \ldots, l\}$; each G'_i is for the search of the i-th net surface $\mathcal{N}_i$. $V' = \bigcup_{i=1}^{l} V'_i$ and $E' = \bigcup_{i=1}^{l} E'_i \cup E'_s$. The surface separation constraints between any two consecutive net surfaces $\mathcal{N}_i$ and $\mathcal{N}_{i+1}$ are enforced in G' by a subset of edges in E'_s, which connect the subgraphs G'_i and G'_{i+1}.

The construction of each subgraph $G'_i = (V'_i, E'_i)$ is similar to that in [13]. For G'_i, every vertex v_k in G corresponds to exactly one vertex $v^i_k \in V'_i$. Each column $Col(v)$ in G associates with a *chain* $v^i_{\kappa-1} \to v^i_{\kappa-2} \to \cdots \to v^i_0$ in G'_i. We then put directed edges into E'_s between G'_i and G'_{i+1}, to enforce the surface separation constraints. For each vertex v^i_k with $k < \kappa - L_i$ on the chain $Ch_i(v)$ in G'_i, a directed edge is put in E'_s from v^i_k to $v^{i+1}_{k+L_i}$ on $Ch_{i+1}(v)$ in G'_{i+1}. On the other hand, each vertex v^{i+1}_k with $k \geq L_i$ on $Ch_{i+1}(v)$ has a directed edge in E'_s to $v^i_{k'}$ on $Ch_i(v)$ with $k' = \max\{0, k - U_i\}$ (note that $v_{k'}$ in G is the $(i+1)$-th downstream vertex of v_k). Note that in this construction, each vertex v^i_k with $k \geq \kappa - L_i$ has no edge to any vertex on $Ch_{i+1}(v)$, and each vertex v^{i+1}_k with $k < L_i$ has no edge to any vertex on $Ch_i(v)$. These vertices of G' are called *deficient vertices*, whose corresponding vertices in G cannot possibly appear in any feasible solution for the LNS problem. By exploiting the geometric properties of the properly ordered graphs, all deficient vertices in G' can be pruned in linear time. We simply denote the graph thus resulted also by G'. Then, for every $v \in V_B$ and $i = 1, 2, \ldots, l$, let $\mu_i(v)$ and $\kappa_i(v)$ be the smallest and largest d-th coordinates of the vertices on the chain $Ch_i(v)$ of G'_i, respectively. We then assign a cost $w(v^i_k)$ for each vertex v^i_k in G'_i: If $k = \mu_i(v)$, then $w(v^i_k) = b(v_k) + \sum_{j=0}^{k} [c_{i-1}(v_j) - c_i(v_j)]$; otherwise, $w(v^i_k) = [b(v_k) - b(v_{k-1})] + [c_{i-1}(v_k) - c_i(v_k)]$. This completes the construction of G'.

3.3 Computing Optimal Layered Net Surfaces for the LNS Problem

The graph G' thus constructed allows us to find l optimal net surfaces in G, by computing a non-empty minimum-cost closed set in G'. Given any closed set $\mathcal{C} \neq \varnothing$ in G', we define l feasible net surfaces, $\mathcal{NS} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l\}$, in G, as follows. Recall that we search for each net $\mathcal{N}_i$ in the subgraph $G'_i = (V'_i, E'_i)$. Let $\mathcal{C}_i = \mathcal{C} \cap V'_i$. For each vertex $v \in V_B$, denote by $\mathcal{C}_i(v)$ the set of vertices of $\mathcal{C}_i$ on the chain $Ch_i(v)$ of G'_i. Based on the construction of G'_i, it is not hard to show that $\mathcal{C}_i(v) \neq \varnothing$. Let $r_i(v)$ be the largest d-th coordinate of the vertices in $\mathcal{C}_i(v)$. Define the function $\mathcal{N}_i$ as $\mathcal{N}_i(v) = r_i(v)$ for every $v \in V_B$. The following lemma is a key to our algorithm.

Lemma 1. *Any closed set $\mathcal{C} \neq \varnothing$ in G' specifies l feasible net surfaces in G whose total cost differs from that of $\mathcal{C}$ by a fixed value $c_l(V)$.*

Next, we argue that any l feasible net surfaces, $\mathcal{NS} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_l\}$, in G correspond to a closed set $\mathcal{C} \neq \varnothing$ in G'. Based on the construction of G', every vertex v_k on the net $\mathcal{N}_i$ corresponds to a vertex v^i_k in G'_i (v^i_k is not a deficient vertex). We construct a closed set $\mathcal{C}_i \neq \varnothing$ in G'_i for each net $\mathcal{N}_i$, as follows. Initially, let $\mathcal{C}_i = \varnothing$. For each vertex $v \in V_B$, we add to $\mathcal{C}_i$ the subset $\mathcal{C}_i(v) = \{v^i_k \;:\; k \leq \mathcal{N}_i(v)\}$ of vertices on $Ch_i(v)$ of G'_i.

