A Min-Max Relation on Packing Feedback Vertex Sets*

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Abstract. Let G be a graph with a nonnegative integral function w defined on $V(G)$. A family $\mathcal{F}$ of subsets of $V(G)$ (repetition is allowed) is called a *feedback vertex set packing* in G if the removal of any member of $\mathcal{F}$ from G leaves a forest, and every vertex $v \in V(G)$ is contained in at most $w(v)$ members of $\mathcal{F}$. The *weight* of a cycle C in G is the sum of $w(v)$, over all vertices v of C. In this paper we characterize all graphs with the property that, for any nonnegative integral function w, the maximum cardinality of a feedback vertex set packing is equal to the minimum weight of a cycle.

1 Introduction

We begin with a brief introduction to the theory of packing and covering. More details on this subject can be found in [6]. A *hypergraph* H is an ordered pair $(V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a set of subsets of V. Members of V and $\mathcal{E}$ are called *vertices* and *edges* of H, respectively. An edge is *minimal* if none of its proper subsets is an edge. A *clutter* is a hypergraph whose edges are all minimal. The *blocker* of hypergraph $H = (V, \mathcal{E})$ is the clutter $b(H) = (V, \mathcal{E}')$, where $\mathcal{E}'$ is the set of all minimal subsets $B \subseteq V$ such that $B \cap A \neq \emptyset$ for all $A \in \mathcal{E}$. We also define $b(H)^{\uparrow} = (V, \mathcal{E}'')$, where $\mathcal{E}''$ consists of all $B \subseteq V$ such that $B \cap A \neq \emptyset$ for all $A \in \mathcal{E}$. It is well known that $b(b(\mathcal{C})) = \mathcal{C} = b(b(\mathcal{C})^{\uparrow})$ holds for every clutter $\mathcal{C}$.

Let I be a set and let α be a function with domain I. Then, for any finite subset S of I, we denote by $\alpha(S)$ the sum of $\alpha(s)$, over all $s \in S$. Let $\boldsymbol{R}_+$ (resp. $\boldsymbol{Z}_+$) denote the sets of nonnegative real numbers (resp. integers). Let M be the $\mathcal{E}$-V incidence matrix of a hypergraph $H = (V, \mathcal{E})$. For any $w \in \boldsymbol{Z}_+^V$,

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let $\nu_w^*(H) = \max\{\boldsymbol{x}^T\mathbf{1} : \boldsymbol{x} \in \boldsymbol{R}_+^{\mathcal{E}}, \boldsymbol{x}^T M \le w^T\}$, $\tau_w^*(H) = \min\{w^T\boldsymbol{y} : \boldsymbol{y} \in \boldsymbol{R}_+^V, M\boldsymbol{y} \ge \mathbf{1}\}$, $\nu_w(H) = \max\{\boldsymbol{x}^T\mathbf{1} : \boldsymbol{x} \in \boldsymbol{Z}_+^{\mathcal{E}}, \boldsymbol{x}^T M \le w^T\}$, $\tau_w(H) = \min\{w^T\boldsymbol{y} : \boldsymbol{y} \in \boldsymbol{Z}_+^V, M\boldsymbol{y} \ge \mathbf{1}\}$. Combinatorially, each vector $\boldsymbol{x} \in \boldsymbol{Z}_+^{\mathcal{E}}$ with $\boldsymbol{x}^T M \le w^T$ can be interpreted as a family $\mathcal{F}$ of edges (repetition is allowed) of H, for which each vertex $v \in V$ belongs at most $w(v)$ members of $\mathcal{F}$. Such a family is called a *w-packing* of H. It is clear that $\nu_w(H)$ is the maximum size of a w-packing of H. Similarly, $\tau_w(H)$ is the minimum of $w(B)$, over all edges B of $b(H)^\uparrow$. Notice from the LP Duality Theorem that

$$\nu_w(H) \le \nu_w^*(H) = \tau_w^*(H) \le \tau_w(H). \tag{1}$$

One of the fundamental problems in combinatorial optimization is to identify scenarios under which either one or two of the above inequalities holds with equality. In particular, H is *ideal* if $\tau_w^*(H) = \tau_w(H)$, for all $w \in \boldsymbol{Z}_+^V$, while H is *Mengerian* if $\nu_w^*(H) = \nu_w(H)$, for all $w \in \boldsymbol{Z}_+^V$. Obviously $b(H)$ is Mengerian iff so is $b(H)^\uparrow$. It is well known that being Mengerian is actually equivalent to $\nu_w(H) = \tau_w(H)$ for all $w \in \boldsymbol{Z}_+^V$ [3]. Thus every Mengerian hypergraph is ideal.

Our present work is a continuation of [1,2]. To clarify our motivation, we summarize the main results in [1]. For any simple graph $G = (V, E)$, let $\mathcal{C}_G = (V, \mathcal{E})$ denote the clutter in which $\mathcal{E}$ consists of $V(C)$, for all induced cycles C of G. A *Θ-graph* is a subdivision of $K_{2,3}$. A *wheel* is obtained from a cycle by adding a new vertex and making it adjacent to all vertices of the cycle. A *W-graph* is a subdivision of a wheel. An *odd ring* is a graph obtained from an odd cycle by replacing each edge $e = uv$ with *either* a cycle containing e or two triangles $uabu, vcdv$ together with two additional edges ac and bd. A subdivision of an odd ring is called an *R-graph*. Let $\mathcal{L}$ be the class of simple graphs G such that no induced subgraph of G is isomorphic to a Θ-graph, a W-graph, or an R-graph.

