Looking for Arbitrage or Term Structures in Frictional Markets

Zhongfei Li[1] and Kai W. Ng[2]

[1] Lingnan (University) College, Sun Yat-Sen University,
Guangzhou 510275, China
lnslzf@zsu.edu.cn

[2] Department of Statistics and Actuarial Science, The University of Hong Kong,
Pokfulam Road, Hong Kong
kaing@hku.hk

Abstract. In this paper we consider a frictional market with finitely many securities and finite and discrete future times. The frictions under consideration include fixed and proportional transaction costs, bid-ask spreads, and taxes. In such a market, we find that whether there exists an arbitrage opportunity does not dependent on the fixed transaction costs. Under a reasonable assumption, the no-arbitrage is equivalent to the condition that the optimal value of some linear programming problem is zero, and to the existence of a so-called consistent term structure. These results permit us to identify and to find arbitrage and consistent term structures in polynomial time. Two linear programming problems are proposed, each of which can identify and find the arbitrage opportunity or the consistent term structure if either exists.

1 Introduction

No-arbitrage is a generally accepted condition in finance. For frictionless financial markets, it is very well understood. The pioneer work of Ross (1976), for example, characterized arbitrage with the existence of positive valuation or pricing operators in discrete time. This approach has been widely adopted in various models, for instance, by Harrison and Kreps (1979), Green and Srivastava (1985), Spremann (1986), and Flåm (2000). In reality, however, financial markets are never short of friction. Investors are required to pay transaction costs, commissions and taxes. Selling and buying prices are differentiated with ask-bid spread. A security is available at a price only for up to a maximum amount. One may buy or sell a stock at an integer number of shares (or an integer number of hundreds of shares). Friction is a de facto matter in financial markets.

Study of arbitrage in frictional markets has attracted more and more attention in recent years and a body of literature has emerged. Garman and Ohlson (1981) extended the work of Ross (1976) to markets with proportional transaction costs and showed that equilibrium prices in markets with proportional transaction costs are equal prices in the corresponding markets with no friction plus a "certain factor". Later, Prisman (1986) studied the valuation of risky

X. Deng and Y. Ye (Eds.): WINE 2005, LNCS 3828, pp. 612–621, 2005.

assets in arbitrage-free economies with taxation. Ross (1987) extended the martingale analysis of no-arbitrage pricing to worlds with taxation in a one-period setting and showed that the absence of arbitrage implies the existence of different shadow prices for income streams that are subject to differing tax treatment. Recently, Dermody and Rockafellar (1991) investigated no-arbitrage pricing of government bonds in the presence of transaction costs and taxes. Dermody and Prisman (1993) extended the results of Garman and Ohlson (1981) to markets with increasing marginal transaction costs and showed the precise relation of the "certain factor" to the structure of transaction costs. Jouini and Kallal (1995) investigated, by means of martingale method, the no-arbitrage problem under transaction costs and short sale constraints respectively. Jaschke (1998) presented arbitrage bounds for the term structure of interests in presence of proportional transaction costs. Ardalan (1999) showed that, in financial markets with transaction costs and heterogeneous information, the no-arbitrage imposes a constraint on the bid-ask spread. Deng, Li and Wang (2000) presented a necessary and sufficient condition for no-arbitrage in a finite-asset and finite-state financial market with proportional transaction costs. This result allows ones to use polynomial time algorithms to look for arbitrage opportunities by applying linear programming techniques. This necessary and sufficient was generalized to the case of multiperiod by Zhang, Xu and Deng (2002).

Although the literature on models with friction is rapidly growing, there are only a few papers dealing with fixed transaction costs and, to the best of our knowledge, rare work on algorithmic study of arbitrage under realistic frictions although it is important, interesting and challenging. In this paper we study the computation issues of arbitrage and term structures with fixed and proportional transaction costs, bid-ask spreads, and taxes.

2 The Market and Preliminaries

Consider a market of n fixed income securities (or bonds) $i = 1, 2, \ldots, n$. Let $0 = t_0 < t_1 < t_2 < \ldots < t_m$ be all the payment dates (or the times to maturities) that can occur, which need not be equidistant. A cash stream is a vector $w = (w_1, w_2, \ldots, w_m)^T$, where ^{T} denotes the transposition of vector or matrix, and w_j is the income received at time t_j and may be positive, zero or negative. Assume that bond i pays the before-tax cash stream $A_i = (a_{1i}, a_{2i}, \ldots, a_{mi})^T$. So we have the $m \times n$ payoff matrix $A = (A_1, A_2, \ldots, A_n)$.

