# Reducing Right-Hand Sides for Termination

Hans Zantema

Department of Computer Science, TU Eindhoven,
P.O. Box 513, 5600 MB, Eindhoven, The Netherlands
h.zantema@tue.nl

**Abstract.** We propose two transformations on term rewrite systems (TRSs) based on reducing right-hand sides: one related to the transformation order and a variant of dummy elimination. Under mild conditions we prove that the transformed system is terminating if and only if the original one is terminating. Both transformations are very easy to implement, and make it much easier to prove termination of some TRSs automatically.

## Preface

Before introducing the technical contents of this paper first I want to spend some personal words. Several years ago, around 1990, I was looking for a new research area. At that time I was employed at Utrecht University, and among other things I was responsible for a seminar in algebraic specification. Only vaguely I was aware of the area of term rewriting providing a way for implementation of algebraic specifications. Just before that a nice booklet appeared, in Dutch, about term rewriting. I liked this booklet, and decided to use it for my seminar. This booklet appeared to be the course material of a course by Jan Willem Klop at the Free University in Amsterdam, only 40 kilometers from Utrecht. I heard that the group around Jan Willem Klop was active in research in term rewriting, and that they had meetings every two or three weeks around this research, called TeReSe: term rewriting seminar. Since I liked the topic as I learned it from the booklet, and still was looking for a new research area, I decided to follow these meetings. There I met Jan Willem Klop and the people of his group: Aart Middeldorp, Fer-Jan de Vries, Roel de Vrijer, Vincent van Oostrom and Femke van Raamsdonk. I liked the meetings and the pleasant atmosphere, and very naturally and smoothly inside this area I found challenges that happened to grow out to my own research topics.

Now fifteen years have been passed, and I may look back to (co-)authoring dozens of papers related to this area. Although I have never been a member of Jan Willem's group, I realize that in the way sketched above Jan Willem and his group have played a crucial role in my development as a scientist. I am very grateful for this. To mark one issue, on several places in the present paper the underlying theory is based on completions of diagrams as you may see from the pictures if you browse through the paper. For sure this way of completion of diagrams, preferably in the setting of abstract reduction systems, is inspired by the way Jan Willem propagated to do so in these TeReSe meetings long ago.

## 1 Introduction

Developing techniques for proving termination of TRSs is a challenging research area already for a long time. In recent years the emphasis in this area has shifted towards implementation: for new techniques to prove termination it is no longer sufficient that they can be used to prove termination of particular TRSs in theory, but also tools should be able to use these techniques to prove termination fully automatically. Several tools have been developed for this goal, and there is a yearly competition in which all of these tools are applied to an extensive set of examples (TPDB [20], the termination problem data base), and compared, see

`http://www.lri.fr/~marche/termination-competition/`.

In this paper we present two transformations on TRSs for which termination of the original TRS can be concluded from termination of the transformed TRS. Since these transformations are very easy to implement and proving termination of the transformed TRS by standard techniques is often much simpler than proving termination of the original TRS, they are very suitable to be used as preprocessing steps before using any of the tools.

Both transformations do not change left-hand sides, and reduce right-hand sides. In the first transformation, related to the *transformation ordering* [3] this is done by rewriting right hand sides using the same TRS. So here it is assumed that at least one right-hand side of a rule is not in normal form. In the second transformation, a variant of *dummy elimination* [8], the right-hand sides are decomposed with respect to a special symbol (a *dummy symbol*) that occurs in a right-hand side but in no left-hand side.

The technique of rewriting right-hand sides was considered before in [12], but there it was required that the whole TRS is non-overlapping (or a mild weakening of it), while our approach does not have such global restrictions. Our approach is based on the transformation ordering from [3] presented in a more abstract setting in [24]. The first approach to implement this was described in [17]. In order to use this technique for rewriting right-hand sides we had to adjust the underlying theory. In this paper all required theory is included.

For our present variant of dummy elimination the main theorem states that the original TRS is terminating if the transformed TRS is terminating, just as in [8]. However, in case of left-linearity we also have the converse, as we prove in this paper. Therefore the new variant is called *complete dummy elimination*, and is often stronger than the earlier version from [8].

For string rewriting the techniques described in this paper have been implemented in TORPA: Termination of Rewriting Proved Automatically [21], a tool developed by the author, which was the winner in the above mentioned competition in the category of string rewriting, both in 2004 and 2005.

For term rewriting the techniques described in this paper have been implemented in the tool TPA: Termination Proved Automatically, written by Adam Koprowski, [15]. In the above mentioned termination competition in 2005 this tool was third among 6 participants in the category of term rewriting, after AProVE [6] and TTT [13].

The organization of this paper is as follows. First in Section 2 the preliminaries are given, both for abstract rewriting and term rewriting. Then in Section 3 the theory and implementation of the technique of rewriting right-hand sides is presented. Next, Section 4 treats complete dummy elimination, first for term rewriting and then for string rewriting. To derive the result for string rewriting from the result for term rewriting in Subsection 4.2 we apply a general theorem. A TRS having no symbols of arity greater than one is transformed to an SRS simply by ignoring all parentheses and variable symbols. The theorem states that the TRS is terminating if and only if the SRS is terminating. Finally, in Section 5 we give some conclusions.

## 2 Preliminaries

### 2.1 Abstract Rewriting

In the following $R$, $S$ and $T$ are arbitrary binary relations on a fixed set. In the applications they will correspond to rewrite relations of TRSs. We write a dot symbol for relational composition, i.e., one has $t(R \cdot S)t'$ if and only if there exists a $t''$ such that $tRt''$ and $t''St'$. We write $R^+$ for the transitive closure of $R$ and $R^*$ for the reflexive transitive closure of $R$, and we write $R^{-1}$ for the inverse of $R$. Further we write $R \subseteq S$ if $tRt'$ implies $tSt'$. Clearly, if $R \subseteq S$ then $R \cdot T \subseteq S \cdot T$ and $T \cdot R \subseteq T \cdot S$.

Using these notations confluence of a relation $R$, written as $\mathsf{CR}(R)$, can be expressed shortly as $(R^{-1})^* \cdot R^* \subseteq R^* \cdot (R^{-1})^*$. Similarly, local confluence of a relation $R$, written as $\mathsf{WCR}(R)$, can be expressed as $R^{-1} \cdot R \subseteq R^* \cdot (R^{-1})^*$.

We write $\infty(t, R)$ if there exists an infinite sequence $tRt_1Rt_2Rt_3R\cdots$. Such an infinite sequence is called an *infinite $R$-reduction.* A relation $R$ is called *terminating* on $t$, written as $\mathsf{SN}(t, R)$, if not $\infty(t, R)$. A relation $R$ is called *terminating*, written as $\mathsf{SN}(R)$, if it is terminating on every $t$, i.e., no infinite $R$-reduction exists at all.

For a terminating relation $R$ we can apply *induction on $R$*, i.e. if for all elements $t$ we can prove

$$(\forall t' : (tRt' \Rightarrow P(t'))) \Rightarrow P(t)$$

then we may conclude that the property $P(t)$ holds for all $t$. The assumption $\forall t' : (tRt' \Rightarrow P(t'))$ is called the *induction hypothesis.*

We write $R/S$ for $S^* \cdot R \cdot S^*$. For instance, $(R/S)^+$ describes a sequence of $R \cup S$-steps containing at least one $R$-step, so

$$(R/S)^+ \;=\; S^* \cdot R \cdot (R \cup S)^* \;=\; (R \cup S)^* \cdot R \cdot S^*.$$

### 2.2 Term Rewriting

Write $\mathsf{Var}(t)$ for the set of variables in a term $t$. A *rewrite rule* is a pair of terms $(\ell, r)$, written as $\ell \to r$, such that $\ell$ is not a variable and $\mathsf{Var}(r) \subseteq \mathsf{Var}(\ell)$. The terms $\ell, r$ are called the *left-hand side* (lhs) and the *right-hand side* (rhs) of the rule $\ell \to r$, respectively. A rule $\ell \to r$ is called *left-linear* if every variable occurs at most once in $\ell$. A rule $\ell \to r$ is called *non-erasing* if $\mathsf{Var}(r) = \mathsf{Var}(\ell)$.

A *term rewrite system* (TRS) is defined to be a set of rewrite rules. A TRS is called left-linear if all its rules are left-linear. A TRS is called non-erasing if all its rules are non-erasing.

A term $t$ rewrites to a term $u$ w.r.t. a TRS $\mathcal{R}$, notation $t \to_{\mathcal{R}} u$, if there is a rule $\ell \to r$ in $\mathcal{R}$, a context $C$ and a substitution $\sigma$ such that $t = C[\ell\sigma]$ and $u = C[r\sigma]$. A TRS $\mathcal{R}$ is said to be terminating, confluent or locally confluent (notation: $\mathsf{SN}(\mathcal{R}), \mathsf{CR}(\mathcal{R}), \mathsf{WCR}(\mathcal{R})$) if the corresponding property holds for the binary relation $\to_{\mathcal{R}}$ on terms. Basic techniques to prove termination of TRSs include recursive path order [5] and polynomial interpretations [4]. More involved techniques in which TRSs are first transformed before basic techniques are applied include semantic labelling [23] and dependency pairs [1, 14, 11]. For an overview of techniques for proving termination of TRSs see [24]. For a general introduction to rewriting see [2, 18].

