Embedding Point Sets into Plane Graphs of Small Dilation

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Abstract. Let S be a set of points in the plane. What is the minimum possible dilation of all plane graphs that contain S? Even for a set S as simple as five points evenly placed on the circle, this question seems hard to answer; it is not even clear if there exists a lower bound > 1. In this paper we provide the first upper and lower bounds for the embedding problem.

1. Each finite point set can be embedded into the vertex set of a finite triangulation of dilation ≤ 1.1247.
2. Each embedding of a closed convex curve has dilation ≥ 1.00157.
3. Let P be the plane graph that results from intersecting n infinite families of equidistant, parallel lines in general position. Then the vertex set of P has dilation $\geq 2/\sqrt{3} \approx 1.1547$.

Keywords: Dilation, geometric network, lower bound, plane graph, spanning ratio, stretch factor.

1 Introduction

Transportation networks like railway systems can be modeled by geometric graphs: stations correspond to vertices, and the tracks between stations are represented by arcs. One measure of the performance of such a network P is given by its *vertex-to-vertex dilation.* For any two vertices, p and q, let $\pi(p,q)$ be a shortest path from p to q in P. Then,

$$\delta_P(p,q) := \frac{|\pi(p,q)|}{|pq|}$$

measures the detour one encounters in using P, in order to get from p to q, instead of traveling straight; here $|.|$ denotes the Euclidean length. The dilation of P is given by

$$\delta(P) := \sup_{p,q \text{ vertices of } P} \delta_P(p,q).$$

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1.1 Problem Statement

Suppose we are given a set of stations, and we want to build a network connecting them whose dilation is as low as possible. In this work we are assuming that bridges cannot be used. That is, where two or more tracks cross each other, a station is required, that must also be considered in evaluating the dilation of the network.

More precisely, we are given a set S of points in the plane, and we are interested in plane graphs $P = (V, E)$ whose vertex set V contains S, such that the dilation $\delta(P)$ is as small as possible. At this point we are not concerned with the algorithmic cost of computing P, nor with its building cost in terms of the total length of all edges in E, or the size of V—only the dilation of P matters. However, to rule out degenerate solutions like the complete graph over all points in the real plane, we require that the vertex set V of P contains only a finite number of vertices in addition to S. This leads to the following definition.

Definition 1. *Let S be a set of points in the plane. Then the* dilation of S *is given by*

$$\Delta(S) = \inf \{ \ \delta(P); P = (V, E) \ \textit{plane graph} \ \& \ S \subseteq V \ \& \ V \setminus S \ \textit{finite} \ \} .$$

The challenge is in computing the dilation of a given point set. Even for a set as simple as S_5, the vertices of a regular 5-gon, $\Delta(S_5)$ is not known. It is not even clear if $\Delta(S_5) > 1$ holds.[1] Figure 1 depicts some attempts to find good embeddings for S_5.

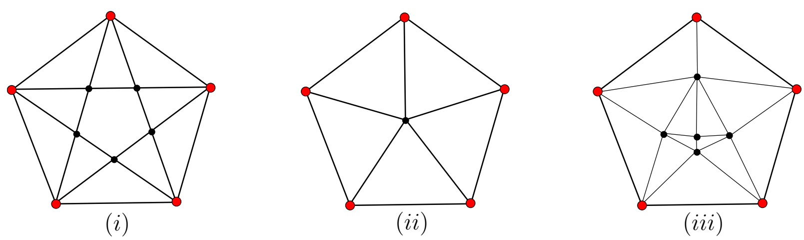

Fig. 1. Embedding five points in regular position into the vertex set of a plane graph of low dilation. (i) Constructing the complete graph results in a new 5-gon. (ii) A star yields dilation ≈ 1.05146. (iii) Using a 4-gon around an off-center point gives dilation ≈ 1.02046, as was shown by Lorenz [17].

[1] While S_5 is not contained in the vertex set of any plane graph of dilation 1, according to the characterization of dilation-free graphs given by D. Eppstein [12], there could be a sequence of plane graphs, each containing S_5, whose dilation shrinks towards 1.