Bond i can be purchased at a current price p_i^a, the so-called ask price. There is also a bid price p_i^b at which bond i can be sold. The difference between these two prices, the so-called bid-ask spread, reflects a type of friction. This friction exists in most economic markets. We form the ask price vector $p^a = (p_1^a, p_2^a, \ldots, p_n^a)^T$ and the bid price vector $p^b = (p_1^b, p_2^b, \ldots, p_n^b)^T$.

The second type of friction considered in this paper is transaction costs including fixed and proportional. We assume that the fixed transaction cost is c_i if bond i is traded and that no fixed transaction cost occurs if no trading of bond i. The c_i is a positive constant regardless of the amount of bond i traded.

Denote $c = (c_1, c_2, \ldots, c_n)^T$ the fixed transaction cost vector. Besides the fixed transaction cost, there is additional transaction cost that is proportional to the amount of the bond traded. Let λ_i^a and λ_i^b be such fees if one dollar of bond i is bought and sold respectively. Here $0 \leq \lambda_i^a, \lambda_i^b < 1, i = 1, 2, \ldots, n$. Denote $\lambda^a = (\lambda_1^a, \lambda_2^a, \ldots, \lambda_n^a)^T$ and $\lambda^b = (\lambda_1^b, \lambda_2^b, \ldots, \lambda_n^b)^T$.

The third type of friction incorporated into our model is taxes. Here we concentrate only on a single investor as a member of just one tax class among many. For all investors in this class, the tax amount at time t_j for holding one unit of bond i in long position is assumed to be t_{ji}^a, and the after-tax income at that time is then $a_{ji} - t_{ji}^a$; whereas the tax amount for holding one unit of bond i in short position is t_{ji}^b as a credit against the obligation to pay a_{ji} at time t_j, and the net after-tax payment to be made is then $a_{ji} - t_{ji}^b$. Let T^a be the $m \times n$ matrix whose entries are t_{ji}^a, and T^b the $m \times n$ matrix whose entries are t_{ji}^b.

Now, the bond market considered in this paper can be described by the 8-tuple $\mathcal{M} = \{p^a, p^b, \lambda^a, \lambda^b, c, A, T^a, T^b\}$.

Every investor in the fixed tax class under consideration will modify his or her position. Let the modification be $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, called also a portfolio, where x_i is the number of units of bond i modified by the investor.

For any portfolio $x = (x_1, x_2, \ldots, x_n)^T$, we let $x_i^+ = \max\{x_i, 0\}$ be the long position taken in bond i (in number of units) and $x_i^- = -\min\{x_i, 0\}$ the short position in bond i, and let $x^+ = (x_1^+, x_2^+, \ldots, x_n^+)^T$ be the vector of buy orders and $x^- = (x_1^-, x_2^-, \ldots, x_n^-)^T$ the vector of sell orders. Then $x_i = x_i^+ - x_i^-$ and $x_i^+ x_i^- = 0$ for $i = 1, 2, \ldots, n$. The complementary constraints $x_i^+ x_i^- = 0$, $i = 1, 2, \ldots, n$ mean that each bond is in either long position or short position.

Define the function $\delta : \mathbb{R} \to \mathbb{R}$ by $\delta(x) = 1$ if $x \neq 0$ or 0 if $x = 0$.

If trading a portfolio $x = (x_1, x_2, \ldots, x_n)^T$, then the investor pays the cost $f(x) := \sum_{i=1}^n (1+\lambda_i^a) p_i^a x_i^+ - \sum_{i=1}^n (1-\lambda_i^b) p_i^b x_i^- + \sum_{i=1}^n c_i \delta(x_i^+ - x_i^-)$ in the present and receive the after-tax gain $g_j(x) := \sum_{i=1}^n (a_{ji} - t_{ji}^a) x_i^+ - \sum_{i=1}^n (a_{ji} - t_{ji}^b) x_i^-$ at future date t_j for $j = 1, 2, \ldots, m$. The after-tax cash stream of gains generated by the portfolio x is then the vector $G(x) := (g_1(x), g_2(x), \ldots, g_m(x))^T$.