The TRS $\mathcal{E}mb$ is defined to consist of all rules of the shape $f(x_1, \ldots, x_n) \to x_i$. A rule $\ell \to r$ is called *self-embedding* if $r \to^*_{\mathcal{E}mb} \ell$. A TRS $\mathcal{R}$ is called *simply terminating* if $\mathcal{R} \cup \mathcal{E}mb$ is terminating. It is obvious that a TRS containing a self-embedding rule is not simply terminating. It is well-known ([22, 24]) that termination of a TRS can not be proved by recursive path order or polynomial interpretations if the TRS is not simply terminating.

Two non-variable terms $t, u$ are said to have an overlap if there are substitutions $\sigma, \tau$ such that either $t'\sigma = u\tau$ for a non-variable subterm $t'$ of $t$, or $t\sigma = u'\tau$ for a non-variable subterm $u'$ of $u$. Two rules $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are said to have a non-trivial overlap if either the rules are distinct and $\ell_1$ and $\ell_2$ have an overlap, or the rules are equal and there are substitutions $\sigma, \tau$ such that $t'\sigma = \ell_1\tau$ for a non-variable proper subterm $t'$ of $\ell_1$. Here properness is essential to exclude the trivial overlap caused by $\ell_1\sigma = \ell_1\tau$ for $\sigma = \tau$. It is well-known that $\mathsf{WCR}(\mathcal{R})$ holds if no two (possibly equal) rules of $\mathcal{R}$ have a non-trivial overlap.

## 3 Rewriting Right-Hand Sides

### 3.1 The Theory

Our theory of rewriting right-hand sides is based on a modification of a commutation property that was the basis of the transformation ordering, [3].

**Lemma 1.** *Let $S, T$ be binary relations satisfying*

1. *$S \cup T$ is terminating,*
2. *$T$ is locally confluent, and*
3. *$T^{-1} \cdot S \subseteq (S/T)^+ \cdot (T^{-1})^*$.*

*Then $(T^{-1})^* \cdot (S/T)^+ \subseteq (S/T)^+ \cdot (T^{-1})^*$.*

*Proof.* We prove by induction on $S \cup T$ that for every $t$ the following holds:

Let $t'(T^{-1})^*t(S/T)^+w$. Then there exists $w'$ satisfying $t'(S/T)^+w'\ (T^{-1})^*w$.

First observe that $(S/T)^+ = T^* \cdot S \cdot (S \cup T)^*$. Since $(S \cup T)^* = T^* \cup (S/T)^+$, we obtain $u, v$ satisfying $tT^*uSv(T^* \cup (S/T)^+)w$. Since $T$ is terminating by *1* and locally confluent by *2*, we have $\mathsf{CR}(T)$ by Newman's Lemma: $T$ is confluent. Since $tT^*t'$, $tT^*u$ and $\mathsf{CR}(T)$ we obtain $u'$ satisfying $t'T^*u'$ and $uT^*u'$. If $u' = u$ then we may choose $w' = w$ indeed satisfying $t'(S/T)^+w'(T^{-1})^*w$ and we are done. In the remaining case we have $u''$ satisfying $uTu''T^*u'$. Applying condition *3* to $u''T^{-1}uSv$ yields $v''$ satisfying $u''(S/T)^+v''(T^{-1})^*v$. Since $tT^*uTu''$ we may apply the induction hypothesis to $u''$, yielding $v'$ satisfying $u'(S/T)^+v'(T^{-1})^*v''$. Now we have $vT^*v'$ and either $vT^*w$ or $v(S/T)^+w$. In the first case $\mathsf{CR}(T)$ yields $w'$ satisfying $v'T^*w'(T^{-1})^*w$, in the second case the induction hypothesis applied to $v$ yields $w'$ satisfying $v'(S/T)^+w'(T^{-1})^*w$. In all cases we have $t'T^*u'(S/T)^+v'(T^* \cup (S/T)^+)w'$, so $t'(S/T)^+w'$, and $wT^*w'$, and we are done. Summarized in a picture:

$$\begin{tikzcd}
t \arrow[r, "T^*"] \arrow[dd, "T^*"'] & u \arrow[r, "S"] \arrow[d, "T"] \arrow[dr, phantom, "\text{cond. 3}"] & v \arrow[r, "T^* \cup (S/T)^+"] \arrow[d, "T^*"] & w \arrow[dd, "T^*"] \\
\mathsf{CR}(T) & u'' \arrow[r, "(S/T)^+"] \arrow[d, "T^*"] \arrow[dr, phantom, "\text{I. H.}"] & v'' \arrow[d, "T^*"] & \text{I. H. and } \mathsf{CR}(T) \\
t' \arrow[r, "T^*"] & u' \arrow[r, "(S/T)^+"] & v' \arrow[r, "T^* \cup (S/T)^+"] & w'
\end{tikzcd}$$

□

**Theorem 2.** *Let $R, S, T$ be binary relations satisfying*

1. *$S \cup T$ is terminating,*
2. *$T$ is locally confluent,*
3. *$T^{-1} \cdot S \subseteq (S/T)^+ \cdot (T^{-1})^*$, and*
4. *$R \subseteq ((S/T)^+ \cdot (T^{-1})^*) \cup T$.*

*Then $R$ is terminating.*

*Proof.* Assume $R$ is not terminating. So there is an infinite $R$-reduction, i.e., a sequence $t_1, t_2, t_3, \ldots$ such that $t_iRt_{i+1}$ for all $i = 1, 2, 3, \ldots$. Write $R' = (S/T)^+ \cdot (T^{-1})^*$. Let $u_1 = t_1$, and define $u_i$ for $i = 2, 3, 4, \ldots$ satisfying $t_iT^*u_i$ in the following way:

- If $t_{i-1}Tt_i$ then choose $u_i$ such that $t_iT^*u_i$ and $u_{i-1}T^*u_i$. This can be done since $T$ is confluent, following from Newman's Lemma and conditions *1, 2*.
- Otherwise, by condition *4* we have $t_{i-1}R't_i$, i.e., there exists $v_i$ satisfying $t_{i-1}(S/T)^+v_i(T^{-1})^*t_i$. By Lemma 1 we may choose $u_i$ such that $u_{i-1}(S/T)^+u_i$ and $v_iT^*u_i$, by which we obtain $t_iT^*u_i$.

A typical initial part of this construction is sketched in the following picture:

$$\begin{tikzcd}
t_1 \arrow[r, "R'"] \arrow[dd, "T^*"'] \arrow[dr, "(S/T)^+"] & t_2 \arrow[r, "T"] \arrow[d, "T^*"] & t_3 \arrow[r, "R'"] \arrow[dd, "T^*"'] \arrow[dr, "(S/T)^+"] & t_4 \arrow[r, "R'"] \arrow[d, "T^*"] \arrow[dr, "(S/T)^+"] & t_5 \arrow[r, "T"] \arrow[d, "T^*"] & t_6 \arrow[r, "R'"] \arrow[dd, "T^*"'] \arrow[dr, "(S/T)^+"] & t_7 \arrow[d, "T^*"] \\
 & v_2 \arrow[d, "T^*"] & & v_4 \arrow[d, "T^*"] & v_5 \arrow[d, "T^*"] & & v_7 \arrow[d, "T^*"] \\
u_1 \arrow[r, "(S/T)^+"] & u_2 \arrow[r, "T^*"] & u_3 \arrow[r, "(S/T)^+"] & u_4 \arrow[r, "(S/T)^+"] & u_5 \arrow[r, "T^*"] & u_6 \arrow[r, "(S/T)^+"] & u_7
\end{tikzcd}$$

Since $T$ is terminating by condition *1*, the second case $t_{i-1}R't_i$ occurs infinitely often. So we have $u_{i-1}(S/T)^+u_i$ for infinitely many values of $i$, while for the other values of $i$ we have $u_{i-1}T^*u_i$. Hence $u_1 \to u_2 \to u_3 \to \cdots$ is an infinite $S \cup T$-reduction, contradicting condition *1*, concluding the proof. □

Theorem 2 is closely related to the underlying theory in the transformation ordering, [3]. A generalization of this underlying theory is expressed in Theorem 6.5.16 in [24]. In fact this Theorem 6.5.16 coincides with the present Theorem 2 where conditions *3, 4* are replaced by

*3'.* $T^{-1} \cdot S \subseteq T^* \cdot S \cdot (S \cup T \cup T^{-1})^*$,
*4'.* $R \subseteq (S/(T \cup T^{-1}))^+$,

respectively. Condition *3'* is a strict weakening of condition *3*. However, conditions *4'* and *4* are incomparable, since in condition *4* it is allowed that some $R$-step is only a single $T$-step and in condition *4'* it is not. In our application this is essential, therefore this new Theorem 2 was developed rather than applying the earlier result. Later on, it was pointed out by Vincent van Oostrom that by applying Theorem 6.5.16 to $R \setminus T$ rather than $R$ then a simple argument shows that not only $\mathsf{SN}(R \setminus T)$ can be concluded, but also $\mathsf{SN}(R)$. In this way a variant of Theorem 6.5.16 is obtained in which condition *4'* is weakened to $R \subseteq T \cup (S/(T \cup T^{-1}))^+$. Now our new Theorem 2 can be obtained as a corollary of this variant of Theorem 6.5.16. For being self-contained we kept the direct proof of Theorem 2; moreover this proof is slightly simpler than the proof of Theorem 6.5.16.