1.2 Related Work

In the context of spanners, the dilation is often called the stretch factor or the spanning ratio of a graph P; see Eppstein's handbook chapter [11], Arikati et al. [3], or the forthcoming monograph [18] by Narasimhan and Smid. However, spanners are usually allowed to contain edge crossings, unlike the plane graphs considered here.

Substantial work has been done on proving upper bounds to the dilation of certain plane graphs. For example, Dobkin et al. [5] and Keil and Gutwin [16] have shown that the Delaunay triangulation of a finite point set has a dilation bounded from above by a small constant. The best upper bound known is 2.42, but a better bound of $\pi/2$ is conjectured to hold. Moreover, there are structural properties of plane graphs, like the good polygon and diamond properties, which imply that the dilation is bounded from above, see Das and Joseph [4]. This result implies that the minimum weight and the greedy triangulations also have a dilation bounded by a constant. Our approach differs from this work, in that the use of extra vertices is allowed. This will lead to an upper bound considerably smaller than $\pi/2$.

Quite recently, a related measure called geometric dilation has been introduced, see [10,8,1,6,9,7], where *all* points of the graph, vertices and interior edge points alike, are considered. The small difference in definition leads to rather different results which do not apply here. For example, plane graphs of minimum geometric dilation tend to have curved edges, whereas for the vertex-to-vertex dilation, straight edges work best.

1.3 New Results

In this paper we provide the first lower and upper bounds to the dilation of point sets, as defined in Definition 1. First, in Section 2, we prove a structural property similar in spirit to the good polygon and diamond properties mentioned above. If a plane graph P contains a face R whose diameter is a weak local maximum, so that each face in a certain neighborhood of R has a diameter at most a few percents larger than R, then the dilation of P can be bounded away from 1. We will derive the following consequence. If C denotes a closed convex curve then $\Delta(C) > 1.00157$ holds for the set of points on C, i.e., each point on curve C is considered a degree 2 vertex; see Figure 2 (i). Another consequence: If P is a plane graph whose faces have diameters bounded from above by some constant then $\delta(P) > 1.00156$ holds.

When looking for plane graphs of low dilation that can accommodate a set of given points, grids come to mind. Even the simple quadratic grid consisting of equidistant vertical and horizontal lines has a vertex-to-vertex dilation of only $\sqrt{2} \approx 1.414$, and we can force its vertex set to contain any finite number of points with rational coordinates, by choosing the cell size appropriately. How to accommodate points with real coordinates is discussed in the full paper. If we use three families of lines, as in the tiling of the plane by equilateral triangles, a smaller dilation of only $2/\sqrt{3} \approx 1.1547$ can be achieved; see Figure 3.

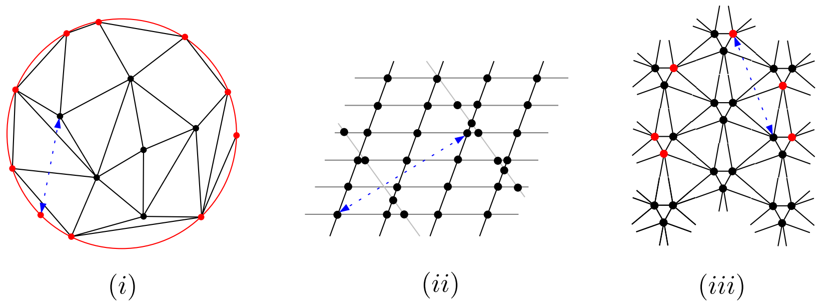

Fig. 2. Results. (i) Each embedding of a closed convex curve has dilation > 1.00157. (ii) Each such arrangement has dilation > 1.1547. (iii) Each finite point set has dilation < 1.1247.

An interesting question is if the dilation can be decreased even further by using lines of more than three different slopes. The answer is somewhat surprising, because parallel highways, a mile apart, for each orientation $2\pi i/n$, would in fact provide very low dilation to long distance traffic. But there are always vertices relatively close to each other, for which the dilation is at least $2/\sqrt{3}$ as we shall prove in Section 3.