When the gains at some dates are positive, no liabilities will be claimed and there exist surplus gains at those dates. These surplus gains can be transformed for use at the succedent dates. Hence we can enlarge the market $\mathcal{M}$ by introducing a dummy bond, indexed by $i = 0$, that costs one dollar at date 0 and immediately pays back the one dollar. This means that $p^a, p^b, \lambda^a, \lambda^b, c, x, A, T^a$, T^b, from now on, replaced by

$$\begin{pmatrix} 1 \\ p^a \end{pmatrix}, \begin{pmatrix} 1 \\ p^b \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda^a \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda^b \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} x_0 \\ x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & T^a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & T^b \end{pmatrix}$$

respectively, where x_0 can be interpreted as the amount we have to deduce from the current income $-f(x)$ to help cover the future obligations $G(x)$. Thus, $f(x) := x_0^+ - x_0^- + \sum_{i=1}^n (1 + \lambda_i^a) p_i^a x_i^+ - \sum_{i=1}^n (1 - \lambda_i^b) p_i^b x_i^- + \sum_{i=1}^n c_i \delta(x_i^+ - x_i^-)$, $g_0(x) := x_0^+ - x_0^-$, $g_j(x) := \sum_{i=1}^n (a_{ji} - t_{ji}^a) x_i^+ - \sum_{i=1}^n (a_{ji} - t_{ji}^b) x_i^-$, $j = 1, 2, \ldots, m$, $G(x) := (g_0(x), g_1(x), \ldots, g_m(x))^T$. Further let $p^+ = (1, (1 + \lambda_1^a) p_1^a, \ldots, (1 +$

$\lambda_n^a)p_n^a)$ and $p^- = (1, (1-\lambda_1^b)p_1^b, \ldots, (1-\lambda_n^b)p_n^b)$. Then, $f(x) = p^+x^+ - p^-x^- + \sum_{i=1}^n c_i\delta(x_i^+ - x_i^-)$ and $G(x) = (A - T^a)x^+ - (A - T^b)x^-$.

For convenience, we use the vector notation $x \geqq y$ to indicate that $x_i \geq y_i$ for all i, and denote by B the lower-triangular $(m+1)\times(m+1)$-matrix whose diagonal and lower-triangular elements all are ones.

Definition 1. *A portfolio x is said to be a strong arbitrage if it has a negative date-0 cost (i.e., $f(x) < 0$) and a non-negative cumulative after-tax cash stream (i.e., $BG(x) \geqq 0$). A portfolio x is said to be a weak arbitrage if it satisfies $f(x) \leq 0$ and $BG(x) \geqq 0$ with at least one strict inequality.*

Definition 2. *A term structure is a discount factor vector u such that $u \in K := \{u \in \mathbb{R}^{m+1} :\ 1 = u_0 \geq u_1 \geq u_2 \geq \ldots \geq u_m \geq 0\}$.*

Lemma 1. *A vector $u = (1, u_1, u_2, \ldots, u_n)^T$ is a term structure if and only if $u^T B^{-1} \geqq 0^T$.*

3 Looking for Arbitrage and Term Structures

Theorem 1. *Whether there exists a strong (weak) arbitrage in the market $\mathcal{M}$ is independent of the fixed transaction costs c. In particular, the market $\mathcal{M}$ excludes strong (weak) arbitrage if and only if the market $\mathcal{M}$ without fixed transaction costs excludes strong (weak) arbitrage.*

Proof. We only prove the case of strong arbitrage. Let c and c' be two different fixed transaction cost vectors and x a strong arbitrage under c. Then $BG(x) \geqq 0$ and $p^+x^+ - p^-x^- + \sum_{i=1}^n c_i\delta(x_i^+ - x_i^-) < 0$. Noting that $\delta(\lambda x) = \delta(x)$ and $G(\lambda x) = G(x)$ for any $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we can take a positive number λ large enough so as to $p^+(\lambda x^+) - p^-(\lambda x^-) + \sum_{i=1}^n c_i'\delta(\lambda x_i^+ - \lambda x_i^-) < 0$ and $BG(\lambda x) \geqq 0$. This immediately means that λx is a strong arbitrage under c'. □

Assumption 1. *$p_i^a \geq p_i^b$ and $t_{ji}^a \geq t_{ji}^b$ for all $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$.*

Assumption 1 demonstrates that, for each bond, the bid price is not higher than the ask price and the tax amount for buying one unit bond is not lower than the tax amount for selling one unit bond. The former condition is usually satisfied. The latter condition may cover a more general case (in many cases the two amounts could well be the same).