Next we apply Theorem 2 to term rewriting. In order to do so we first give a lemma analyzing how applications of non-overlapping rewrite rules commute, similar to the well-known critical pair lemma.

**Lemma 3.** *Let $\ell_i \to r_i$ be rewrite rules for $i = 1, 2$ for which $\ell_1$ and $\ell_2$ do not have an overlap, and having rewrite relations $\to_1, \to_2$, respectively. Let $C$ be a context and $\sigma, \tau$ be substitutions such that $C[\ell_1\sigma] = \ell_2\tau$. Then $\ell_2 = C_2[x]$ for* *some context $C_2$ and some variable $x$, for which the two reducts $C[r_1\sigma]$ and $r_2\tau$ of $C[\ell_1\sigma] = \ell_2\tau$ satisfy*

$$C[r_1\sigma] \;\to_1^{n-1} \cdot \to_2 \;\cdot\; \leftarrow_1^k \; r_2\tau,$$

*where $n, k$ are the numbers of occurrences of $x$ in $\ell_2, r_2$, respectively.*

*Proof.* Since there is no overlap between $\ell_1$ and $\ell_2$ we can write $C = C_2[D]$ where $\ell_2 = C_2[x]$ for contexts $C_2, D$ and some variable $x$ for which $x\tau = D[\ell_1\sigma]$. Define $\tau'$ by $x\tau' = D[r_1\sigma]$, and $y\tau' = y\tau$ for $y \neq x$. By applying the reduction $x\tau \to_1 x\tau'$ to the occurrences of $x\tau$ corresponding to the other $n-1$ occurrences of $x$ in $\ell_2 = C_2[x]$, we obtain $C[r_1\sigma] \to_1^{n-1} \ell_2\tau'$. Conversely we obtain $r_2\tau \to_1^k r_2\tau'$ since $x$ occurs $k$ times in $r_2$. Combining these observations yields

$$C[r_1\sigma] \;\to_1^{n-1}\; \ell_2\tau' \;\to_2\; r_2\tau' \;\leftarrow_1^k\; r_2\tau,$$

proving the lemma. □

To sketch a typical example of how Lemma 3 applies, consider $\ell_1 \to r_1$ to be the rule $a \to b$ and $\ell_2 \to r_2$ to be the rule $f(x,x) \to g(x,x,x)$. Let $C = f(\Box, a)$ and $x\tau = a$; since $\ell_1$ does not contain variables, $\sigma$ plays no role. Indeed we have $C[\ell_1\sigma] = f(a,a) = \ell_2\tau$, and

$$C[r_1\sigma] = f(b,a) \;\to_1\; f(b,b) \;\to_2\; g(b,b,b) \leftarrow_1^3 \; g(a,a,a) = r_2\tau.$$

Now we are ready to give the main theorem.

**Theorem 4.** *Let $\mathcal{R}$ be a TRS for which a rhs is not in normal form, i.e., $\mathcal{R}$ contains a rule $\ell \to r$ and a rule of the shape $\ell' \to C[\ell\sigma]$. Assume that*

- *$\ell \to r$ is left-linear,*
- *$\ell \to r$ is non-erasing,*
- *$\mathsf{WCR}(\{\ell \to r\})$, and*
- *there is no overlap between $\ell$ and the lhs of any rule of $\mathcal{R} \setminus \{\ell \to r\}$.*

*Let $\mathcal{R}'$ be obtained from $\mathcal{R}$ by replacing the rule $\ell' \to C[\ell\sigma]$ by $\ell' \to C[r\sigma]$. Then $\mathcal{R}$ is terminating if and only if $\mathcal{R}'$ is terminating.*

*Proof.* The 'only if'-part is immediate from the observation that every $\mathcal{R}'$-step can be mimicked by one or two $\mathcal{R}$-steps: an $\mathcal{R}'$-step applying the rule $\ell' \to C[r\sigma]$ is mimicked by first applying the $\mathcal{R}$-rule $\ell' \to C[\ell\sigma]$ and then the $\mathcal{R}$-rule $\ell \to r$, all other $\mathcal{R}'$-steps are $\mathcal{R}$-steps themselves. It remains to prove the 'if'-part.

First note that the rules $\ell \to r$ and $\ell' \to C[\ell\sigma]$ are distinct, since otherwise the rule $\ell \to C[C[\ell\sigma]\sigma]$ contained in $\mathcal{R}'$ is not terminating.

We apply Theorem 2 for the binary relations

- $T$ being the rewrite relation of the single rule $\ell \to r$,
- $S$ being the rewrite relation of $\mathcal{R}' \setminus \{\ell \to r\}$, and
- $R$ being the rewrite relation of the TRS $\mathcal{R}$.

Termination of the TRS $\mathcal{R}$ is proved by checking all four conditions of Theorem 2.

Condition *1* holds since $S \cup T$ is the rewrite relation of $\mathcal{R}'$ and $\mathcal{R}'$ is assumed to be terminating.

Condition *2* holds by assumption.

For proving condition *3* assume that $uT^{-1}tSv$, i.e., a term $t$ rewrites by $T$ to $u$ and by $S$ to $v$. We distinguish three cases:

– The $T$-redex is in parallel with the $S$-redex. Then we have

$$(u, v) \in S \cdot T^{-1} \subseteq (S/T)^+ \cdot (T^{-1})^*.$$

– The $T$-redex is above the $S$-redex. Then we apply Lemma 3 for $\ell_2 \to r_2$ being $\ell \to r$, so $\to_2 = T$, and $\to_1 \subseteq S$. This yields

$$v = C[r_1\sigma] \to_1^{n-1} \cdot \to_2 \cdot \leftarrow_1^k r_2\tau = u,$$

where $n, k$ are the numbers of occurrences of $x$ in $\ell, r$, respectively. Since $\ell \to r$ is left-linear and non-erasing, we have $k > 0$ and $n = 1$. So

$$(v, u) \in \to_2 \cdot \leftarrow_1^+,$$

hence $(u, v) \in \to_1^+ \cdot \leftarrow_2 \subseteq S^+ \cdot T^{-1} \subseteq (S/T)^+ \cdot (T^{-1})^*$.

– The $S$-redex is above the $T$-redex. Then we apply Lemma 3 for $\ell_1 \to r_1$ being $\ell \to r$, so $\to_1 = T$, and $\to_2 \subseteq S$. This yields

$$u = C[r_1\sigma] \to_1^* \cdot \to_2 \cdot \leftarrow_1^* r_2\tau = v,$$

so $(u, v) \in \to_1^* \cdot \to_2 \cdot \leftarrow_1^* \subseteq T^* \cdot S \cdot (T^{-1})^* \subseteq (S/T)^+ \cdot (T^{-1})^*$.

In all cases we proved $(u, v) \in (S/T)^+ \cdot (T^{-1})^*$, concluding condition *3*.

Condition *4* is verified by considering all three possibilities for an $\mathcal{R}$-rewrite step $t \to u$.

– If $t \to u$ is an application of the rule $\ell \to r$ then $tTu$.
– If $t \to u$ is an application of the rule $\ell' \to C[\ell\sigma]$ then $tS \cdot T^{-1}u$ where the $S$-step is an application of the rule $\ell' \to C[r\sigma]$.
– If $t \to u$ is an application of another rule, then this rule is in $\mathcal{R}' \setminus \{\ell \to r\}$, so $tSu$.

In all three cases we conclude $(t, u) \in (S \cdot (T^{-1})^*) \cup T \subseteq ((S/T)^+ \cdot (T^{-1})^*) \cup T$, concluding condition *4*. □

In the dependency pair framework [1, 14, 11] it often occurs that proving innermost termination is simpler than proving termination, and therefore conditions have been investigated for which termination can be concluded from innermost termination. In a similar way Theorem 4 can be seen as a theorem stating that full termination can be concluded from termination with respect to a particular strategy: $R'$-rewriting can be seen as $R$-rewriting following the strategy that the application of a rule for which the rhs is not in normal form is always followed by reduction of this rhs.

One may wonder whether all conditions of Theorem 4 are essential. Indeed they are, as is shown by the following four examples.

In the first example ([12], Example 5) let $\mathcal{R}$ consist of the two rules

$$f(x) \to a, \;\; b \to f(b).$$

Let $\ell \to r$ be the first rule, applicable to the rhs of the second rule. Then $\mathcal{R}'$ consisting of the rules $f(x) \to a$, $b \to a$ is terminating, while $\mathcal{R}$ is not, and all conditions of Theorem 4 hold except for non-erasingness of $\ell \to r$.