Yet in Section 4 we introduce a way of getting below the $2/\sqrt{3}$ bound offered by the equitriangular tiling depicted in Figure 3. One can modify this tiling by replacing each vertex with a triangle, and by connecting neighboring triangles as shown in Figure 2 (iii). The resulting graph has a dilation less than 1.1247. We can scale, and slightly deform this graph, so that its vertex set contains any given finite set of points; then we cut off the unbounded part which does not host any point. These operations increase the dilation by some factor that can be made arbitrarily small. Thus we obtain that $\Delta(S) < 1.1247$ holds, for every finite point set S.

Finally, in Section 5, we address some of the questions left open and discuss future work.

2 A Lower Bound

First, we introduce some notations. Let P be a plane graph, and let R be a face of P with boundary $\partial(R)$. As usual, let

$$\mathrm{diam}(R) = \sup\{\ |ef|\ ;\ e, f \text{ vertices of } R\}$$

be the *diameter* of R; unbounded faces have infinite diameter. Now let R be a bounded face of P. For any positive number r, the *r-neighborhood of face R* is defined as the set of all faces of P that have non-empty intersection with a disk of radius r centered at the midpoint of a segment ef, where e, f are vertices on $\partial(R)$ satisfying $|ef| = \mathrm{diam}(R)$; see Figure 4. If there are more than one pair e, f of vertices of this kind we break ties arbitrarily. One should observe that the r-neighborhood of a bounded face may include unbounded faces of P.

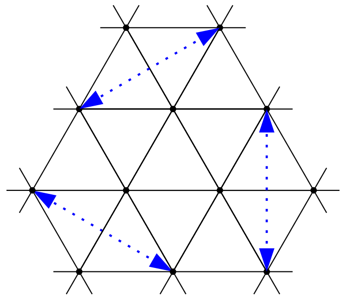

Fig. 3. The tiling by equilateral triangles is of dilation $2/\sqrt{3} \approx 1.1547$

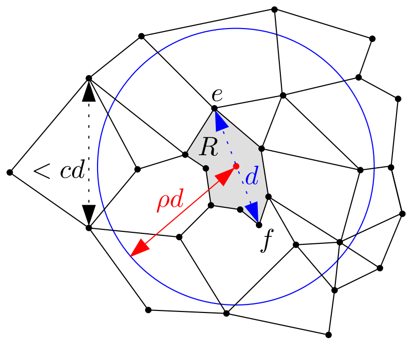

Fig. 4. No face intersected by the disk is of diameter $> cd$

The results of this section are based on the following lemma.

Lemma 1. *For each parameter $c \in [1, 1.5)$ there exist numbers $\rho > 1$ and $\delta > 1$ such that the following holds. Suppose that the plane graph P contains a bounded face R of diameter d, such that all faces in the ρd-neighborhood of R have diameter less than cd.*[2] *Then the dilation of P is at least δ.*

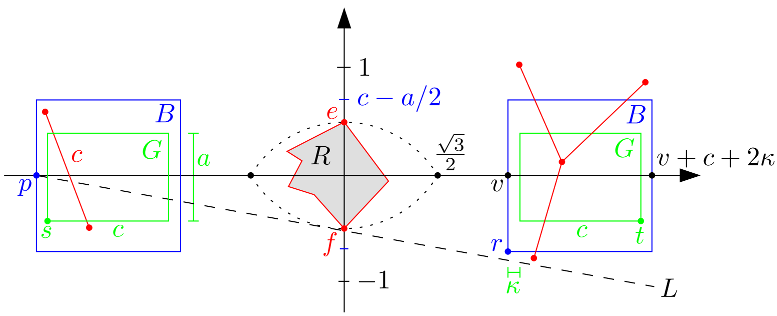

Fig. 5. Constructing a lower bound

Proof. We may assume that face R is of diameter $d = 1$. Also, we assume that ef is vertical and that its midpoint equals the origin; see Figure 5. As no vertex of $\partial(R)$ has a distance > 1 from e or from f, face R is completely contained in the lune spanned by e and f.[3]

Now we place two axis-parallel boxes of width c and height a symmetrically on the X-axis, at a distance of $v + \kappa$ to either side of the origin; these boxes are denoted by G in Figure 5. The parameters a, v, and κ will be chosen later.