Consider the following linear programming problem:

$$(LP) \qquad \begin{cases} \text{minimize} & p^+x^+ - p^-x^- \\ \text{subject to} & B(A - T^a)x^+ - B(A - T^b)x^- \geqq 0,\ x^+,\ x^- \geqq 0. \end{cases}$$

Theorem 2. *Under Assumption 1, the market $\mathcal{M}$ excludes*

(1) strong arbitrage if and only if the optimal value of (LP) is zero;

(2) weak arbitrage if and only if the optimal value of (LP) is zero and its every optimal solution satisfies the equality constraint $B(A-T^a)x^+ - B(A-T^b)x^- = 0$.

Proof. We need only to consider the market $\mathcal{M}$ without fixed transaction costs by Theorem 1 and show only assertion (2). Let x be a portfolio such that $f(x) \leq 0$ and $BG(x) \geqq 0$. Then the corresponding (x^+, x^-) satisfies $p^+x^+ - p^-x^- \leq 0$ and is a feasible solution to problem (LP). Since the optimal value of (LP) is zero, it follows that $p^+x^+ - p^-x^- = 0$ and hence (x^+, x^-) is an optimal solution to (LP). Since every optimal solution of (LP) satisfies the equality constraint $B(A - T^a)x^+ - B(A - T^b)x^- = 0$, we have $f(x) = 0$ and $BG(x) = 0$. Thus, by the definition, the market excludes weak arbitrage. Conversely, assume that the market excludes weak arbitrage. Let (x^+, x^-) be a feasible solution of (LP). For $i = 0, 1, \dots, n$, let $y_i^{\pm} = x_i^{\pm} - \min\{x_i^+, x_i^-\}$. Then $y_i^{\pm} \geqq 0$, $y_i^+ y_i^- = 0$, and, by Assumption 1, it can be checked that

$$p^+y^+ - p^-y^- \leq p^+x^+ - p^-x^-,$$

$$B(A - T^a)y^+ - B(A - T^b)y^- \geqq B(A - T^a)x^+ - B(A - T^b)x^- \geqq 0.$$

Hence, it must holds that $p^+x^+ - p^-x^- \geq 0$ for otherwise $y = y^+ - y^-$ would be a weak arbitrage. This means that the objective function of (LP) is nonnegative at any feasible solution. On the other hand, it is clear that $(x^+, x^-) = (0, 0)$ is feasible to (LP) and at which the objective function vanishes. Hence, the optimal value of (LP) is zero. Furthermore, if $(\hat{x}^+, \hat{x}^-)$ is an optimal solution to (LP), then $p^+\hat{x}^+ - p^-\hat{x}^- = 0$ and $B(A - T^a)\hat{x}^+ - B(A - T^b)\hat{x}^- \geqq 0$. Hence, $B(A - T^a)\hat{x}^+ - B(A - T^b)\hat{x}^- = 0$ because, otherwise, the market would have a weak arbitrage $\hat{y} = \hat{y}^+ - \hat{y}^-$ where $\hat{y}_i^{\pm} = \hat{x}_i^{\pm} - \min\{\hat{x}_i^+, \hat{x}_i^-\}$. □

The dual linear programming problem of (LP) is given by

$$(DP) \quad \begin{cases} \text{maximize} & y^T 0 \\ \text{subject to} & y^T B(A - T^a) \leqq p^+, \ -y^T B(A - T^b) \leqq -p^-, \ y \geqq 0. \end{cases}$$

Let $u = B^T y$. Then $u \in K$ by Lemma 1 and (DP) can be written as

$$(DP)' \quad \begin{cases} \text{maximize} & u^T 0 \\ \text{subject to} & u^T(A - T^a) \leqq p^+, \ u^T(A - T^b) \geqq p^-, \ u \in K. \end{cases}$$

Theorem 3. *Under Assumption 1, the market $\mathcal{M}$ excludes*

(1) strong arbitrage iff there exists a term structure u that satisfies

$$u^T(A - T^a) \leqq p^+ \quad \text{and} \quad u^T(A - T^b) \geqq p^-; \tag{1}$$

(2) weak arbitrage iff there exists a term structure u that satisfies (1) and

$$1 = u_0 > u_1 > u_2 > \ldots > u_m > 0. \tag{2}$$

Proof. Conclusion (1) immediately follows from Theorem 2 and the duality theory of linear programming. Now we show assertion (2) and consider only the market $\mathcal{M}$ without fixed transaction costs by Theorem 1.