Theorem 1. [1] *The following are equivalent for every simple graph G:* (i) *$\mathcal{C}_G$ is Mengerian;* (ii) *$\mathcal{C}_G$ is ideal;* (iii) *$G \in \mathcal{L}$.*

Fulkerson [4] proved that a hypergraph is ideal iff its blocker is ideal. Therefore, the equivalence of (ii) and (iii) in Theorem 1 implies the following

Corollary 1. *$b(\mathcal{C}_G)^\uparrow$ is ideal if and only if $G \in \mathcal{L}$.*

At this point, Guenin [5] suggested a natural question: When is $b(\mathcal{C}_G)^\uparrow$ Mengerian? In general, the blocker of a Mengerian hypergraph does not have to be Mengerian (see [6]). However, the following theorem, our main result in this paper, says that $\mathcal{C}_G$, $b(\mathcal{C}_G)^\uparrow$, and hence $b(\mathcal{C}_G)$ are always Mengerian together.

Theorem 2. *$b(\mathcal{C}_G)^\uparrow$ is Mengerian if and only if $\mathcal{C}_G$ is.*

Let $G = (V, E)$ be a simple graph and let $w \in \boldsymbol{Z}_+^V$. A subset of V is called an *feedback vertex set* (FVS) in G if it meets every cycle in G. Since the edge set of $b(\mathcal{C}_G)^\uparrow$ is exactly the set of feedback vertex sets (FVSs) in G, we also call a w-packing of $b(\mathcal{C}_G)^\uparrow$ a *w-packing of FVSs in G*, or simply an *FVS packing.* The min-max relation in our main result can be restated as follows: if $G = (V, E)$ is a simple graph, then the maximum cardinality of a w-packing of FVSs equals the minimum weight of a cycle in G for any $w \in \boldsymbol{Z}_+^V$ iff $G \in \mathcal{L}$.

The rest of the paper is devoted to the proof of Theorem 2. In section 2, we prove some results on how Mengerian hypergraphs can be put together to get a larger Mengerian hypergraph. Then, in Section 3, we explain results from [1], which describe how graphs in $\mathcal{L}$ can be constructed from some "prime" graphs by "summing" operations. Finally, we establish Theorem 2 in Section 4 by showing that all prime graphs have the required Mengerian property.

2 Sums of Hypergraphs

The purpose of this section is to prove a few lemmas, which claim that being Mengerian is preserved under some natural summing operations.

Let $H = (V, \mathcal{E})$ be a hypergraph and $w \in \boldsymbol{Z}_+^V$. It is easy to see that $\tau_w(b(H)^\uparrow) = \min_{A\in\mathcal{E}} w(A)$. Denoting $r_w(H) = \min_{A\in\mathcal{E}} w(A)$, we have

$$b(H)^\uparrow \text{ is Mengerian iff } b(H)^\uparrow \text{ has a } w\text{-packing of size } r_w(H), \forall\, w \in \boldsymbol{Z}_+^V. \quad (1)$$

Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. If $|V_1 \cap V_2| \in \{0, 1\}$, then $(V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2)$ is called the $|V_1 \cap V_2|$-*sum* of H_1 and H_2. If $V_1 \cap V_2 = \{x_1, x_2\}$ and for $i = 1, 2$, H_i has an edge $A_i = \{x_1, x_2, y_i\}$ that is the only edge containing y_i, then $((V_1 \cup V_2) - \{y_1, y_2\}, (\mathcal{E}_1 \cup \mathcal{E}_2) - \{A_1, A_2\})$ is called the 2-*sum* of H_1 and H_2. If $V_1 \cap V_2 = \{x_1, x_2, x_3\}$ and $A = \{x_1, x_2, x_3\}$ is an edge of H_1 and H_2, then $(V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2)$ is called the 3-*sum* of H_1 and H_2 over A. The notations given here will be used implicitly in the proofs of Lemma 1 and Lemma 2 below.

Lemma 1. *Let H be a k-sum ($k \in \{0, 1, 2\}$) of H_1 and H_2. If both $b(H_1)^\uparrow$ and $b(H_2)^\uparrow$ are Mengerian, then so is $b(H)^\uparrow$.*

Proof. The conclusion is obvious when $k \in \{0, 1\}$. We consider the case of $k = 2$. Suppose $H = (V, \mathcal{E})$. By (1), it suffices to show that $(*)$ $b(H)^\uparrow$ has a w-packing of size $r_w(H)$ for all $w \in \boldsymbol{Z}_+^V$. Suppose otherwise, $(*)$ were false for some $w \in \boldsymbol{Z}_+^V$ with $w(V)$ minimum. Let $r = r_w(H)$.

(1.1) $w(v) \le r$ for all $v \in V$.

Suppose (1.1) fails. Then $w' \in \boldsymbol{Z}_+^V$ with $w'(v) = \min\{r, w(v)\}$ for all $v \in V$ satisfies $r_{w'}(H) = r$ and $w'(V) < w(V)$, and therefore $b(H)^\uparrow$ has a w'-packing of size r, which is also a w-packing of $b(H)^\uparrow$, a contradiction. So (1.1) holds.