[2] This implies that only bounded faces can be included in the ρd-neighborhood of R.

[3] The lune spanned by e and f equals the intersection of the two circles of radius $|ef|$ centered at e and f, respectively.

Suppose that all faces of P that intersect the disk of radius $\rho := v + c + \kappa$ about the origin have a diameter less than c.

Now, let us consider such a box, G. As its width equals c, there cannot exist two points on the left and on the right vertical side of G, respectively, that are contained in the same face R' of the graph, because this would imply $\mathrm{diam}(R') \geq c$. Thus, the vertical sides of G must be separated by P, i. e., there must be a sequence of edges, or a single edge, cutting through the upper and lower horizontal sides of G. In the first case, box G must contain a vertex of P, as shown on the right hand side in the figure. In the second case, the edge that crosses G top-down must itself be of length $< c$, because it belongs to a face intersected by the disk of radius ρ. We enclose G in the smallest axis-parallel box B which contains all line segments of length c that cross both horizontal sides of G. The outer box B is of height $2c - a$, its width exceeds the width c of G by $\kappa = \kappa(a, c)$ on either side, to include all slanted segments. The analysis of $\kappa(a, c)$ can be found in the full paper. By construction, both the upper and the lower half of B must contain a vertex of P in the second case, as shown on the left hand side of Figure 5.

Now we discuss how to choose the parameters v and a such as to guarantee a dilation of $\delta > 1$ in either possible case. First, we let $v > \sqrt{3}/2$ so that the boxes B are disjoint from the lune; consequently, every shortest path in P connecting vertices in the two boxes B has to go around the face R.

Case 1. Each of the boxes G contains a vertex of P. If a is less than $|ef| = 1$ then these vertices cause a dilation of at leas t a certain value $\delta > 1$ which depends only on a, c, v and κ.

Case 2. Each of the boxes B contains two vertices of P, one above, and one below the X-axis. Let p and r denote vertices in the upper part of the left and in the lower part of the right box B, respectively. We assume w. l. o. g. that the shortest path in P connecting them runs below vertex f, so that its length is at least $|pf| + |fr|$. If we make sure that, even at the extreme position depicted in Figure 5, vertex r lies above the line L through p and f, a dilation $\delta > 1$ is guaranteed. The equation of L is given by

$$Y = -\frac{1}{2(v + c + 2\kappa)}\, X - \frac{1}{2}.$$

Thus, we must ensure that

$$-(c - \frac{a}{2}) > -\frac{1}{2(v + c + 2\kappa)}\, v - \frac{1}{2}$$

or, equivalently,

$$a > 2c - 1 - \frac{v}{v + c + 2\kappa}$$

holds. Together with the condition $1 > a$ from Case 1 we obtain

$$3 - 2c > a + 2 - 2c > \frac{c + 2\kappa}{v + c + 2\kappa} > 0. \qquad (*)$$

Case 3. There exist at least one vertex of P in the left box G, and at least two vertices in the right box B, above and below the X-axis. Since the vertex in the box G must reside in either the upper or the lower part of the enclosing box B, Case 2 applies.

Clearly, the above conditions can be fulfilled for each given $c \in [1, 1.5)$. First, we pick $a \in (2c-2, 1)$, which guarantees $3-2c > a+2-2c > 0$. Then we choose $v > \sqrt{3}/2$ so large that the second inequality in condition (*) is satisfied. This proves Lemma 1.