Sufficiency. Assume that there exists a term structure u that satisfies (1) and (2). By (1), the market excludes strong arbitrage. Suppose to the contrary that there is a weak arbitrage, which is a portfolio x with zero cost $f(x) = p^+x^+ - p^-x^-$ and non-negative and non-zero cash stream $BG(x) = B(A - T^a)x^+ - B(A - T^b)x^-$. Let $y = (B^{-1})^T u$. Then (2) implies that y is strictly positive and (1) leads to $0 = f(x) = p^+x^+ - p^-x^- \geq u^T(A - T^a)x^+ - u^T(A - T^b)x^- = y^T B(A - T^a)x^+ - y^T B(A - T^b)x^- = y^T (BG(x)) > 0$, a contradiction.

Necessity. Assume that the market excludes weak arbitrage and hence strong arbitrage. By (1), there exists a term structure that satisfies (1). Suppose to the contrary that this term structure does not satisfy (2). Then any solution of the dual problem (DP) has at least one component equal to zero. Since the arithmetical mean of finite many solutions of (DP) is also a solution, all the solutions of (DP) have at least one common component that is equal to zero. That is, there exists an index j corresponding to a date to maturity such that $y_j = 0$ for all solutions y of (DP). Consequently, the linear programming problem

$$(LP)_j \quad \begin{cases} \text{maximize} & y^T e_j \\ \text{subject to} & y^T B(A - T^a) \leqq p^+,\ -y^T B(A - T^b) \leqq -p^-,\ y \geqq 0 \end{cases}$$

has the optimal value of zero, where e_j is the vector with all zero components except for the j-th equal to one. Hence, $(LP)_j$'s dual problem

$$(DP)_j \quad \begin{cases} \text{minimize} & p^+x^+ - p^-x^- \\ \text{subject to} & B(A - T^a)x^+ - B(A - T^b)x^- \geqq e_j,\ x^+, x^- \geqq 0 \end{cases}$$

has the optimal value of zero. An optimal solution of $(DP)_j$ is in fact a portfolio with a zero cost and a non-negative cumulative cash stream whose component at date j is at least one, and hence a weak arbitrage, a contradiction. □

Since linear programming is known to be solvable in polynomial time, We conclude that: *Under Assumption 1, the followings are polynomially solvable: 1) to identify whether there exists a strong (weak) arbitrage, 2) to identify whether there exists a consistent term structure (that satisfies (2)), 3) to find a strong (weak) arbitrage if it does exist, and 4) to find a consistent term structure (that satisfies (2)) if it does exist.*

Computationally this justifies the general belief that any arbitrage opportunity will be short-lived since it will be identified very quickly, be taken advantage of, and subsequently bring economic states to a no-arbitrage one.

We already knew from Theorem 3 that when u is a term structure, each positive component of

$$\varepsilon_i^a = \sum_{j=1}^{m} (a_{ji} - t_{ji}^a)u_j - (1 + \lambda_i^a)p_i^a, \quad i = 1, 2, \ldots, n \tag{3}$$

$$\varepsilon_i^b = (1 - \lambda_i^b)p_i^b - \sum_{j=1}^{m} (a_{ji} - t_{ji}^b)u_j, \quad i = 1, 2, \ldots, n \tag{4}$$

points to a bond i which, at best, remains "wrongly" priced. Furthermore, when $\varepsilon_i^a > 0$, bond i is worthy of being bought because it exhibits positive net return (taking the fixed transaction cost as zero; see Theorem 1 for the reason); when $\varepsilon_i^b > 0$, bond i should be sold.