Let $i = 1, 2$, define $w_i \in \boldsymbol{Z}_+^V$ with $w_i(y_i) = \max\{0, r - w(x_1) - w(x_2)\}$ and $w_i(v) = w(v)$ for all $v \in V_i - \{y_i\}$. Then $r_{w_i}(H_i) \ge r$, and by (1), $b(H_i)^\uparrow$ has a w_i-packing $\mathcal{B}_i$ of size r. Choosing such $\mathcal{B}_i$ with maximum $\sum_{B\in\mathcal{B}_i} |B|$, we have

(1.2) For any $j \in \{1, 2\}$, x_j is contained in exactly $w(x_j)$ members of $\mathcal{B}_i$.

Suppose $B_1 \cap \{x_1, x_2\} = B_2 \cap \{x_1, x_2\}$ for some $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. Let χ_1, χ_2, and χ be the characteristic vectors of B_1, B_2, and $B = (B_1 \cup B_2) - \{y_1, y_2\}$, which are considered as subsets of V_1, V_2, and V, respectively. Define $w_1' =$

$w_1-\chi_1$, $w'_2 = w_2-\chi_2$, and $w' = w-\chi$. For $i = 1, 2$, since $b(H_i)^{\uparrow}$ has a w'_i-packing $\mathcal{B}_i - \{B_i\}$ of size $r-1$, it follows from (1) that $r_{w'_i}(H_i) = \tau_{w'_i}(b(H_i)^{\uparrow}) \geq r-1$. Therefore $r_{w'}(H) \geq r-1$. Since $w'(V) < w(V)$, $b(H)^{\uparrow}$ has a w'-packing $\mathcal{B}'$ of size $r-1$, which yields a w-packing $\mathcal{B}' \cup \{B\}$ of $b(H)^{\uparrow}$. This contradiction gives

(1.3) $B_1 \cap \{x_1, x_2\} \neq B_2 \cap \{x_1, x_2\}$, for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

It can be deduced from (1.2) and (1.3) that $w(x_1) + w(x_2) < r$. Recalling $w_i(A_i) = r$, we have $|B_i \cap A_i| = 1$, for all $B_i \in \mathcal{B}_i$, $i = 1, 2$, which, together with (1.2), implies a contradiction to (1.3). The lemma is proved. □

Lemma 2. *Let H is be a 3-sum of H_1 and H_2 over $A = \{x_1, x_2, x_3\}$. For $i = 1, 2$ and $1 \leq j < k \leq 3$, let H_{ijk} be obtained from H_i by adding a new vertex x_{ijk} and a new edge $A_{ijk} = \{x_{ijk}, x_j, x_k\}$. If all $b(H_{ijk})^{\uparrow}$ are Mengerian, then so is $b(H)^{\uparrow}$.*

Proof. Let $H = (V, \mathcal{E})$. As in the proof of Lemma 1, we shall prove: $(*)$ $b(H)^{\uparrow}$ has a w-packing of size $r_w(H)$ for all $w \in \boldsymbol{Z}_+^V$. Suppose that $(*)$ were false for a $w \in \boldsymbol{Z}_+^V$ with $w(V)$ minimum. Writing $r = r_w(H)$, we have

(2.1) $w(v) \leq r$ for all $v \in V$.

Let $1 \leq i \leq 2$, $1 \leq j < k \leq 3$, $V_{ijk} = V_i \cup \{x_{ijk}\}$, and define $w_{ijk} \in \boldsymbol{Z}_+^{V_{ijk}}$ with $w_i(x_{ijk}) = \max\{0, r - w(x_j) - w(x_k)\}$ and $w_{ijk}(v) = w(v)$ for all $v \in V_i$. Then $b(H_{ijk})^{\uparrow}$ has a w_{ijk}-packing $\mathcal{B}_{ijk}$ of size r. Choosing such $\mathcal{B}_{ijk}$ with $\sum_{B \in \mathcal{B}_i} |B|$ as *large* as possible, we have

(2.2) For any $1 \leq h \leq 3$, $1 \leq i \leq 2$, and $1 \leq j < k \leq 3$, x_h is contained in exactly $w(x_h)$ members of $\mathcal{B}_{ijk}$.
(2.3) $B \cap A \neq B' \cap A$, for all $B \in \mathcal{B}_{1jk}$ and $B' \in \mathcal{B}_{2j'k'}$ with $1 \leq j < k \leq 3$ and $1 \leq j' < k' \leq 3$.

(2.4) $w(x_j) + w(x_k) > r$ for all $1 \leq j < k \leq 3$.

Suppose otherwise. By symmetry, we assume $w(x_1) + w(x_2) \leq r$. Then, for $i = 1, 2$, $w_{i12}(A_{i12}) = r$, and hence no member of $\mathcal{B}_{i12}$ can contain $\{x_1, x_2\}$.

If $w(x_1) + w(x_3) > r$, then, by (2.2), some B_{i12} in $\mathcal{B}_{i12}$ $(i = 1, 2)$ contains both x_1 and x_3, which implies $B_{112} \cap A = \{x_1, x_3\} = B_{212} \cap A$ contradicting (2.3). Hence $w(x_1) + w(x_3) \leq r$, and by symmetry, $w(x_2) + w(x_3) \leq r$. Therefore $|B \cap \{x_j, x_k\}| \leq 1$ for all $B \in \mathcal{B}_{ijk}$.