It is quite straightforward to derive quantitative results from the above construction, by adjusting the values of the parameters a and v. The following numerical values for ρ and δ have been obtained using Maple.

c	1.0	1.001	1.1	1.2	1.3	1.4
δ	1.00157	1.00156	1.00043	1.000092	1.000012	1.00000056
ρ	1.923	1.925	2.46	3.9	6.9	16.5

As a first consequence, we get the following result.

Theorem 1. *Let P be a infinite graph whose faces cover the whole plane and have a diameter bounded from above by some constant. Then $\delta(P) > 1.00156$ holds for its dilation.*

Proof. By assumption, $d^* := \sup\{\text{diam}(R) : R \text{ face of } P\}$ is finite. For each $\epsilon > 0$ there exists a face R of P such that $\text{diam}(R) > (1 - \epsilon)d^*$ holds. By the assumption on graph P, *all* faces R' of P, in particular those in any neighborhood of R, satisfy

$$\text{diam}(R') \leq d^* < \frac{1}{1 - \epsilon}\,\text{diam}(R) \leq 1.001\,\text{diam}(R),$$

if ϵ is small enough. Thus, graph P has dilation at least 1.00156.

The second consequence of Lemma 1 is a lower bound to the Δ function.

Theorem 2. *Let C denote the set of points on a closed convex curve. Then $\Delta(C) > 1.00157$ holds for its dilation.*

Proof. (Sketch) If curve C intersects or encircles a box G, we can argue as before. If C passes between G and ef it becomes even easier to provide two points of high dilation.

For the circle one can find an embedding of dilation $(1 + \epsilon)/\sin 1 \approx 1.188$ by placing a single vertex at the center and adding many equidistant radial segments.

3 A Lower Bound for Line Arrangements

A lower bound much stronger than 1.00157 can be shown for graphs that result from intersecting n families F_i of infinitely many equidistant parallel lines. Each family is defined by three parameters, its orientation α_i, the distance w_i in X-direction between consecutive lines, and the offset distance e_i from the origin to the first line in positive X-direction. We say that such families are in *general position* if the numbers w_i^{-1} are linearly independent over the rationals[4].

Theorem 3. *Given n families F_i, $2 \leq i \leq n$, each consisting of infinitely many equidistant parallel lines. Suppose that these families are in general position. Then their intersection graph P is of dilation at least $2/\sqrt{3}$.*

One should observe that this lower bound is attained by the equitriangular grid shown in Figure 3.

Proof. For $n \leq 3$ the claim can be proven quite easily without assuming general position, as shown in the full paper.

Now let $n > 3$. We shall prove the existence of a face R of P that represents so large a barrier between two vertices p and p' of P that even the Euclidean shortest path from p to p' around R is of dilation at least $2/\sqrt{3}$. In fact, we shall provide such a face and two vertices that are *symmetric* about the same center point, which greatly helps with our analysis; see Figure 6. To this end we use the general position assumption, and apply Kronecker's theorem [2] on simultaneous approximation in its following form.

Theorem 4. *(Kronecker) Let L be a line in $\mathbf{R}^n$ that passes through the origin and through some point $(y_1, \ldots, y_n)$ whose coordinates are linearly independent over the rationals. For each point $t \in \mathbf{R}^n$, and for each $\epsilon > 0$, there is an integer translate $t + m$, $m \in \mathbf{Z}^n$, of t whose $\epsilon-$neighborhood is visited by L.*

In other words, line L is dense on the torus $\mathbf{R}^n/\mathbf{Z}^n$. Proofs can be found in, e. g., Apostol [2] or Hlawka [15].

Since the families of lines are in general position, the real numbers $y_i := w_i^{-1}, 1 \leq i \leq n$, do not satisfy a linear equation with rational coefficients. Kronecker's theorem, applied to $t_i := e_i/w_i + 1/2$ and $\epsilon > 0$ yields the existence of integers m_i and of a real number x satisfying

$$|\frac{e_i}{w_i} + \frac{1}{2} + m_i - x\ \frac{1}{w_i}| < \epsilon$$

that is,

$$|e_i + \frac{1}{2}\ w_i + m_i\ w_i - x| < \epsilon\ w_i.$$

Consequently, the point $(x, 0)$ lies, for each family F_i, halfway between two neighboring lines, so that it is center of symmetry for some face R in P—up to an error that can be made arbitrarily small since the numbers w_i are fixed.