Expressions (3) and (4) also hold when the summation $\sum_{j=1}^m$ is replaced by $\sum_{j=0}^m$ and hold for $i = 0$ with $\varepsilon_0^a = \varepsilon_0^b = 0$. Thus, the vectors $\varepsilon^a = (\varepsilon_0^a, \varepsilon_1^a, \ldots, \varepsilon_n^a)$ and $\varepsilon^b = (\varepsilon_0^b, \varepsilon_1^b, \ldots, \varepsilon_n^b)$ represent a direction in which existing portfolios should move. This insight inspires us to consider the *minimal pricing error*, i.e. the optimal value of the linear programming problem

$$(LP1) \quad \begin{cases} \text{minimize} & \varepsilon^a \mathbf{1} + \varepsilon^b \mathbf{1} \\ \text{subject to} & u^T(A - T^a) \leqq p^+ + \varepsilon^a, \ u^T(A - T^b) \geqq p^- - \varepsilon^b \\ & \varepsilon^a \geqq 0, \varepsilon^b \geqq 0, u \in K \end{cases}$$

where $\mathbf{1}$ is the vector whose components are all ones. Clearly, $\varepsilon_0^a = \varepsilon_0^b = 0$ in any optimal solution of $(LP1)$.

Replacing u with $z = u^T B^{-1}$ in $(LP1)$, the dual problem of $(LP1)$ is

$$(DP1) \quad \begin{cases} \text{maximize} & -p^+x^+ + p^-x^- \\ \text{subject to} & B(A - T^a)x^+ - B(A - T^b)x^- \geqq 0 \\ & x^+ \leqq \mathbf{1}, \ x^- \leqq \mathbf{1}, \ x^+, x^- \geqq 0. \end{cases}$$

The optimal value of $(DP1)$ can be interpreted as the *maximal arbitrage profit* per unit transaction volume in each bond.

Since $(DP1)$ has a bounded feasible solution set which is nonempty (indeed $x^+ = x^- = 0$ is a feasible solution), it must have an optimal solution. By the duality theory of linear programming, $(LP1)$ and $(DP1)$ both have optimal solutions and their optimal values equal. In other words, *the minimal pricing error equals the maximal arbitrage profit.*

This fact implies that arbitrage possibilities are kept small by traders who try to exploit them. The way traders construct their intended arbitrage transactions determines in which sense the pricing error is kept small.

Theorem 4. *Let $(\overline{x}^+, \overline{x}^-; \overline{u}, \overline{\varepsilon}^a, \overline{\epsilon}^b)$ be a primal-dual optimal solution to $(DP1)$ and r^* the optimal value. Under Assumption 1, if $r^* = 0$, then $\overline{u}$ is a consistent term structure and the market $\mathcal{M}$ excludes strong arbitrage (and weak arbitrage if further $\overline{u}$ satisfies (2)); if $r^* \neq 0$, then $(\overline{x}^+, \overline{x}^-)$ provides a strong arbitrage and the associated arbitrage profit is r^*.*

Proof. Let $r^* = 0$. Then the minimal pricing error, i.e. the optimal value of $(LP1)$ is zero. Since $(\overline{u}, \overline{\varepsilon}^a, \overline{\epsilon}^b)$ is an optimal solution of $(LP1)$, we have $\overline{\varepsilon}^a = \overline{\epsilon}^b = 0$. This implies that $\overline{u}$ is a consistent term structure. Now let $r^* \neq 0$. Then $r^* > 0$ because $(x^+, x^-) = (0, 0)$ is a feasible solution of the maximization problem $(DP1)$ with objective value 0. Hence, $(\overline{x}^+, \overline{x}^-)$ is a strong arbitrage with arbitrage profit r^*. □

Since $(DP1)$ always has an optimal solution and (LP) does not, solving $(DP1)$ is more convenient than solving (LP).

When applying Theorem 4 and finding that $r^* = 0$ and $\overline{u}$ does not satisfy (2), to identify the existence of weak arbitrage and to find a one if it exists in an analogous manner we can use the result stated below.

Consider the linear programming problem $(DP)_j$ and its dual problem $(LP)_j$ in which we let $u = B^T y$ (i.e., the problem $(DP)'$ with a replacement of the objective function by $u^T B^{-1} e_j$).