In the second example let $\mathcal{R}$ consist of the two rules

$$a \to b, \;\; a \to a.$$

Let $\ell \to r$ be the first rule, applicable to the rhs of the second rule. Then $\mathcal{R}'$ consisting of two copies of the rule $a \to b$ is terminating, while $\mathcal{R}$ is not, and all conditions of Theorem 4 hold except for the non-overlappingness condition.

In the third example let $\mathcal{R}$ consist of the three rules

$$f(f(x)) \to g(x), \quad h(x) \to f(f(x)), \quad g(f(a)) \to h(h(a)).$$

Let $\ell \to r$ be the first rule, applicable to the rhs of the second rule. Then $\mathcal{R}'$ consisting of the three rules

$$f(f(x)) \to g(x), \quad h(x) \to g(x), \quad g(f(a)) \to h(h(a))$$

is terminating by recursive path order using the precedence $f > h > g$. However, $\mathcal{R}$ is not terminating due to the reduction

$$h(h(a)) \to h(f(f(a))) \to f(f(f(f(a)))) \to f(g(f(a))) \to f(h(h(a))).$$

All conditions of Theorem 4 hold except for $\mathsf{WCR}(\{\ell \to r\})$.

In the last example let $\mathcal{R}$ consist of the four rules

$$f(x,x) \to g(x), \quad a \to b, \quad a \to c, \quad f(b,c) \to f(a,a).$$

Let $\ell \to r$ be the first rule, applicable to the rhs of the last rule. Then $\mathcal{R}'$ is terminating, while $\mathcal{R}$ is not due to the reduction $f(b,c) \to f(a,a) \to f(a,c) \to f(b,c)$. All conditions of Theorem 4 hold except for left-linearity of $\ell \to r$.

A possible generalization of Theorem 4 would be in weakening the restriction of non-overlap to a restriction of critical pairs having a particular kind of common reduct. Moreover, even in case this critical pair condition does not hold one can think of extending $T$ by the normalized versions of the corresponding critical pairs, introducing a kind of completion as in [3, 17]. However, in this paper we want to concentrate on very simple criteria not involving branching choices as is introduced in searching for common reducts of critical pairs in typically non-confluent TRSs.

A variant of Theorem 4 was given by Gramlich in [12]:

**Theorem 5.** *Let $\mathcal{R}$ be a non-overlapping TRS for which a rhs is not in normal form, i.e., $\mathcal{R}$ contains a rule $\ell \to r$ and a rule of the shape $\ell' \to C[\ell\sigma]$. Assume that $\ell \to r$ is non-erasing. Let $\mathcal{R}'$ be obtained from $\mathcal{R}$ by replacing the rule $\ell' \to C[\ell\sigma]$ by $\ell' \to C[r\sigma]$. Then $\mathcal{R}$ is terminating if and only if $\mathcal{R}'$ is terminating.*

So here we do not have a left-linearity requirement for $\ell \to r$ any more, but the full TRS $\mathcal{R}$ is required to be non-overlapping, while our Theorem 4 only requires non-overlappingness involving the rule $\ell \to r$. In typical applications to TRSs describing arithmetic and having a rule $p(s(x)) \to x$ to be applied to some rhs, Theorem 5 is only applicable if the TRS is non-overlapping. So this approach fails as soon as overlapping combinations of usual rules like

$$x - 0 \to x, \quad s(x) - s(y) \to x - y, \quad x - x \to 0$$

occur. In fact, in Gramlich's paper the requirement of non-overlappingness is slightly weakened, but not overcoming these drawbacks. Our Theorem 4 still applies directly if the TRS contains rules of this shape. Therefore we think that in practice our Theorem 4 is more powerful than Theorem 5.

## 3.2 Implementation

We propose to use Theorem 4 as a pre-processing phase for any tool for proving termination of TRSs as follows. Let $\mathcal{R}$ be any finite TRS for which termination has to be proved.

**Basic procedure:**
Check if $\mathcal{R}$ can be written as

$$\mathcal{R} = \mathcal{R}_0 \cup \{\ell \to r,\ \ell' \to C[\ell\sigma]\}$$

where $\ell \to r$ is left-linear and non-erasing, and has no non-trivial overlap with any rule of $R$.

If so, then replace $\mathcal{R}$ by $\mathcal{R}_0 \cup \{\ell \to r,\ \ell' \to C[r\sigma]\}$, and start again.

Applying this basic procedure is straightforward. From Theorem 4 it follows that for every step the replaced TRS is terminating if and only if the original TRS is terminating; note that local confluence of the single rule $\ell \to r$ follows from the property that $\ell \to r$ has no non-trivial overlap with itself. So for any number of steps the resulting TRS is terminating if and only if the original TRS is terminating.

In case $\mathcal{R}$ is terminating, then the basic procedure is terminating too. This can be seen as follows. Assume that the procedure goes on forever, respectively yielding TRSs $\mathcal{R}_1 = \mathcal{R}, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \ldots$. Since every $\mathcal{R}_{i+1}$-step can be mimicked by one or two $\mathcal{R}_i$-steps, as we saw in the 'only if'-part of the proof of Theorem 4, we conclude that $\to_{\mathcal{R}_i} \subseteq \to_{\mathcal{R}}^{+}$ for every $i = 1, 2, 3, \ldots$. Since $\mathcal{R}_{i+1}$ is obtained from $\mathcal{R}_i$ by applying $\to_{\mathcal{R}_i}$ to a rhs of $\mathcal{R}_i$, we can also obtain $\mathcal{R}_{i+1}$ from $\mathcal{R}_i$ by applying $\rightarrow_{\mathcal{R}}^{+}$ to one of the rhs's. Since there are only finitely many rules, but infinitely many steps from $\mathcal{R}_i$ to $\mathcal{R}_{i+1}$, there is some rhs of the original TRS $\mathcal{R}$ on which $\rightarrow_{\mathcal{R}}^{+}$ is applied infinitely often, contradicting termination of $\mathcal{R}$.

Unfortunately, in case $\mathcal{R}$ is not terminating then it can be the case that the basic procedure does not terminate. For instance, if $\mathcal{R}$ consists of the two rules $a \rightarrow a,\ b \rightarrow a$ then the basic procedure can be applied again yielding $\mathcal{R}$ after one step. This process may go on forever. More general, if $\mathcal{R}$ is of the shape $\mathcal{R}_0 \cup \{\ell \rightarrow r\}$ in which $r$ admits an infinite $\mathcal{R}_0$-reduction, then the basic procedure applied to $\mathcal{R}$ may go on forever. A simple way to get a robust implementation of the basic procedure that terminates on every TRS is to put some upper bound on the total number of steps of the basic procedure.

String rewriting can be seen as a particular case of term rewriting in which all symbols have arity 1. Since in this case variables and parentheses are redundant, they are usually omitted.

A tool for automatically proving termination of string rewriting is called TORPA: Termination of Rewriting Proved Automatically [21]. This tool has been developed by the author. After having ideas in mind for years the actual implementation started in July 2003. Earlier versions of TORPA have been described in [25] (version 1.1), in [26] (version 1.2) and in [27] (version 1.3). The extensive paper [27] also contains a full treatment of all the underlying theory.

Our basic procedure for rewriting right-hand sides was implemented for string rewriting in the newest version of TORPA, version 1.4. This version of the TORPA tool participated in the termination competition in 2005, and was the winner among the eight participants in the string rewriting category, see

`http://www.lri.fr/~marche/termination-competition/2005/`.

In the text generated by TORPA our technique is called *transformation order*. As an example we consider the string rewriting system consisting of the following five rules

$$f0 \rightarrow s0, \quad d0 \rightarrow 0, \quad ds \rightarrow ssdps, \quad fs \rightarrow dfps, \quad ps \rightarrow e,$$

where $e$ represents the empty string. This system describes computation of powers of 2: think of $s$ being successor, $p$ being predecessor, $d$ being doubling and $f$ being exponentiation.[1] It is easy to observe that $fs^n0$ rewrites to its normal form $s^{2^n}0$ for every $n = 0, 1, 2, \ldots$. The normal form of $f^n0$ has super-exponential size and requires a super-exponential number of steps to be computed. Note that the system is not simply terminating: both the third and the fourth rule are self-embedding. TORPA yields the following termination proof:

[1] The fact that the constant 0 may be treated here as a unary symbol will be justified by Theorem 12.

```
TORPA 1.4 is applied to the string rewriting system
f 0  -> s 0
d 0  -> 0
d s  -> s s d p s
f s  -> d f p s
p s  -> e
Choose polynomial interpretation f: lambda x.x+1, rest identity
remove: f 0  -> s 0
Remaining rules:
    d 0  -> 0
    d s  -> s s d p s
    f s  -> d f p s
    p s  -> e

Transformation order: apply rule 4 on rhs of rule 2, result:
d 0  -> 0
d s  -> s s d
f s  -> d f p s
p s  -> e

Transformation order: apply rule 4 on rhs of rule 3, result:
d 0  -> 0
d s  -> s s d
f s  -> d f
p s  -> e
%

Choose polynomial interpretation p: lambda x.x+1, rest identity
remove: p s  -> e
Remaining rules:
    d 0  -> 0
    d s  -> s s d
    f s  -> d f

Terminating by recursive path order with precedence:
  d>s  f>d
```

For term rewriting our basic procedure has been implemented in the tool TPA, written by Adam Koprowski, [15].