[4] This means, if $\sum_{i=1}^{n} a_i\ w_i^{-1} = 0$ holds for rational coefficients a_i then each a_i must be zero.

From now on, we assume that R is symmetric about the origin, and that its longest diagonal, d, is vertical and of length 2. Let us assume that the dilation of graph P is less than $2/\sqrt{3}$. We shall derive a contradiction by proving that there exist two families of lines that contribute to the boundary of R and have symmetric intersection vertices p, p' that cause a dilation $> 2/\sqrt{3}$ in the presence of the barrier R. Since R is symmetric, it is sufficient to provide *one* such vertex p satisfying

$$\frac{|pa| + |pb|}{2|p|} \geq 2/\sqrt{3}$$

where a and b are the endpoints of diagonal d, and $|p| = |p0|$ denotes the distance from p to the origin, as shown in Figure 6.

To this end, consider the locus E of all points where equality holds in the above inequality, see Figure 7. E satisfies the quartic equation $(X^2 + Y^2 - \frac{3}{2})^2 = \frac{9}{4}(1 - Y^2)$. We want to show that the right part of its interior—referred to as an *ear*, due to its shape—contains a vertex p of P.

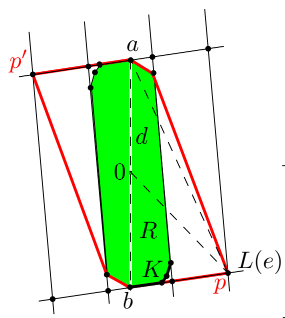

Fig. 6. A symmetric face acting as barrier

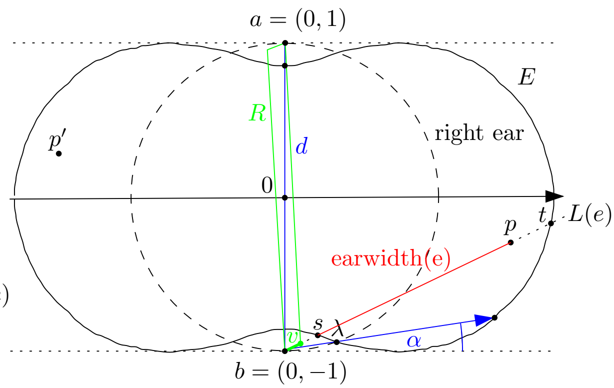

Fig. 7. The locus E of all points that cause dilation $2/\sqrt{3}$ in the presence of line segment ab. The halfline $L(e)$ extends the edge $e = bv$ to the right.

Since d is the longest diagonal of the symmetric face R, the vertices of R are contained in the circle spanned by d. On the other hand, R itself is of dilation $< 2/\sqrt{3}$, by assumption. Hence, the vertices of R must be outside of the locus curve. This leaves only the small caps at a, b for the remaining vertices of R, and $\partial(R)$ must contain two long edges.

First, assume that R is a parallelogram, as shown in Figure 7. Its shape is determined by the position of its lower right vertex v in the bottom cap. Consider the edge e from vertex b to v. Its extension beyond v, $L(e)$, intersects the right ear in a segment of length earwidth(e). In Figure 7 we have earwidth$(e) = |st|$, while the length of the intersection of $L(e)$ with the lower cap equals $|bs|$. The ratio

$|st|/|bs|$ takes on its minimum value $3.11566\ldots > 1$ at the angle $\alpha \approx 9.74°$, when $L(e)$ hits the intersection point, λ, of the locus curve with the circle. Because of

$$\text{earwidth}(e) \;>\; 3.11|bs| \;\geq\; 3.11|bv| \;>\; |bv|$$

the segment of $L(e)$ passing through the ear must contain a vertex p of the two line families bounding R. This proves Theorem 3 in case face R is a parallelogram.