Obviously $w(A) \geq r_w(H) = r$. Furthermore $w(A) > r$ as $w(A) = r$ implies a contradiction to (2.3). It follows from (2.2) that each $\mathcal{B}_{ijk}$ has an edge B_{ijk} with $|B_{ijk} \cap A| \geq 2$. Thus, by (2.3), for $1 \leq j < k \leq 3$, $\{B_{1jk} \cap A, B_{2jk} \cap A\} = \{\{x_j, x_\ell\}, \{x_k, x_\ell\}\}$ where $\ell \in \{1, 2, 3\} - \{j, k\}$. Without loss of generality, let $B_{112} \cap A = \{x_1, x_3\}$ and $B_{212} \cap A = \{x_2, x_3\}$. By (2.3), $B_{113} \cap A \neq \{x_2, x_3\}$. Thus $B_{113} = \{x_1, x_2\}$ and $B_{213} \cap A = \{x_2, x_3\}$. Now $B_{223} \cap A \in \{\{x_1, x_2\}, \{x_1, x_3\}\}$ violates (2.3), which proves (2.4).

(2.5) $|B \cap A| \leq 2$ for all $B \in \mathcal{B}_{ijk}$, where $1 \leq i \leq 2$ and $1 \leq j < k \leq 3$.

Suppose otherwise. Without loss of generality, we assume that some $\mathcal{B}_{1j_0k_0}$ has a member B_0 with $B_0 \supseteq A$. It follows from (2.3) that $|B \cap A| \leq 2$ for all

$B \in \mathcal{B}_{212} \cup \mathcal{B}_{213} \cap \mathcal{B}_{223}$. Let $1 \le j < k \le 3$. Since, by (2.4), $w(x_j) + w(x_k) > r$, it follows from (2.2) that $\mathcal{B}_{2jk}$ has a member B_{2jk} that contains both x_j and x_k, implying $B_{2jk} \cap A = \{x_j, x_k\}$. Therefore, by (2.3), $|B \cap A| \ne 2$ for all $B \in \mathcal{B}_{112} \cup \mathcal{B}_{113} \cup \mathcal{B}_{123}$.

Let $\{j, k, \ell\} = \{1, 2, 3\}$ with $j < k$, and let $\mathcal{B}'_{1jk}$ consist of members of $\mathcal{B}_{1jk}$ that contains x_ℓ. By (2.2), $|\mathcal{B}'_{1jk}| = w(x_\ell)$. As, by (2.4), $w_{1jk}(x_{1jk}) = 0$, we have $|B \cap \{x_j, x_k\}| \ge 1$ and hence $B \supseteq A$ for all $B \in \mathcal{B}'_{1jk}$. Consequently, $w(x_j) \ge w(x_\ell)$ and $w(x_k) \ge w(x_\ell)$. Since j, k, ℓ were chosen arbitrarily, it follows that $w(x_1) = w(x_2) = w(x_3)$, $\mathcal{B}'_{1jk} = \mathcal{B}_{1jk}$ and thus $w(x_1) = w(x_2) = w(x_3) = r$. On the other hand, since by (2.3), $|B \cap A| \le 2$ for all $B \in \mathcal{B}_{212}$, we deduce from (2.2) that $r = |\mathcal{B}_{212}| \ge w(A)/2 = 3r/2$, a contradiction, which proves (2.5).

Finally, let $i \in \{1, 2\}$. By (2.4), $w(x_1) + w(x_2) > r$, which, together with (2.2), implies that $\mathcal{B}_{i12}$ has a member B_{i12} that contains both x_1 and x_2. Now by (2.5), we must have $B_{i12} = \{x_1, x_2\}$, contradicting (2.3) and establishing the lemma. □

3 Graphical Structures

In this section, we summarize some results form [1] that describe how graphs in $\mathcal{L}$ can be constructed from "prime" graphs.

All graphs considered are undirected, finite, and simple, unless otherwise stated. Let $G = (V, E)$ be a graph. For any $U \subseteq V$ or $U \subseteq E$, let $G \backslash U$ be the graph obtained from G by deleting U, and let $G[U]$ be the subgraph of G induced by U; when U is a single to $\{u\}$, we simply write $G \backslash u$ instead of $G \backslash \{u\}$. A *rooted graph* consists of a graph G and a specified set R of edges such that each edge of R belongs to a triangle and each triangle in G contains at most one edge from R. By *adding pendent triangles* to the rooted graph G we mean the following operation: to each edge uv in R, we introduce a new vertex t_{uv} and two new edges ut_{uv} and vt_{uv}. The readers are referred to [1] for the definitions of sums of graphs.

Lemma 3. [1] *For any graph $G \in \mathcal{L}$, at least one of the following holds.*

(i) *G is a k-sum of two smaller graphs, for $k \in \{0, 1, 2, 3\}$;*
(ii) *G is obtained from a rooted 2-connected line graph by adding pendent triangles.*

Let G be a k-sum $(k = 0, 1, 2, 3)$ of graphs G_1 and G_2, then $H = \mathcal{C}_G$ is the k-sum of $H_1 = \mathcal{C}_{G_1}$ and $H_2 = \mathcal{C}_{G_2}$, and each hypergraph H_{ijk} defined in Lemma 2 is precisely $\mathcal{C}_{G_{ijk}}$, where G_{ijk} is the graph defined in the following lemma.