Theorem 5. *Assume that the market $\mathcal{M}$ excludes strong arbitrage and that Assumption 1 holds. Let $(\overline{x}^{j+}, \overline{x}^{j-}; \overline{u}^j)$ be a primal-dual optimal solution of $(DP)_j$ and r^{j*} its optimal value. If $r^{j*} = 0$ for some $j \in \{0, 1, \cdots, m\}$, then $(\overline{x}^{j+}, \overline{x}^{j-})$ is a weak arbitrage; If $r^{j*} \neq 0$ for all $j \in \{0, 1, \cdots, m\}$, then $\sum_{j=0}^{m} \overline{u}^j/(m+1)$ is a consistent term structure that satisfies (2) and the market $\mathcal{M}$ excludes weak arbitrage.*

Proof. It directly follows from the proof of Theorem 3 (2). □

4 Applications

As a byproduct, the linear programming problem $(DP)_j$ can be used to valuate the Arrow security (or state price) at date (or state) j in the market $\mathcal{M}$ without fixed transaction costs. The optimal value of $(DP)_j$ is the minimal price of a portfolio whose cumulative cash stream is at least e_j. When the market is complete, the minimal cost is just the price of the Arrow security. When the market is incomplete, the cumulative cash stream of the minimal cost portfolio is in fact strictly greater than e_j for some j and hence the minimal cost is the supremum of the price of the Arrow security for state j.

As an application of Theorem 3, we can check the market efficiency, i.e. determine the maximal range of oscillation of bid and ask prices that exclude strong arbitrage. The minimal ask price for the i-th bond can be obtained by finding term structures that satisfy (1):

$$\begin{cases} \text{minimize} & u^T(A - T^a)e_i \\ \text{subject to} & u^T(A - T^a) \leqq p^+,\ u^T(A - T^b) \geqq p^-,\ u \in K \end{cases}$$

Analogously, the maximal bid price for the i-th bond can be obtained by solving the linear programming problem

$$\begin{cases} \text{maximize} & u^T(A - T^b)e_i \\ \text{subject to} & u^T(A - T^a) \leqq p^+,\ u^T(A - T^b) \geqq p^-,\ u \in K. \end{cases}$$

Of cours, if we further want to maintain the absence of weak arbitrage, to get the minimal ask price and the maximal bid price for the i-th bond we need only to substitue the last constraint $u \in K$ with (2).

Finally, as a numerical example we apply the method developed in this paper to a simple economy where $p^+ = (1, 1, 5/2)$, $p^- = (1, 1/2, 2)$, and

$$A - T^a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad A - T^b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solving problem $(DP1)$ yields that a primal-dual optimal solution of $(DP1)$ is $(\overline{x}^+, \overline{x}^-; \overline{u}, \overline{\varepsilon}^a, \overline{\epsilon}^b) = (1, 0, 0; 1, 0, 0; 1, 1, 1/3, 1/6; 0, 0, 0; 0, 0, 0)$ and its optimal value is $r^* = 0$. By Theorem 4, there exists no strong arbitrage and $\overline{u} = (1, 1, 1/3, 1/6)$ is a consistent term structure. However, the consistent term structure $\overline{u}$ does not satisfy (2). To investigate the existence of weak arbitrages, problem $(DP)_j$ has to be solved for all indexes $j = 0, 1, 2, 3$. It turns out that a primal-dual optimal solution $(\overline{x}^{j+}, \overline{x}^{j-}; \overline{u}^j)$ to $(DP)_j$ and its optimal value r^{j*} are as below:

$$\begin{array}{llll} j = 0 : \overline{x}^{0+} = (1, 0, 0), & \overline{x}^{0-} = (0, 0, 1/4), & \overline{u}^0 = (1, 1/2, 1/2, 0), & r^{0*} = 1/2; \\ j = 1 : \overline{x}^{1+} = (4/3, 0, 0), & \overline{x}^{1-} = (0, 0, 1/3), & \overline{u}^1 = (1, 1, 1/3, 1/4), & r^{1*} = 2/3; \\ j = 2 : \overline{x}^{2+} = (0, 0, 1/4), & \overline{x}^{2-} = (0, 0, 0), & \overline{u}^2 = (1, 5/8, 5/8, 0), & r^{2*} = 5/8; \\ j = 3 : \overline{x}^{3+} = (0, 0, 1/3), & \overline{x}^{3-} = (0, 0, 0), & \overline{u}^3 = (1, 5/6, 5/6, 5/6), & r^{3*} = 5/6. \end{array}$$

Because none of these problems has the optimal value of zero, Theorem 5 implies that there exists no weak arbitrage and $\sum_{j=0}^{3} \overline{u}^j / 4 = (1, 71/96, 55/96, 13/48)$ is a consistent term structure that satisfies (2).