It is interesting to observe that in the limiting case $\alpha = 0$, when R degenerates into its diagonal d, the largest possible dilation $2/\sqrt{3}$ is only attainable by picking p as the bottommost point of the ear. This shows why these arguments would not work for any lower bound larger than $2/\sqrt{3}$.

Now let R be a general symmetric convex polygon. We may assume that its two long edges have non-positive slope. Let K denote the convex boundary chain of R that starts at vertex b and leads to the right until it hits the rightmost long edge of R. In this situation the following holds.

Lemma 2. *There exists an edge e in chain K such that the line $L(e)$ passing through e has the following property. The intersection of $L(e)$ with the the right ear of the locus curve is at least twice as long as the vertical projection of chain K onto $L(e)$.*

The proof of Lemma 2 requires some technical effort; we skip it due to space limitations. Consider Figure 8. Let $\text{cut}(e)$ denote the length of the segment of $L(e)$ that is cut out by the extensions of the long edges of face R, and let $\text{proj}_K(e)$ be the length of the vertical projection of chain K onto $L(e)$. By translating $L(e)$ to the endpoint of K, we can see $\text{cut}(e) \leq 2\ \text{proj}_K(e)$, so Lemma 2 implies

$$\text{cut}(e) \;\leq\; 2\ \text{proj}_K(e) \;\leq\; \text{earwidth}(e).$$

This guarantees the existence of a vertex of P in the interior of the locus curve and completes the proof of Theorem 3 in the general case.

4 An Upper Bound to the Dilation of Finite Point Sets

First, we show how to modify the equitriangular grid, H, displayed in Figure 3, in order to decrease its dilation. The construction is shown in Figure 9.

We replace each vertex v of H with an equilateral triangle T that has one vertex on each of the three lines passing through v in H. The distance, a, between the vertices and the center v of T is a parameter of our construction. Next, we connect by an edge each vertex of T to the two visible vertices of its neighboring triangle T'. Afterwards, all vertices and edges of the old graph H are removed. Let $H_A = H_A(a)$ denote the resulting graph.

Theorem 5. *Within the family $H_A(a)$, the minimum dilation of* $1.1246\ldots$ *is attained for* $a \approx 0.2486$.

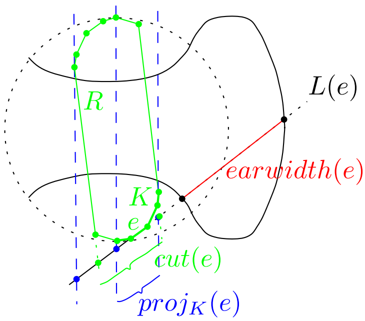

Fig. 8. Vertically projecting the convex chain K onto the line $L(e)$. The intersection of $L(e)$ with the right ear of the locus curve is of length earwidth(e).

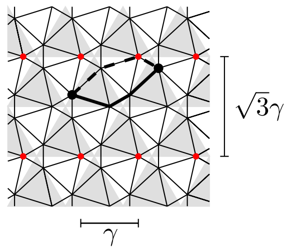

Fig. 9. The new graph H_A of dilation ≈ 1.1246

Proving this result requires considerable technical effort. For example, it is in general not true that the dilation of an unbounded graph is attained by vertices that are close to each other, as a rectangular grid of irrational aspect ratio shows. After introducing global and local coordinates for the vertices of H_A, one has to distinguish 45 shortest path types. Closer inspection leads to four functions of the integer coordinates i, j, whose maximum must be minimized by a suitable choice of parameter a.

A vertex pair causing maximum dilation is shown in Figure 9, together with two shortest paths connecting the two vertices. We obtain the following consequence of Theorem 5.

Theorem 6. *Each finite point set S is of dilation $\Delta(S) < 1.1247$.*

The proof uses a technique introduced in [8] which is also based on the approximation of reals by rationals. It allows us to scale H_A, and distort it carefully, without affecting the dilation by more than a factor arbitrary close to 1, so that the points of S can be accommodated in a finite part of the graph that contains, for any two vertices, the shortest path connecting them in the original graph H_A.