Lemma 4. [1] *Let $G \in \mathcal{L}$ be a k-sum of two smaller graphs. Then*

(i) *If $k \in \{0, 1, 2\}$, then G is a k-sum of two smaller graphs that belong to $\mathcal{L}$.*
(ii) *If G is a 3-sum of G_1 and G_2 over a triangle $x_1x_2x_3x_1$, then all G_{ijk} $(1 \le i \le 2,\ 1 \le j < k \le 3)$) are in $\mathcal{L}$, where G_{ijk} is obtained from G_i by adding a new vertex x_{ijk} and two new edges $x_{ijk}x_j$ and $x_{ijk}x_k$.*

Two distinct edges are called *in series* if they form a minimal edge cut. Every edge is also considered as being series with itself. Being in series is an equivalence relation. Each equivalence class is called a *series family.* A series family is *nontrivial* if it has at least two edges. A graph G is *weakly even* if, for every nontrivial series family F of G with $|F|$ odd, there are two distinct edges xy and xz such that they are the only two edges of G that are incident with vertex x. A graph is *subcubic* if the degree of each vertex is at most three. A graph is *chordless* if every cycle of the graph in an induced cycle. Let K_4^- be obtained from K_4 by deleting an edge, W_4^- be obtained a wheel on five vertices by deleting a rim edge, and $K_{2,3}^+$ be obtained from $K_{2,3}$ by adding an edge between the two vertices of degree three. As usual, $L(G)$ stands for the line graph of G.

Lemma 5. [1] *Suppose $G \in \mathcal{L}$ is not a 2-sum of two smaller graphs. If G is obtained from a rooted 2-connected line graph $L(Q)$ by adding pendent triangles, where Q has no isolated vertices, then the following statements hold:* (i) *if Q has a triangle, then $G \in \{K_3, K_4^-, W_4^-, K_{2,3}^+\}$;* (ii) *$Q$ is connected, subcubic, and chordless;* (iii) *every cut edge of Q is a pendent edge; (iv) Q is weakly even.*

Lemma 6. [1] *If Q is subcubic and chord chordless, then every noncut edge belongs to a nontrivial series family.*

A path with end vertices u and v is called a *u-v path.* If a vertex v has degree three, then the subgraph formed by the three edges incident with v is called a *triad with center v*. In the next lemma, the sum of the indices is taken mod t.

Lemma 7. [1] *Suppose Q is connected and subcubic, and all its cut edges are pendent edges. If $F = \{e_1, \ldots, e_t\}$ is a nontrivial series family of Q, then $Q\backslash F$ has precisely t components $Q_1, \ldots, Q_t$. The indices can be renamed such that each e_i is between $V(Q_i)$ and $V(Q_{i+1})$. In addition, if $|V(Q_i)| = 2$, then the only edge in $E(Q_i)$ is a pendent edge of Q and it forms a triad with e_{i-1} and e_i; if $|V(Q_i)| > 2$, and u and v are the ends of e_{i-1} and e_i in Q_i, then $u \neq v$ and Q_i has two internally vertex-disjoint u-v paths.*

Let $G = (V, E)$ be a graph. The degree of a vertex $v \in V$ is denoted by $d_G(v)$. A 2-*edge coloring of G* is an assignment of two colors to every edge in E. We say that a color is *represented* at vertex v if at least one edge incident with v is assigned that color.

Lemma 8. *Let $G = (V, E)$ be a graph and let $U \subseteq V$. Suppose $G[U]$ is bipartite and $d_G(u) \geq 2$ for all $u \in U$. Then G has a 2-edge coloring such that both colors are represented at every vertex in U.*

Let G' be a connected subgraph of G. Then the *contraction* of G' in G is obtained from $G\backslash E(G[V(G')])$ by identifying all vertices in $V(G')$. This is the same as the ordinary contraction except we also delete the resulting loops.

Lemma 9. *Let $G = (V, E)$ be subcubic, chordless, and weakly even. If $G' = (V', E')$ is obtained from G by repeatedly contracting induced cycles, and $U = (V' - V) \cup \{v \in V \cap V' : d_G(v) = 3\}$, then $G'[U]$ is bipartite.*

4 Packing Feedback Vertex Sets

The goal of this section is to prove Theorem 2, the main result of this paper. The major part of our proof consists of the following two lemmas.

Lemma 10. *Let G be obtained from a rooted 2-connected line graph $L(Q)$ by adding pendent triangles, where Q is triangle-free and satisfies* (ii)-(iv) *in Lemma 5. Let $\mathcal{C}$ be a collection of induced cycles in G, which include all triangles in G. Suppose $S \subseteq V(G)$ with $|S \cap V(C)| \geq 2$ for every $C \in \mathcal{C}$. Then S can be partitioned into R and B such that $R \cap V(C) \neq \emptyset \neq B \cap V(C)$ for every $C \in \mathcal{C}$.*

Proof. Let us call a pair (R, B) satisfying the conclusion of the lemma a *certificate* for $(G, \mathcal{C}, S)$. Suppose the lemma is false. Then we can choose a counterexample $\Omega = (G, \mathcal{C}, S)$ such that (a) $|\mathcal{C}|$ is minimized; (b) subject to (a), $t_\Omega = |C \in \mathcal{C} : |V(C)| = 3$ and $V(C) \subseteq S\}|$ is minimized; (c) subject to (a) and (b), $d_\Omega = |\{v : v \in V(G)$ and $d_G(v) = 4\}|$ is minimized. Clearly we have
(10.1) $|\mathcal{C}| \geq 2$.

By (a)-(c), we shall define $\Omega' = (G', \mathcal{C}', S')$ such that Ω' satisfies the hypothesis of the lemma with G', $\mathcal{C}'$, S' in place of G, $\mathcal{C}$, S, respectively, and Ω' has a certificate (R', B'), from which we deduce contradiction to the assumption that Ω has no certificate.

(10.2) If $x \in V(G)$ belongs to a triangle T of G and $d_G(x) = 2$, then $x \notin S$; in particular $S \in E(Q)$.