Lemma 2. *Any set $\mathcal{NS}$ of l feasible net surfaces in G defines a closed set $\mathcal{C} \neq \varnothing$ in G' whose cost differs from that of $\mathcal{NS}$ by a fixed value.*

By Lemmas 1 and 2, we compute a minimum-cost closed set $\mathcal{C}^* \neq \varnothing$ in G', which specifies l optimal net surfaces in G. Note that G' has $O(l \cdot n)$ vertices and $O(l \cdot n \cdot \frac{m_B}{n_B})$ edges, where $n = |V|$ is the number of vertices in G, and $n_B = |V_B|$ and $m_B = |E_B|$ for the net model B. By using the minimum s-t cut algorithm in [5] to compute a minimum-cost closed set in G', we have the following result.

Theorem 1. *The LNS problem can be solved in* $O(l^2 n^2 \frac{m_B}{n_B} \log(\frac{l \cdot n \cdot n_B}{m_B}))$ *time.*

4 Algorithms for the Net Surface Volume (NSV) Problems

This section presents our algorithms for several optimal net surface volume (NSV) problems. Specifically, instead of looking for multiple inter-related net surfaces as in Section 3, for a given d-D voxel grid $\Gamma = [0..N-1]^d$ of $n = N^d$ cells, with each cell $\mathbf{x}(x_0, x_1, \ldots, x_{d-1}) \in \Gamma$ having an arbitrary real "volume" value $vol(\mathbf{x})$, we seek multiple surfaces that enclose a well-shaped region $R \subseteq \Gamma$, such that the *volume* $vol(R)$ of R, $vol(R) = \sum_{\mathbf{x} \in R} vol(\mathbf{x})$, is minimized (or maximized). Note that even the case of the NSV problem on finding an optimal simple polygon in a weighted 2-D grid is in general NP-hard [1].

We consider two classes of regions, called *weakly watershed-monotone regions* and *watershed-monotone shells*, defined as follows. For any integers $0 \leq i < d$ and $0 \leq c < N$, let $\Gamma_i(c)$ denote all voxels of Γ whose x_i-coordinate is c (note that $\Gamma_i(c)$ is orthogonal to the x_i-axis). A region R in Γ is said to be x_i-*monotone* if for any line l parallel to the x_i-axis, the intersection $R \cap l$ is either empty or a continuous segment. Further, we say that R is *watershed-monotone with respect to* $\Gamma_i(c)$ if (1) R is x_i-monotone, and (2) for any line l orthogonal to $\Gamma_i(c)$, if the intersection $R \cap l \neq \varnothing$, then $R \cap l$ intersects a voxel of $R \cap \Gamma_i(c)$. (Intuitively, the intersection of R and $\Gamma_i(c)$ is equal to the projection of R onto $\Gamma_i(c)$, and is like a "watershed" of R.) If for every $i = 0, 1, \ldots, d-1$, R is watershed-monotone to a $\Gamma_i(c_i)$ for an integer $0 \leq c_i < N$, then we say that R is *watershed-monotone.* A region $R \subseteq \Gamma$ is *weakly watershed-monotone* if R is watershed-monotone to every axis in a set of $d-1$ axes of Γ and is monotone (but need not be watershed-monotone) to the remaining axis. Clearly, watershed-monotone regions are a subclass of weakly watershed-monotone regions. Suppose R is watershed-monotone with respect to some $\Gamma_i(c_i)$, for each $i = 0, 1, \ldots, d-1$; then it is easy to see that $\cap_{i=0}^{d-1} \Gamma_i(c_i) \neq \varnothing$. A voxel in $\cap_{i=0}^{d-1} \Gamma_i(c_i)$ is called a *kernel voxel* of R. Our second region class is called the *watershed-monotone shells.* For any two watershed-monotone regions R_1 and R_2 such that R_1 and R_2 have a common kernel voxel $\mathbf{c}$ and $R_2 \subseteq R_1$, the region R in Γ bounded between R_1 and R_2, i.e., $R = R_1 - R_2$, is a *watershed-monotone shell.*