5 Conclusion

We have introduced the notion of the dilation of a set of points, and proven a non-trivial lower bound to the dilation of the points on a closed curve. The big challenge is in proving a similar lower bound for a finite set of points like, e.g., S_5. As to the arrangements of lines, we conjecture that our lower bound holds without the assumption of general position. Another interesting question is the following. What is the lowest possible dilation of a graph whose faces cover the whole plane and have bounded diameter? Our results place this value into the interval $(1.00157, 1.1247)$. Any progress on the upper bound might lead to an improvement of Theorem 6.

References

1. P. Agarwal, R. Klein, Ch. Knauer, S. Langerman, P. Morin, M. Sharir, and M. Soss. Computing the Detour and Spanning Ratio of Paths, Trees and Cycles in 2D and 3D. Manuscript, submitted for publication, 2005.
2. T. M. Apostol. Dirichlet Series in Number Theory, 2nd ed. Springer-Verlag, 1997. Pp. 148–155.
3. S. R. Arikati, D. Z. Chen, L. P. Chew, G. Das, and M. Smid. Planar Spanners and Approximate Shortest Path Queries Among Obstacles in the Plane. European Symposium on Algorithms, 514–528, 1996.
4. G. Das and D. Joseph. Which Triangulations Approximate the Complete Graph? Proc. Int. Symp. Optimal Algorithms, Springer LNCS 401, 168–192, 1989.
5. D.P. Dobkin, S.J. Friedman, and K.J. Supowit. Delaunay Graphs Are Almost as Good as Complete Graphs. *Discrete & Computational Geometry* 5:399-407 (1990).
6. A. Dumitrescu, A. Grüne, and G. Rote. On the Geometric Dilation of Curves and Point Sets. Manuscript, 2004.
7. A. Dumitrescu, A. Ebbers-Baumann, A. Grüne, R. Klein, and G. Rote. On Geometric Dilation and Halving Chords. WADS'05
8. A. Ebbers-Baumann, A. Grüne, and R. Klein. On the Geometric Dilation of Finite Point Sets. 14th International Symposium ISAAC 2003, Kyoto. In T. Ibaraki, N. Katoh, and H. Ono (Eds.) Algorithms and Computation, Proceedings, pp. 250–259, LNCS 2906, Springer-Verlag, 2003. To appear in *Algorithmica*.
9. A. Ebbers-Baumann, A. Grüne, and R. Klein. Geometric Dilation of Closed Planar Curves: New Lower Bounds. To appear in *Computational Geometry: Theory and Applications*.
10. A. Ebbers-Baumann, R. Klein, E. Langetepe, and A. Lingas. A Fast Algorithm for Approximating the Detour of a Polygonal Chain. *Computational Geometry: Theory and Applications*, 27:123–134, 2004.
11. D. Eppstein. Spanning trees and spanners. In Handbook of Computational Geometry, J.-R. Sack and J. Urrutia, editors, pp. 425-461. Elsevier, 1999.
12. D. Eppstein. The Geometry Junkyard. http://www.ics.uci.edu/∼eppstein/junkyard/dilation-free/.
13. D. Eppstein and D. Hart. Shortest Paths in an Arrangement with k Line Orientations. Proceedings 10th Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 310-316, 1999.
14. S. Fekete, R. Klein, and A. Nüchter. Online Searching with an Autonomous Robot. Workshop on Algorithmic Foundations of Robotics, 2004.
15. E. Hlawka. Theorie der Gleichverteilung. BI Wissenschaftsverlag.
16. J.M. Keil and C.A. Gutwin. The Delaunay Triangulation Closely Approximates the Complete Euclidean Graph. *Discrete Comput. Geom.* 7, 1992, pp. 13-28.
17. D. Lorenz. On the Dilation of Finite Point Sets. Diploma Thesis. Bonn, 2005.
18. G. Narasimhan and M. Smid. Geometric Spanner Networks. Cambridge University Press, to appear.