Otherwise, $\Omega' = (G \backslash x, \mathcal{C} - \{T\}, S - \{x\})$ has a certificate (R', B'), and therefore either $(R' \cup \{x\}, B')$ or $(R', B' \cup \{x\})$ is a certificate for Ω. So (10.2) holds.

(10.3) If x is a pendent edge of Q, then $x \notin S$.

By (10.1) and (10.2), we may assume that x are contained in both a triangle T in $L(Q)$ and a pendent triangle T' in G. Since $\Omega' = (G \backslash ((V(T') \cap E(Q)) - \{x\}), \mathcal{C} - \{T, T'\}, S - \{x\})$ has a certificate (R', B'), we have $x \notin S$ as otherwise either $(R' \cup \{x\}, B')$ or $(R', B' \cup \{x\})$ is a certificate for Ω. Thus (10.3) holds.

Given $x \in E(Q)$, Q_x is obtained from Q by subdividing x with a new vertex w_x. There is a natural 1-1 correspondence between triangles in $L(Q)$ and triangles in $L(Q_x)$. Additionally, $L(Q_x)$ can be rooted the same way as $L(Q)$ was rooted. Let G_x be obtained from the rooted $L(Q_x)$ by adding pendent triangles. For every $C \in \mathcal{C}$, we define cycle C_x in G_x as follows: if C is a triangle, then C_x is a triangle in G_x that naturally corresponds to C; if C has length at least four, then $C_x = C$ when C avoids x, and $C_x = L(D_x)$ when $C = L(D)$ for cycle D through x in Q, and D_x is obtained from D by subdividing x with w_x. Set $\mathcal{C}_x = \{C_x : C \in \mathcal{C}_x\}$. An edge in Q is called *maximum* if its both ends have degree 3.

(10.4) Every maximum edge of Q belong to S.

If $x \notin S$ for some maximum edge x, then the certificate for $\Omega' = (G_x, \mathcal{C}_x, S)$ is a certificate for Ω. Hence (10.4) holds.

(10.5) $t_\Omega = 0$. That is, $|S \cap V(T)| = 2$ for all triangles T of G.

If (10.5) fails for T, then, by (10.2), T is a triad in Q with center u and contains edge $x = uv$ which is not a root edge. As, by (10.3), x is not a pendent edge, $\Omega' = (G_x, \mathcal{C}_x, (S - \{x\}) \cup \{w_x v\})$ has a certificate (R', B'). Now replacing $w_x v$ with x in (R', B') results in a certificate for Ω. This contradiction proves (10.5).

(10.6) $G = L(Q)$.

If G has a pendent triangle T, then, by (10.2) and (10.5), the certificate for $\Omega' = (G, \mathcal{C} - \{T\}, S)$ is a certificate for Ω. So we have (10.6).

By (10.5), every triad T of Q contains precisely two edges in S. Let S_T be the set of these two edges. Let $\mathcal{D}$ be the set of cycles D of Q such that $L(D) \in \mathcal{C}$. If $S_T \subseteq E(D)$ for some triad T and $D \in \mathcal{D}$, then the certificate for $\Omega' = (G, \mathcal{C} - \{L(D)\}, S)$ is a certificate for Ω. Therefore we have

(10.7) $|S_T \cap E(D)| < 2$ for all triads T of Q and all cycles $D \in \mathcal{D}$.

(10.8) No cycle in $\mathcal{D}$ contains a maximum edge.

Suppose some $D \in \mathcal{D}$ contains a maximum edge x. Then by Lemma 6 and Lemma 5(iii), x is contained in a nontrivial series family $F = \{e_1, \ldots, e_t\}$ of Q. Let components $Q_1, \ldots, Q_t$ of $Q \backslash F$ be indexed as in Lemma 7. It can be deduced from Lemma 7 and (10.3), (10.7) that $|V(Q_i)| \neq 2$ for all i. Notice that $I = \{i : 1 \leq i \leq t \text{ and } |V(Q_i)| > 2\}$ is of size at least two.

In case of $|I| = t$, (10.4) implies $F \subseteq S$. Let $Z_1 = Q \backslash V(Q_2)$ and $Z_2 = Q_2$. For $i = 1, 2$, let Q_i' be obtained from Q by contracting Z_{3-i} into a vertex z_i, and then adding a pendent edge f_i at z_i, let $G_i = L(Q_i')$, $\mathcal{C}_i = \{C \in \mathcal{C}, V(C) \subseteq V(G_i)\} \cup \{f_i e_1 e_2 f_i\}$, and $S_i = (S \cap E(Z_i)) \cup \{e_1, e_2\}$. Each $(G_i, \mathcal{C}_i, S_i)$ has a certificate (R_i, B_i), which gives a certificate $(R_1 \cup R_2, B_1 \cup B_2)$ for Ω. In case of $|I| < t$, suppose $1 \notin I$. For every $i \in I$, let $Q_i' = Q[E(Q_i) \cup E(D)]$, $G_i = L(Q_i')$, $\mathcal{C}_i = \{C \in \mathcal{C} : V(C) \subseteq E(Q_i) \cup \{e_{i-1}, e_i\}\} \cup \{L(D)\}$ and $S_i = S \cap E(Q_i) \cup \{e_{i-1}, e_i\}$. Then every $(G_i, \mathcal{C}_i, S_i)$, $i \in I$ has a certificate (R_i, S_i) such that for all $\{i, i+1\} \subseteq I$, if $S_i \cap S_{i+1} \neq \emptyset$, then e_i belongs to either $R_i \cap R_{i+1}$ or $B_i \cap B_{i+1}$. It follows that $(\cup_{i \in I} R_i, S - \cup_{i \in I} R_i)$ is a certificate for Ω. The contradiction establishes (10.8).