Theorem 2. *(1) The optimal weakly watershed-monotone region problem is solvable in* $O(dn^2 \log \frac{n}{d})$ *time. (2) The optimal watershed-monotone shell problem is solvable in* $O(dn^2 \log \frac{n}{d})$ *time.*

5 Algorithm for the Bipartite LNS (BLNS) Problem

In this section, we consider the layered net surface problem on a more general *ordered* multi-column graph. Recall that any two adjacent columns of a *properly* ordered multi-column graph are in proper order (Section 2). We now define the *reverse order* on two adjacent columns $Col(v)$ and $Col(u)$ in a d-D multi-column graph $G = (V, E)$ generated by a $(d-1)$-D net model $B = (V_B, E_B)$: If for any two vertices v_k and v_{k+1} in $Col(v)$, $Bottom(I(v_k, u)) \geq Bottom(I(v_{k+1}, u))$ and $Top(I(v_k, u)) \geq Top(I(v_{k+1}, u))$, and if the same holds for any two vertices u_k and u_{k+1} of $Col(u)$ on $Col(v)$, then we say that $Col(u)$ and $Col(v)$ are in *reverse order*. If every two adjacent columns in G are in either proper order or reverse order, then we call G a d-D *ordered* multi-column graph. Further, for two (L, U)-separate nets $\mathcal{N}_1$ and $\mathcal{N}_2$ in G, if two adjacent columns $Col(v)$ and $Col(u)$ are in reverse order, then $L \leq \mathcal{N}_1(v) - \mathcal{N}_2(v) \leq U$ and $L \leq \mathcal{N}_2(u) - \mathcal{N}_1(u) \leq U$. In this section, we assume that each vertex in G has only an on-surface cost.

We can prove that the LNS problem on a general d-D ordered multi-column graph ($d \geq 3$) is NP-hard, by reducing to it the minimum vertex cover problem that is known to be NP-complete [4].

Next, we consider the LNS problem on a d-D ordered multi-column graph $G = (V, E)$ with a special net model $B = (V_B, E_B)$, defined as follows. First, remove from B all reverse edges; the remaining B is a set $\mathcal{CC}$ of connected components with proper edges only. Then, contract each connected component of $\mathcal{CC}$ into a single vertex. Finally, for each (removed) reverse edge $(v, u) \in E_B$, say, v in $C' \in \mathcal{CC}$ and u in $C'' \in \mathcal{CC}$ ($C' = C''$ is possible), add an edge between the contracted vertices of C' and C''. The resulting graph is called the *p-contracted graph* of B. The **bipartite LNS** (BLNS) problem is defined on a d-D ordered multi-column graph with a net model B whose p-contracted graph is *bipartite*. Let $n = |V|$, $m_B = |E_B|$, and $n_B = |V_B|$.

Theorem 3. *The general BLNS problem can be solved in* $O(l^2 n^2 \frac{m_B}{n_B} \log(\frac{l \cdot n \cdot n_B}{m_B}))$ *time, where* l *is the number of sought net surfaces.*

6 Implementation and Experiments

To further examine the behavior and performance of our LNS algorithm, we implemented it in standard C++ templates. After the implementation, we extensively experimented with 3-D synthetic and real medical image data, and compared with a previously validated slice-by-slice 2-D segmentation approach based on graph search techniques [14]. Our LNS program was tested on an AMD Athlon MP 2000+ Dual CPU workstation running MS Windows XP.

The experiments showed that our LNS algorithm and software are computationally efficient and produce highly accurate and consistent segmentation results. The average execution time of our simultaneous 3-surface detection algorithm on images of size $200 \times 200 \times 40$ is 401.3 seconds. An accuracy assessment on images of physical phantom tubes revealed that the overall signed errors for the inner and outer diameters derived from the tube boundaries were (mean $\pm$

standard deviation) $-0.36 \pm 2.47\%$ and $-0.08 \pm 1.35\%$, respectively. Our LNS approach was tested on segmenting both the inner and outer airway wall surfaces in CT images, in which outer wall surfaces are very difficult to detect due to their blurred and discontinuous appearance and the presence of adjacent blood vessels. The CT images had a nearly isotropic resolution of $0.7 \times 0.7 \times 0.6mm^3$. The currently used 2-D dynamic programming method is unsuitable for the segmentation of the outer airway wall. Our new approach produces good segmentation results for both airway wall surfaces in a robust manner. Comparing to manual tracing on 39 randomly selected slices, our LNS technique yielded signed border positioning errors of -0.01 ± 0.15mm and 0.01 ± 0.17mm for the inner and outer wall surfaces, respectively.

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