5 Conclusion

In this paper, we discussed strong and weak arbitrages and consistent term structures in fractional markets with fixed and proportional transaction costs, bid-ask spreads, and taxes. We concluded that the existence of strong (weak) arbitrages is independent of the fixed transaction costs and that no strong (weak) arbitrage is equivalent to the fact that the optimal value of some linear programming problem is zero (and its very optimal solution makes the inequality constraints becoming equality constraints) and to the existence of consistent term structures (that satisfies (2)). These characterizations extend some known results in discrete time security markets. Further, two linear programming problems are constructed and used to identify and find a strong (weak) arbitrage and a consistent term structure (that satisfies (2)). The computation of the method can be completed in polynomial time by using linear programming techniques.

The results demonstrated in this work are not limited to the model in this paper. For example, the described multi-period setting with one single outcome state per period may be interpreted as a one-period investment problem with n assets and m different outcome states. The methods dealt with them are much the same. It is also interesting to consider computational issues in a more general setting of friction or/and time (period). Such extensions require more sophisticated tools and are worthy of investigation further in future.

Acknowledgements

This work is partially supported by Program for New Century Excellent Talents in University of China (NCET-04-0798), a grant of Foundation for the Author of National Excellent Doctoral Dissertation of China (No. 200267), and a grant of the National Natural Science Foundation of China (No. 70471018).

References

Ardalan, K. (1999): The No-Arbitrage Condition and Financial Markets with Transaction Costs and Heterogeneous Information: the Bid-Ask Spread, *Global Finance Journal* 10, 83–91.

Deng, X. T, Z. F. Li, and S. Y. Wang (2000): On Computation of Arbitrage for Markets with Friction, in *Computing and Combinatorics*, Du, D. Z., P. Eades, V. Estivill-Castro, X. Lin, and A. Sharma, eds. *Lecture Notes in Computer Science* 185. New York: Springer, 309–319.

Dermody, J. C., and E. Z. Prisman (1993): No Arbitrage and Valuation in Market with Realistic Transaction Costs, *Journal of Financial and Quantitative Analysis* 28, 65–80.

Dermody, J. C., and R. T. Rockafellar (1991): Cash Stream Valuation in the Face of Transaction Costs and Taxes, *Mathematical Finance* 1, 31–54.

Flåm, S. C. (2000): Looking For Arbitrage, *International Review of Economics and Finance*, 9, 1–9.

Garman, M. B., and J. A. Ohlson (1981): Valuation of Risky Assets in Arbitrage-Free Economies with Transactions Costs, *Journal of Financial Economics* 9, 271–280.

Green, R. C., and S. Srivastava (1985): Risk Aversion and Arbitrage, *The Journal of Finance* 40, 257–268.

Harrison, J. M., and D. M. Kreps (1979): Martingales and Arbitrage in Multiperiod Securities Markets, *J. Econom. Theory* 20, 381-408.

Ingersoll, J. E. (1987): *Theory of Financial Decision Making.* Savage, Maryland: Rowman & Littlefield, 1987.

Jaschke, S. R. (1998): Arbitrage Bounds for the Term Structure of Interest Rates, *Finance and Stochastics* 2, 29–490.

Jouini, E., and H. Kallal (1995): Martingales and Arbitrage in Securities Markets with Transaction Costs. *Journal of Economic Theory* 66, 178–197.

Prisman, E. Z. (1986): Valuation of risky assets in arbitrage free economies with fictions, *The Journal of Finance* 41, 545–560.

Ross, S. A. (1976): Return, Risk and Arbitrage. In *Risk and Return in Finance*, I. Friend, , and J. L. Bicksler, ed. Cambridge, Mass., Ballinger, 189–218

Ross, S. A. (1987): Arbitrage and Martingales with Taxation, *Journal of Political Economy* 95, 371–391.

Spremann, K. (1986): The Simple Analytics of Arbitrage, in *Capital Market Equilibria*, G. Bamberg, and K. Spremann, ed. New York: Springer-Verlag, 189–207.

Zhang, S. M., C. L., Xu, and X. T. Deng (2002): Dynamic Arbitrage-Free Asset Pricing with Proportional Transaction Costs. *Mathematical Finance* 12, 89–97.