For each $D \in \mathcal{D}$, edges of Q that have precisely one end in $V(D)$ are called *connectors* of D. The combination of (10.3) and (10.7) implies

(10.9) Every $D \in \mathcal{D}$ has at least two connectors.

(10.10) Cycles in $\mathcal{D}$ are pairwise vertex-disjoint.

Suppose otherwise, D and D' are distinct cycles in $\mathcal{D}$ that share a common vertex. As the certificate (R', B') for $\Omega' = (G, \mathcal{C} - \{L(D)\}, S)$ cannot be a certificate for Ω, we may assume $S \cap E(D) \subseteq R'$, and by (10.7), all connectors of D belong to B'. Observe that D' contains at least two connector x_1, x_2 of D. For $i = 1, 2$, let y_i be the edge in $D \cap R'$ that has a common end with x_i. By (10.8), $y_1 \neq y_2$, and it can be verified that $((R' - \{y_1\}) \cup \{x_1\}, (B' - \{x_1\}) \cup \{y_1\})$ is a certificate for Ω. Hence we have (10.10).

Let $Q_{\mathcal{D}}$ be obtained from Q by contracting D, for every $D \in \mathcal{D}$, into a vertex v_D. Let $U = \{v_D : D \in \mathcal{D}\} \cup \{v \in V(Q) - \cup_{D \in \mathcal{D}} V(D) : d_Q(v) = 3\}$ and let

$Q' = Q_{\mathcal{D}}[S']$, where $S' \subseteq E(Q_{\mathcal{D}})$ is the set of edges corresponding to those in $S - \cup_{D\in\mathcal{D}} E(D)$. By Lemma 9, $Q'[U]$ is bipartite, and by (10.5), (10.7), (10.9), $d_{Q'}(u) \geq 2$ for all $u \in U$. Thus Lemma 8 guarantees a 2-edge coloring of Q' in which both colors are represented at every vertex in U. Let R' and B' be the two color classes. We view $S' = R' \cup B'$ as a subset of S. Then by (10.8), R' and B' can be easily extended to be R and B, respectively, such that (R, B) forms a certificate for Ω. The contradiction completes the proof of the lemma. □

Lemma 11. *Let G be obtained from a rooted 2-connected line graph $L(Q)$ by adding pendent triangles, where Q is triangle-free and satisfies* (ii)-(iv) *in Lemma 5. Then $b(\mathcal{C}_G)^{\uparrow}$ is Mengerian.*

Proof. Let $w \in \boldsymbol{Z}_+^V$ and $r = r_w(\mathcal{C}_G)$. By (1), we only need to show that $b(\mathcal{C}_G)^{\uparrow}$ has w-packing of size r. We may assume that $r \geq 2$, and $w(v) \leq r$ for all $v \in V$. Let $\mathcal{C}'$ consist of all triangles in G and $\mathcal{C}''$ consist of all other cycles in G. For any $F \subseteq V$, let $\alpha(F)$ and $\beta(F)$ be the number of cycles in $\mathcal{C}'$ and $\mathcal{C}''$, respectively, that F meets. Clearly, there is a collection $\mathcal{F}$ of subsets of V such that (a) $|\mathcal{F}| = r$; and (b) every $v \in V$ is contained in exactly $w(v)$ members of $\mathcal{F}$. We chose such an $\mathcal{F}$ such that (c) $\alpha(\mathcal{F}) = \sum_{F\in\mathcal{F}} \alpha(F)$ is maximum, and (d) subject to (c), $\beta(\mathcal{F}) = \sum_{F\in\mathcal{F}} \beta(F)$ is maximum. We prove that every member of $\mathcal{F}$ is an FVS of G, and thus $\mathcal{F}$ is a w-packing of $b(\mathcal{C}_G)^{\uparrow}$ of size r.

(11.1) $F \cap V(C) \neq \emptyset$, for all $F \in \mathcal{F}$ and $C \in \mathcal{C}'$.
Suppose otherwise, $F_0 \cap V(C_0) = \emptyset$ for some $F_0 \in \mathcal{F}$ and $C_0 \in \mathcal{C}'$. It follows that $|F_1 \cap V(C_0)| \geq 2$ for some $F_1 \in \mathcal{F}$. Let $F_0 \Delta F_1 = (F_0 - F_1) \cup (F_1 - F_0)$, $F_{01}^Q = (F_0 \Delta F_1) \cap E(Q)$ and $F_{01}^G = (F_0 \Delta F_1) - E(Q)$. Let $\mathcal{C}_0'$ be the set of all cycles $C \in \mathcal{C}'$ with $V(C) \cap (F_0 \cap F_1) = \emptyset$ and $|V(C) \cap F_{01}^Q| \geq 2$. For each $C \in \mathcal{C}_0'$, certain triad in Q contains all members of $V(C) \cap F_{01}^Q$. Let U be the set of the centers of all these triads. For each pendent triangle $C \in \mathcal{C}_0'$, we perform the following operations on Q. Let x, y be the two edges in $V(C) \cap F_{01}^Q$, let u be their common end, and let $z = uv$ be the other edge incident with u. We replace z with $u'v$, where u' is a new vertex. Let Q' be the resulting graph, after performing this operation over all pendent triangles $C \in \mathcal{C}_0'$. Let $Q'' = Q'[F_{01}^Q]$. By Lemma 9, $Q''[U]$ is bipartite, and by Lemma 8, Q'' has a 2-edge coloring so that both colors are represented at each vertex of U. Let R_0 and R_1 denote the two color classes. For each $z \in V(G) - E(Q)$, let T_z denote the pendent triangle of G that contains z. Let $S_0 = \{z \in F_{01}^G : |V(T_z) \cap R_0| < |V(T_z) \cap R_1|\}$ and $S_1 = F_{01}^G - S_0$. For $i = 0, 1$, let $F_i' = (F_1 \cap F_0) \cup R_i \cup S_i$. Let $\mathcal{F}' = (\mathcal{F} - \{F_0, F_1\}) \cup \{F_0', F_1'\}$. Then $\mathcal{F}'$ satisfies (a) and (b), and $\alpha(\mathcal{F}') > \alpha(\mathcal{F})$ contradicts (c), yielding (11.1).