Let $\mathcal{R}$ be any TRS and let $\mathcal{R}'$ be the result of applying the basic procedure to $\mathcal{R}$. From many examples we observe that proving termination of $\mathcal{R}'$ is much simpler than proving termination of $\mathcal{R}$ directly. We should like to have evidence that it is never the other way around. Since the notion of 'simpler' depends on the unspecified set of techniques to be used, it is hard to make this claim solid. However, by construction we have $\to_{\mathcal{R}'} \subseteq \to^+_{\mathcal{R}}$, and the lhs's of $\mathcal{R}$ are equal to the lhs's of $\mathcal{R}'$. Under these conditions for all techniques known by us it is very unlikely that proving termination of $\mathcal{R}'$ may be harder than proving termination of $\mathcal{R}$. In particular, since $\to_{\mathcal{R}' \cup \mathcal{E}mb} \subseteq \to^+_{\mathcal{R} \cup \mathcal{E}mb}$, simple termination of $\mathcal{R}$ implies simple termination of $\mathcal{R}'$.

Therefore applying our basic procedure as a pre-processing before trying any other tool for proving termination will often increase the power of the tool, and probably never decrease it.

One may wonder whether it is natural to have rhs's that are not in normal form. Of course this is hard to answer since there is no precise definition of *natural*. To our knowledge the most extensive list of termination problems in term rewriting is TPDB, the Termination Problem Data Base [20]. This database was used in the above mentioned competition. Restricted to term rewriting (excluding string rewriting, which is a separate category) it contains 773 TRSs for which the problem of termination has been posed. They are from a wide scala of origins and application areas. Therefore we think it makes sense to consider these TRSs to get an impression of the applicability of our technique. It turns out that among these 773 TRSs there are 98 TRSs for which not only a rhs is not in normal form, but also the extra conditions are satisfied. So for these TRSs our basic procedure is applicable. For several of them, proving termination of the transformed system is much simpler than proving termination of the original system. Of course again the meaning of 'simpler' has not been defined precisely, but it sounds reasonable to consider a proof only using basic techniques like recursive path order and linear polynomial interpretations, simpler than a proof using a combination of dependency pairs, argument filtering and the same basic techniques.

For instance, consider the classical TRS describing computation of factorials consisting of the following rules

$$\begin{array}{rlrl} p(s(x)) & \to x & *(0,y) & \to 0 \\ \mathit{fact}(0) & \to s(0) & *(s(x),y) & \to +(*(x,y),y) \\ \mathit{fact}(s(x)) & \to *(s(x),\mathit{fact}(p(s(x)))) & +(x,0) & \to x \\ & & +(x,s(y)) & \to s(+(x,y)). \end{array}$$

This TRS is `D33/21.trs` in the TRS category of TPDB. In the 2005 competition only two (AProVE [6] and TTT [13]) of the six participating tools were able to prove termination of this TRS. Note that the TRS is not simply terminating since the rule $\mathit{fact}(s(x)) \to *(s(x),\mathit{fact}(p(s(x))))$ is self-embedding, so only using recursive path order and polynomial interpretations will fail. However, by applying our basic procedure this self-embedding rule is replaced by $\mathit{fact}(s(x)) \to *(s(x),\mathit{fact}(x))$, which is replaced again by $\mathit{fact}(s(x)) \to +(*(x,\mathit{fact}(x)),\mathit{fact}(x))$. Termination of the resulting TRS is easily concluded by the recursive path order using the precedence $\mathit{fact} > * > + > s$. This is exactly the proof as it is found automatically by the tool TPA within a fraction of a second.

Using our basic procedure, the tool TPA was able to prove termination of 6 more TRSs in TPDB than without it, including this factorial system.

## 4 Complete Dummy Elimination

In the basic procedure based on Theorem 4 rhs's are rewritten. So a part of such an rhs matches with an lhs. In this section we consider the opposite: we consider rhs's containing symbols that do not occur in lhs's at all. These symbols are called *dummy symbols*. Again we keep the lhs's and reduce the rhs's, but this reduction is done completely different than before: the dummy symbol now is used to split up the rhs into several smaller rhs's, each generating a rule in the transformed TRS, with its lhs kept unchanged. This approach was studied before in [8] and was called *dummy elimination*. An earlier version already appeared in [22]. The main theorem states that if the TRS after applying dummy elimination is terminating, then the original TRS is terminating too. In general the converse is not true. Here we present a modification of dummy elimination for which the same property holds, but for which in case of left-linearity also the converse holds (the transformed TRS is terminating if and only if the original TRS is). Due to this completeness result our new variant is called *complete dummy elimination*. Moreover, complete dummy elimination is more powerful to be used in tools for automatically proving termination of TRSs or SRSs.

## 4.1 Complete Dummy Elimination for Term Rewriting

Before giving precise definitions first we give a very simple example sketching the general idea. Consider the TRS $\mathcal{R}$ consisting of the single rule

$$f(g(x)) \;\to\; f(a(g(x))).$$

Here the symbol $a$ is a dummy symbol: it does not occur in any lhs. Intuitively this means that this dummy symbol does not play an essential role in further reductions of the term, and further reductions can be localized as either affecting the part above the dummy symbol or affecting the part below it. This can be formalized by decomposing the rhs's into smaller terms in which the dummy acts as a separator. In this case this means that the term $f(h(g(x)))$ is decomposed into two terms $f(\diamond)$ and $b(g(x))$, where $\diamond$ is a fresh constant and $b$ is a fresh unary symbol. The lhs's remain the same. The result is the transformed system $DE_a(\mathcal{R})$, in this example consisting of the two rules

$$\begin{aligned} f(g(x)) &\to f(\diamond) \\ f(g(x)) &\to b(g(x)). \end{aligned}$$

The main result states that $DE_a(\mathcal{R})$ is terminating if and only if $\mathcal{R}$ is terminating. So termination of $\mathcal{R}$ can be proved by proving termination of $DE_a(\mathcal{R})$, which is straightforward by recursive path order choosing the precedence $f > b$, $g > \diamond$.

In order to give a precise definition for complete dummy elimination we need some auxiliary definitions. We fix one dummy symbol $a$ of a TRS $\mathcal{R}$. Let $n$ be the arity of $a$. Choose a fresh constant $\diamond_a$ and a fresh unary symbol $b_a$, i.e., $\diamond_a$ and $b_a$ do not occur in $\mathcal{R}$. As long as $a$ is fixed, we omit the subscripts, simply writing $b$ and $\diamond$. For any term $t$ we define inductively a term $\mathsf{cap}_a(t)$ and a set of terms $\mathsf{dec}_a(t)$:

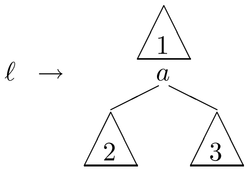
$$\begin{array}{lll} \mathsf{cap}_a(x) & = x & \text{for all } x \in \mathsf{Var}, \\ \mathsf{cap}_a(f(t_1,\ldots,t_k)) & = f(\mathsf{cap}_a(t_1),\ldots,\mathsf{cap}_a(t_k)) & \text{for all } f,\ f \neq a \\ \mathsf{cap}_a(a(t_1,\ldots,t_n)) & = \diamond & \end{array}$$

$$\begin{array}{lll}\mathsf{dec}_a(x) & = \emptyset & \text{for all } x \in \mathsf{Var},\\ \mathsf{dec}_a(f(t_1,\ldots,t_k)) & = \bigcup_{i=1}^{k} \mathsf{dec}_a(t_i) & \text{for all } f,\ f \neq a\\ \mathsf{dec}_a(a(t_1,\ldots,t_n)) & = \bigcup_{i=1}^{n} (\mathsf{dec}_a(t_i) \cup \{b(\mathsf{cap}_a(t_i))\}). & \end{array}$$

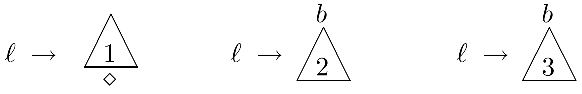
Roughly speaking we decompose a term $t$ by using the symbol $a$ as a separator, where occurrences of $a$ are replaced by $\diamond$ and arguments of $a$ are marked by the symbol $b$. Now the term $\mathsf{cap}_a(t)$ is the topmost part of this decomposition, while $\mathsf{dec}_a(t)$ is the set of all other parts in this decomposition. Now we define the TRS $DE_a(\mathcal{R})$ for any TRS $\mathcal{R}$ having $a$ as a dummy symbol by

$$DE_a(\mathcal{R}) = \{\ell \to u \mid u = \mathsf{cap}_a(r) \vee u \in \mathsf{dec}_a(r) \text{ for a rule } \ell \to r \in \mathcal{R}\}.$$

The transformation $DE_a$ is called *complete dummy elimination.* For instance, applying $DE_a$ on a rule of the shape



where the binary dummy symbol $a$ occurs only once in the rhs, yields the three rules



**Theorem 6.** *Let $a$ be a dummy symbol in a TRS $\mathcal{R}$ for which $DE_a(\mathcal{R})$ is terminating. Then $\mathcal{R}$ is terminating too.*

Before proving this theorem we give an example slightly more complicated than the one given above, and we recall the earlier dummy elimination theorem. Let the TRS $\mathcal{R}$ consist of the two rules

$$\begin{array}{l} f(g(x)) \to f(a(g(a(x,f(x))),g(f(x))))\\ g(f(x)) \to g(g(a(f(x),g(g(x))))). \end{array}$$

Then $DE_a(\mathcal{R})$ consists of the rules

$$\begin{array}{ll} f(g(x)) \to f(\diamond) & f(g(x)) \to b(g(f(x)))\\ f(g(x)) \to b(g(\diamond)) & g(f(x)) \to g(g(\diamond))\\ f(g(x)) \to b(x) & g(f(x)) \to b(f(x))\\ f(g(x)) \to b(f(x)) & g(f(x)) \to b(g(g(x))). \end{array}$$

Indeed Theorem 6 is helpful for proving termination of $\mathcal{R}$: termination of $DE_a(\mathcal{R})$ is easily proved by recursive path order, choosing the precedence $f > g > b > \diamond$.