(11.2) For any $x \in V$, if G' is a block of $G \backslash x$, then there exists a triangle-free graph Q', which satisfies (ii)-(iv) in Lemma 5, such that G' is obtained from $L(Q')$ by adding pendent triangles.

We may assume that $|V(G')| \geq 3$, and for each $z \in V(G') - E(Q)$, the pendent triangle T_z containing z is contained in G'. Let $Q_1 = Q[V(G') \cap E(Q)]$. We may assume that some $T_z \backslash z$ is not contained in any triangle of $L(Q_1)$ for otherwise $Q' = Q_1$ is as desired. Let Z be the set of all such z. Construct Q' from Q_1 by adding $|Z|$ pendent edges such that $L(Q')$ is isomorphic to $G'\backslash(V(G') - E(Q) - Z)$. It can be deduced from Lemma 5 that Q' is as desired. Thus we have (11.2).

Now we prove that each member of $\mathcal{F}$ is an FVS of G. Suppose otherwise. By (11.1) and (b), we have $F_0, F_1 \in \mathcal{F}$ and $C_0 \in \mathcal{C}''$ such that $F_0 \cap V(C_0) = \emptyset$ and $|F_1 \cap V(C_0)| \geq 2$. Suppose that $G_1, \ldots, G_k$ are all blocks of $G\backslash(F_0 \cap F_1)$. By (11.2), G_i is obtained from $L(Q_i)$ by adding pendent triangles, where Q_i is triangle-free and satisfies (ii)-(iv) in Lemma 5. Let $i \in \{1, \ldots, k\}$. Let $S_i = (F_0 \Delta F_1) \cap V(G_i)$ and let $\mathcal{C}_i$ be the set of cycles C of G_i with $|V(C) \cap S_i| \geq 2$. By (11.1), Lemma 10 applies and provides a partition (R_i, B_i) of S_i such that each cycle in $\mathcal{C}_i$ meets both R_i and B_i. By interchanging R_i with B_i if necessary, it can be assumed that if any distinct S_i and S_j have a common vertex v then either $v \in R_i \cap R_j$ or $v \in B_i \cap B_j$. Let $F_0' = (F_0 \cap F_1) \cup (R_1 \cup \cdots \cup R_k)$, $F_1' = (F_0 \cap F_1) \cup (B_1 \cup \cdots \cup B_k)$, and $\mathcal{F}' = (\mathcal{F} - \{F_0, F_1\}) \cup \{F_0', F_1'\}$. Then $\mathcal{F}'$ satisfies (a) and (b), $\alpha(\mathcal{F}') \geq \alpha(\mathcal{F})$, and $\beta(\mathcal{F}') > \beta(\mathcal{F})$, contradicting to (d). The lemma is established. □

Proof of Theorem 2. Since every Mengerian hypergarph is ideal, the "only if" part follows from Corollary 1 and Theorem 1. To establish the "if" part, we only need to show that, if $G \in \mathcal{L}$ then $b(\mathcal{C}_G)^{\uparrow}$ is Mengerian. We apply induction on $|V(G)|$. The base case $|V(G)| = 1$ is trivial, so we proceed to the induction step. By Lemma 4 and Lemma 1, 2, we may assume that G cannot be represented as a k-sum ($k = 0, 1, 2, 3$) of two smaller graphs, for otherwise we are done by induction. Then we conclude from Lemma 3 that G is obtained from a rooted 2-connected line graph $L(Q)$ by adding pendent triangles. It can be assumed that Q has no isolated vertices. If Q has a triangle, then we are done by Lemma 5(i) and (1) since for any $K = (V, E) \in \{K_3, K_4^-, W_4^-, K_{2,3}^+\}$ and $w \in \boldsymbol{Z}_+^V$, it is not hard to find a w-packing of FVSs in K of size equal to the minimum weight of a cycle in K. So we may assume that Q is a triangle-free and satisfies (ii)-(iv) in Lemma 5. Now the result follows from Lemma 11. □

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References

1. G. Ding and W. Zang, Packing cycles in graphs, *J. Combin. Theory Ser. B* **86** (2003), 381-407.
2. G. Ding, Z. Xu, and W. Zang, Packing cycles in graphs, II, *J. Combin. Theory Ser. B* **87** (2003), 244-253.
3. J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, *Annals of Discrete Math.*, **1** (1977), 185-204.
4. D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Mathematical Programming* **1** (1971), 168-194.
5. B. Guenin, Oral communication, 2000.
6. A. Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, Sringer-Verlag, Berlin, 2003.