In the version of dummy elimination from [8] the symbol $b$ was omitted. More precisely, for a TRS $\mathcal{R}$ having $a$ as a dummy symbol the TRS $E(\mathcal{R})$ was defined exactly as $DE_a(\mathcal{R})$, with the only difference that $\mathsf{dec}_a(a(t_1,\ldots,t_n))$ was defined to be $\bigcup_{i=1}^n(\mathsf{dec}_a(t_i)\cup\{\mathsf{cap}_a(t_i)\})$ rather than $\bigcup_{i=1}^n(\mathsf{dec}_a(t_i)\cup\{b(\mathsf{cap}_a(t_i))\})$. As a consequence, the TRS $E(\mathcal{R})$ is obtained from $DE_a(\mathcal{R})$ by removing all symbols $b$ from it. As the main result we recall:

**Theorem 7.** *Let $a$ be a dummy symbol in a TRS $\mathcal{R}$ for which $E(\mathcal{R})$ is terminating. Then $\mathcal{R}$ is terminating too.*

For a proof of Theorem 7 we refer to [8] or [7], where a slightly more general version has been treated. An alternative proof has been given in [16], where even the restriction of the dummy not occurring in lhs's has been weakened slightly. A generalization of this result to rewriting modulo equations has been given in [9].

Now we give the proof of Theorem 6.

*Proof.* Let $a$ be a dummy symbol of arity $n$ in a TRS $\mathcal{R}$ for which $DE_a(\mathcal{R})$ is terminating. Assume $\mathcal{R}$ is not terminating, so admits an infinite reduction. We define a transformation $\Phi$ on terms and TRSs replacing every $a$ by $a(b(-),\ldots,b(-))$, more precisely:

$$\begin{array}{lll}
\Phi(x) & = x & \text{for all } x \in \mathsf{Var},\\
\Phi(f(t_1,\ldots,t_k)) & = f(\Phi(t_1),\ldots,\Phi(t_k)) & \text{for all } f \text{ with } f \neq a,\\
\Phi(a(t_1,\ldots,t_n)) & = a(b(\Phi(t_1)),\ldots,b(\Phi(t_n))), & \\
\Phi(\mathcal{R}) & = \{\Phi(\ell) \to \Phi(r) \mid \ell \to r \in \mathcal{R}\}. &
\end{array}$$

From this definition it is straightforwardly proved that if $t \to_{\mathcal{R}} u$, then $\Phi(t) \to_{\Phi(\mathcal{R})} \Phi(u)$. So the assumed infinite $\mathcal{R}$ reduction transforms by $\Phi$ to an infinite $\Phi(\mathcal{R})$ reduction.

On the other hand the symbol $a$ is still a dummy symbol in $\Phi(\mathcal{R})$. By construction we have $E(\Phi(\mathcal{R})) = DE_a(\mathcal{R})$, which was assumed to be terminating. Hence by Theorem 7 we conclude termination of $\Phi(\mathcal{R})$, contradiction. □

We want to use complete dummy elimination in proving termination automatically is as follows: if termination of $\mathcal{R}$ has to be proved, and $\mathcal{R}$ has a dummy symbol $a$, then apply $DE_a$ to $\mathcal{R}$, and proceed with the search for termination proofs on $DE_a(\mathcal{R})$. For this approach to be useful we should also like to have the converse of Theorem 6: $\mathcal{R}$ is terminating only if $DE_a(\mathcal{R})$ is terminating. In other words, apart from soundness Theorem 6, we also want completeness. This is seen as follows: if $\mathcal{R}$ is terminating but $DE_a(\mathcal{R})$ is not, then trying to prove termination of $DE_a(\mathcal{R})$ will fail. For instance, let $\mathcal{R}$ consist of the two rules

$$f(g(x)) \to g(f(f(x))),\ \ g(f(x)) \to g(a(g(g(x)))).$$

Then indeed $\mathcal{R}$ is terminating, but trying to prove this by proving termination of $E(\mathcal{R})$ consisting of the three rules

$$f(g(x)) \to g(f(f(x))),\ \ g(f(x)) \to g(\diamond),\ \ g(f(x)) \to g(g(x))$$

will fail since $E(\mathcal{R})$ is not terminating due to

$$f(f(g(x))) \to_{E(\mathcal{R})} f(g(f(f(x)))) \to_{E(\mathcal{R})} f(g(g(f(x)))) \to_{E(\mathcal{R})} g(\underbrace{f(f(g}(f(x))))).$$

Note that Theorem 4 does not apply here due to an overlap between the rules. We conclude that dummy elimination $E$ rather than $DE_a$ is not complete.

Next we show that in case of left-linearity, the desired completeness, and hence the 'if and only if' property holds for $DE_a$. First we need two lemmas.

Let $\mathcal{B}l$ be the TRS defined to consist of all rules of the shape $f(x_1, \ldots, x_n) \to \diamond$, for all symbols $f$ of arity $n \geq 0$. This TRS is used for blocking reductions.

**Lemma 8.** *Let $\mathcal{R}$ be a left-linear TRS in which the constant $\diamond$ and the unary symbol $b$ do not occur in any lhs.*

1. *If $t \to_{\mathcal{B}l}^* u$ and $\infty(u, \mathcal{R})$, then $\infty(t, \mathcal{R})$.*
2. *If $\infty(C[b(t)], \mathcal{R})$ for any context $C$ and any term $t$, then either $\infty(C[\diamond], \mathcal{R})$ or $\infty(t, \mathcal{R})$.*

*Proof.* Part *1.*

Let $t, u, v$ be terms satisfying $t \to_{\mathcal{B}l} u \to_{\mathcal{R}} v$. Then $u$ is obtained from $t$ by replacing any subterm by $\diamond$. So the redex of $u \to_{\mathcal{R}} v$ is either above or parallel to this occurrence of $\diamond$. Since $\mathcal{R}$ is left-linear and $\diamond$ does not occur in the lhs of the corresponding rule in $\mathcal{R}$, the TRS $\mathcal{R}$ could also be applied directly to $t$ yielding $t \to_{\mathcal{R}} \cdot \to_{\mathcal{B}l}^* v$. Hence we conclude $\to_{\mathcal{B}l} \cdot \to_{\mathcal{R}} \subseteq \to_{\mathcal{R}} \cdot \to_{\mathcal{B}l}^*$. Using this property one easily proves $\to_{\mathcal{B}l}^* \cdot \to_{\mathcal{R}} \subseteq \to_{\mathcal{R}} \cdot \to_{\mathcal{B}l}^*$, applying induction on the number of $\to_{\mathcal{B}l}$-steps. Using this inclusion the infinite $\mathcal{R}$-reduction starting in $u$ is transformed to an infinite $\mathcal{R}$-reduction starting in $t$, as is sketched in the following picture:

$$\begin{tikzcd}
u \arrow[r, "\mathcal{R}"] & \cdot \arrow[r, "\mathcal{R}"] & \cdot \arrow[r, "\mathcal{R}"] & \cdot \arrow[r, "\mathcal{R}"] & \cdot & \cdots & \cdots \\
t \arrow[u, "\mathcal{B}l^*"'] \arrow[r, "\mathcal{R}"] & \cdot \arrow[u, "\mathcal{B}l^*"'] \arrow[r, "\mathcal{R}"] & \cdot \arrow[u, "\mathcal{B}l^*"'] \arrow[r, "\mathcal{R}"] & \cdot \arrow[u, "\mathcal{B}l^*"'] \arrow[r, "\mathcal{R}"] & \cdot \arrow[u, "\mathcal{B}l^*"'] & \cdots & \cdots
\end{tikzcd}$$

Part *2.* We prove the more general claim for multiple hole contexts:

> Let $C$ be a multi-hole context for which $\mathsf{SN}(C[\diamond, \ldots, \diamond], \mathcal{R})$ and $\infty(C[b(t_1), \ldots, b(t_n)], \mathcal{R})$. Then $\infty(t_i, \mathcal{R})$ for some $i = 1, \ldots, n$.

We prove this for all contexts $C$ satisfying $\mathsf{SN}(C[\diamond, \ldots, \diamond], \mathcal{R})$, by induction on $\to_{\mathcal{R}}$ restricted to reducts of $C[\diamond, \ldots, \diamond]$. Consider the infinite $\mathcal{R}$-reduction starting in $C[b(t_1), \ldots, b(t_n)]$. If all redex positions are below $C$ then every step is in one of the $n$ displayed subterms $b(t_1), \ldots, b(t_n)$ of $C[b(t_1), \ldots, b(t_n)]$, so at least one of the $b(t_i)$ is rewritten infinitely often. Since $b$ does not occur in any lhs, the same holds for $t_i$ and we are done. In the remaining case the infinite $\mathcal{R}$-reduction is of the shape

$$C[b(t_1),\ldots,b(t_n)] \to_{\mathcal{R}}^* C[b(u_1),\ldots,b(u_n)] \to_{\mathcal{R}} D[b(v_1),\ldots,b(v_k)] \to_{\mathcal{R}}^* \cdots,$$

where $t_i \to_{\mathcal{R}}^* u_i$ for every $i$, and for every $j$, $1 \leq j \leq k$, there exists $i$ such that $v_j = u_i$. Since $\mathcal{R}$ is left-linear and $b$'s do not occur in lhs's, we conclude $C[\diamond,\ldots,\diamond] \to_{\mathcal{R}} D[\diamond,\ldots,\diamond]$. Left-linearity is essential here, for instance for $\mathcal{R}$ consisting of $f(x,x) \to a$, $a \to a$ both this claim and the lemma do not hold for $n = 1, t_1 = c, C = f(b(c),\Box)$. Since $C[\diamond,\ldots,\diamond] \to_{\mathcal{R}} D[\diamond,\ldots,\diamond]$ we may apply the induction hypothesis to $D[b(v_1),\ldots,b(v_k)]$, yielding $j$ satisfying $\infty(v_j,\mathcal{R})$. Since there exists $i$ such that $v_j = u_i$ we obtain $t_i \to_{\mathcal{R}}^* u_i = v_j \to_{\mathcal{R}}^\infty$, so $\infty(t_i,\mathcal{R})$. □

**Lemma 9.** *Let $t$ be any term and $a$ any symbol. Then*

1. *$t \to_{\mathcal{B}l}^* \mathsf{cap}_a(t)$, and*
2. *for every $v \in \mathsf{dec}_a(t)$ the terms $t, v$ can be written as $t = C[t']$ and $v = b(v')$, where $t' \to_{\mathcal{B}l}^* v'$.*

*Proof.* By induction on the structure of $t$, straightforward from the definitions of $\mathsf{cap}_a$ and $\mathsf{dec}_a$. □

**Theorem 10.** *Let $\mathcal{R}$ be a left-linear terminating TRS having a dummy symbol $a$. Then $DE_a(\mathcal{R})$ is terminating.*

*Proof.* We prove that $\mathsf{SN}(t, DE_a(\mathcal{R}))$ for every term $t$, by induction on $\mathcal{R}$. So the induction hypothesis states that $\mathsf{SN}(w, DE_a(\mathcal{R}))$ for every term $w$ satisfying $t \to_{\mathcal{R}}^+ w$.

Assume that $t$ admits an infinite $DE_a(\mathcal{R})$-reduction

$$t \to_{DE_a(\mathcal{R})} u \to_{DE_a(\mathcal{R})} \cdot \to_{DE_a(\mathcal{R})} \cdots .$$

For the step $t \to_{DE_a(\mathcal{R})} u$ we distinguish two cases, implied by the definition of $DE_a(\mathcal{R})$.

- $t = C[\ell\sigma]$ and $u = C[\mathsf{cap}_a(r)\sigma]$ for some context $C$, some substitution $\sigma$ and some rule $\ell \to r$ in $\mathcal{R}$. Let $v = C[r\sigma]$. By part *1* of Lemma 9 we conclude that $r \to_{\mathcal{B}l}^* \mathsf{cap}_a(r)$, so $v = C[r\sigma] \to_{\mathcal{B}l}^* C[\mathsf{cap}_a(r)\sigma] = u$. Since $\infty(u, DE_a(\mathcal{R}))$ and $DE_a(\mathcal{R})$ is left-linear and has no $\diamond$ or $b$ symbols in lhs's, we may apply part *1* of Lemma 8, yielding $\infty(v, DE_a(\mathcal{R}))$, contradicting the induction hypothesis.
- $t = C[\ell\sigma]$ and $u = C[v\sigma]$ for some context $C$, some substitution $\sigma$, $v \in \mathsf{dec}_a(r)$, and some rule $\ell \to r$ in $\mathcal{R}$. By part *2* of Lemma 9 we obtain $r = C'[r']$ and $v = b(v')$, where $r' \to_{\mathcal{B}l}^* v'$. By part *2* of Lemma 8 we may distinguish two cases based on the infinite $DE_a(\mathcal{R})$-reduction of $u = C[v\sigma] = C[b(v'\sigma)]$:
  - $\infty(C[\diamond], DE_a(\mathcal{R}))$. Since $C[r\sigma] \to_{\mathcal{B}l} C[\diamond]$ we conclude $\infty(C[r\sigma], DE_a(\mathcal{R}))$ from part *1* of Lemma 8.
  - $\infty(v'\sigma, DE_a(\mathcal{R}))$. Since $r' \to_{\mathcal{B}l}^* v'$ we may apply part *1* of Lemma 8, yielding $\infty(r'\sigma, DE_a(\mathcal{R}))$. Since $C[r\sigma] = C[C'\sigma[r'\sigma]]$ we obtain $\infty(C[r\sigma], DE_a(\mathcal{R}))$.

In both cases we obtain $\infty(w, DE_a(\mathcal{R}))$ for $w = C[r\sigma]$ satisfying $t = C[\ell\sigma] \to_{\mathcal{R}} C[r\sigma] = w$, contradicting the induction hypothesis.

□

Left-linearity is essential in Theorem 10 as is shown by the following example. Let $\mathcal{R}$ consist of the single rule

$$f(x, x) \ \to \ f(a(c), a(d)).$$

Then $\mathcal{R}$ is terminating, but $DE_a(\mathcal{R})$ is not since it contains the non-terminating rule $f(x, x) \to f(\diamond, \diamond)$.

For proving termination of a TRS $\mathcal{R}$ containing a dummy symbol automatically we propose always to try proving termination of $DE_a(\mathcal{R})$ first. For non-left-linear TRSs this may fail even if $\mathcal{R}$ is terminating as was shown by the above example. However, even then it may be a good strategy first to search for some time for a termination proof of $DE_a(\mathcal{R})$, since often termination proofs for $DE_a(\mathcal{R})$ are substantially simpler than direct termination proofs for $\mathcal{R}$.

Just like we did for the technique of rewriting right-hand sides we investigated on how many problems in the termination problem data base TPDB [20] the technique of complete dummy elimination is directly applicable. It turns out that among the 773 TRSs there are 65 TRSs for which it is. Apart from these 65 it may occur that after some transformation complete dummy elimination is applicable, but this latter figure of course depends on details of the tool.

In case a TRS contains more than one dummy symbol it is a natural question how to proceed. It turns out that just like in earlier versions of dummy elimination the order of applying the corresponding $DE$ operations does not influence the result, e.g., if both $a_1$ and $a_2$ are dummy symbols in $\mathcal{R}$, then $DE_{a_1}(DE_{a_2}(\mathcal{R})) = DE_{a_2}(DE_{a_1}(\mathcal{R}))$. In constructing this combined dummy elimination we can apply it for all dummy symbols in one run, introducing a fresh constant $\diamond_a$ and a fresh unary symbol $b_a$ for every dummy symbol $a$. So in case a TRS contains more than one dummy symbol we propose always to proceed by this simultaneous dummy elimination.

The best tool at the moment for proving TRS termination is AProVE [6]. We give two examples now showing that our $DE$-strategy is able to enhance the 2005 version of AProVE. The first TRS consists of two rules

$$f(f(g(g(x)))) \to g(g(g(f(f(f(x)))))), \quad f(x) \to a(x, x).$$

AProVE fails to prove termination of this TRS. However, after applying $DE_a$ the resulting TRS consisting of the rules

$$f(f(g(g(x)))) \to g(g(g(f(f(f(x)))))), \quad f(x) \to \diamond, \quad f(x) \to b(x)$$

is proved to be terminating by AProVE in a fraction of a second.

As the second example consider the TRS consisting of the rules

$$\begin{array}{lr}
f(g(x)) \to f(h(h(a(h(h(g(f(x)))))))) & f(g(x)) \to g(g(f(h(x)))) \\
f(h(x)) \to h(g(f(x))) & g(h(x)) \to h(g(x)) \\
h(f(x)) \to g(g(h(h(a(f(x)))))) & f(x) \to g(g(h(x))).
\end{array}$$

Again AProVE fails to prove termination of this TRS, but by applying $DE_a$ the two rules containing $a$ in their rhs's are replaced by

$$f(g(x)) \to f(h(h(\diamond))), \quad f(g(x)) \to b(h(h(g(f(x))))),$$

$$h(f(x)) \to g(g(h(h(\diamond)))), \quad h(f(x)) \to b(f(x)),$$

resulting in a TRS for which termination is proved easily, e.g., by recursive path order choosing the precedence $f > g > h > b > \diamond$. In the tool TPA complete dummy elimination has been implemented, and indeed for this TRS TPA finds the proof just sketched in a fraction of a second.

In particular this last example is of interest with respect to the following. In [10] it was proved that if $E(\mathcal{R})$ is DP simply terminating then $\mathcal{R}$ is DP simply terminating too. Here roughly speaking DP simple termination means that termination can be proved by the dependency pair technique using argument filtering and a simplification order. As a theorem this is correct, but in [10] it is literally claimed that it implies that *using dummy elimination as a preprocessing step to the dependency pair technique does not have any advantage.* However, our latter example convincingly shows the converse with respect to the present AProVE implementation: here no dependency pair transformation was required, but if the dependency pair transformation had been applied to the resulting system, a straightforward termination proof only using recursive path order would have been found easily too. The difference between $E(\mathcal{R})$ and $DE_a(\mathcal{R})$ is only in the symbol $b$, and does not play any role: if it is omitted then the same proof holds.

## 4.2 Complete Dummy Elimination for String Rewriting

For term rewriting we believe that the operation $DE_a$ is the most natural and most powerful variant of dummy elimination, due to the combination of Theorem 6 and Theorem 10. However, for string rewriting there is a drawback: due to the introduction of the constant $\diamond_a$ for a string rewriting system (SRS) $\mathcal{R}$, being a TRS over a signature only containing unary symbols, the transformed system $DE_a(\mathcal{R})$ is not an SRS any more.

This can be solved by defining a variant $DE'_a$ of $DE_a$, where the only difference is that $\diamond_a$ is a unary symbol rather than a constant. In this way a symmetry between $\diamond_a$ and $b_a$ is introduced. To express this symmetry in the notation, we will write $a_\$$ instead of $\diamond_a$, and ${}_\$a$ instead of $b_a$. As usual, we will identify a term $a_1(a_2(\cdots(a_n(x))\cdots))$ with the string $a_1a_2\cdots a_n$, by simply ignoring all parentheses and the variable symbol. So the single variable $x$ in term notation is written as the empty string $\lambda$ in string notation. Now for a dummy symbol $a$ in an SRS $\mathcal{R}$ we define

$$DE'_a(\mathcal{R}) = \{\ell \to u \mid u = \mathsf{cap}'_a(r) \vee u \in \mathsf{dec}'_a(r) \text{ for a rule } \ell \to r \in \mathcal{R}\},$$

where

$$\begin{aligned} \mathsf{cap}'_a(\lambda) &= \lambda \\ \mathsf{cap}'_a(fs) &= f\mathsf{cap}'_a(s) \text{ for all symbols } f \text{ with } f \neq a \text{ and all strings } s \end{aligned}$$

$$\begin{aligned}
\mathsf{cap}'_a(as) &= a_\$ \\
\mathsf{dec}'_a(\lambda) &= \emptyset \\
\mathsf{dec}'_a(fs) &= \mathsf{dec}'_a(s) \text{ for all symbols } f \text{ with } f \neq a \text{ and all strings } s \\
\mathsf{dec}'_a(as) &= \mathsf{dec}'_a(s) \cup \{{}_\$a(\mathsf{cap}'_a(s))\}.
\end{aligned}$$

Applied on string rewriting, the transformation $DE'_a$ is called *complete dummy elimination*, just as $DE_a$ applied on term rewriting.

First we show how $DE'_a$ acts on a well-known simple standard example. Let the SRS $\mathcal{R}$ consist of the single self-embedding rule $bb \to bab$. Termination of $DE'_a(\mathcal{R})$ consisting of the two rules

$$b\,b \;\to\; b\,a_\$,\;\; b\,b \;\to\; {}_\$a\,b,$$

is trivial by counting the number of $b$-symbols.

The main theorem about $DE'_a$ is the following.

**Theorem 11.** *Let $\mathcal{R}$ be an SRS having a dummy symbol $a$. Then $\mathcal{R}$ is terminating if and only if $DE'_a(\mathcal{R})$ is terminating.*

In order to prove Theorem 11 we need a general theorem relating termination of TRSs over constants and unary symbols, and SRSs.

The function $\phi$ is defined on terms over constants and unary symbols, yielding strings, is defined as follows:

$$\phi(x) = \lambda,\;\; \phi(c) = c,\;\; \phi(f(t)) = f\phi(t)$$

for all variables $x$, all constants $c$ and all unary symbols $f$. A TRS $\mathcal{R}$ over constants and unary symbols is mapped to an SRS $\phi(\mathcal{R})$ as follows:

$$\phi(\mathcal{R}) \;=\; \{\, \phi(\ell) \to \phi(r) \mid \ell \to r \in \mathcal{R}\,\}.$$

**Theorem 12.** *Let $\mathcal{R}$ be a TRS over constants and unary symbols. Then $\mathcal{R}$ is terminating if and only if $\phi(\mathcal{R})$ is terminating.*

For the proof of this theorem we refer to [19].

The impact of Theorem 12 goes far beyond dummy elimination. In fact Theorem 12 states that proving termination of string rewriting is equivalent to termination of term rewriting as long as no symbols of arity higher than one occur.

Now we are ready to prove Theorem 11.

*Proof.* (of Theorem 11)
Since an SRS is left-linear, by Theorem 6 and Theorem 10 we conclude that $\mathcal{R}$ is terminating if and only if $DE_a(\mathcal{R})$ is terminating. By Theorem 12 this holds if and only if $\phi(DE_a(\mathcal{R}))$ is terminating. By construction $\phi(DE_a(\mathcal{R}))$ and $DE'_a(\mathcal{R})$ coincide, up to renaming of $\diamond_a$ to $a_\$$ and $b_a$ to ${}_\$a$. □

The transformation $DE'_a$ for string rewriting has been implemented in TORPA, version 1.4. As an example, we give the result of TORPA on the same example we considered before:

```
f g  -> f h h a h h g f
f h  -> h g f
h f  -> g g h h a f
f g  -> g g f h
g h  -> h g
f  -> g g h

Apply dummy elimination, result:
f g  -> f h h a$
f g  -> $a h h g f
f h  -> h g f
h f  -> g g h h a$
h f  -> $a f
f g  -> g g f h
g h  -> h g
f  -> g g h


Choose polynomial interpretation f: lambda x.x+1, rest identity
remove: h f  -> g g h h a$
remove: f  -> g g h
Choose polynomial interpretation
f: lambda x.4x
g: lambda x.x+5
h: lambda x.x+2
a$: lambda x.x+1
$a: lambda x.x+1
remove: f g  -> $a h h g f
remove: f h  -> h g f
remove: h f  -> $a f
remove: f g  -> g g f h
Choose polynomial interpretation g: lambda x.x+1, rest identity
remove: f g  -> f h h a$
Choose polynomial interpretation:
g: lambda x.10x,  rest lambda x.x+1
remove: g h  -> h g
Terminating since no rules remain.
```

## 5 Conclusions

We described two techniques to transform a given TRS to another one, in such a way that termination of the given TRS can be concluded from termination of the transformed one, and proving termination of the transformed TRS is often easier than proving termination of the given TRS directly.

Both techniques are easy to implement, and have the nice property that no choice has to be made, so never an explosion of the search space will be caused, and no heuristics have to be developed. On the other hand both techniques have a drawback: they are only applicable for a restricted class of TRSs. For rewriting right-hand sides a rhs is required not to be in normal form, and for complete dummy elimination a dummy symbol is required, i.e., a symbol occurring in a rhs but in no lhs. However, both our techniques may be applied not only as a preprocessor, but also in proofs consisting of several transformations of TRSs. If in a proof search the remaining proof obligation is finding a termination proof for some TRS, then both our techniques may be applied, even if they are not applicable to the original TRS.

One may wonder when to apply these techniques. Our proposal is: whenever you can. For rewriting right-hand sides we proved that the original TRS is terminating if and only if the transformed TRS is terminating, and we are not aware of TRSs for which termination of the transformed TRS is harder to prove than termination of the original TRS, while the converse often occurs. For left-linear TRSs the same can be said for complete dummy elimination. So the only situation where the effect may be negative is for complete dummy elimination for non-left-linear TRSs. Indeed for the single rule $f(x,x) \rightarrow f(a(c),a(d))$ we saw that complete dummy elimination should not be applied, since then the transformed TRS is not terminating while the original one is.

Also combinations of both techniques described in this paper make sense: one easily constructs artificial examples on which both rewriting right-hand sides and complete dummy elimination are applicable, and then they can be applied both. We believe that it does not make sense to investigate which order of application of these techniques is preferred, since examples where this makes a difference are really artificial.

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