Exchange Market Equilibria with Leontief's Utility: Freedom of Pricing Leads to Rationality

Yinyu Ye*

Stanford University, Stanford, CA 94305, USA

Abstract. This paper studies the equilibrium property and algorithmic complexity of the exchange market equilibrium problem with more general utility functions: piece-wise linear functions, which include Leontief's utility functions. We show that the Fisher model again reduces to the general analytic center problem, and the same linear programming complexity bound applies to approximating its equilibrium. However, the story for the Arrow-Debreu model with Leontief's utility becomes quite different. We show that, for the first time, that solving this class of Leontief exchange economies is equivalent to solving a known linear complementarity problem whose algorithmic complexity status remains open.

1 Introduction

This paper studies the equilibrium property and algorithmic complexity of the Arrow-Debreu competitive equilibrium problem. In this problem, players go to the market with initial endowments of commodities and utility functions. They sell and buy commodities to maximize their individual utilities under a market clearing price. Arrow and Debreu [1] have proved the existence of equilibrium prices when utility functions are concave and commodities are divisible. From then on, finding an efficient algorithm for computing a price equilibrium has became an attractive research area; see [2, 4, 6, 7, 8, 9, 10, 15, 16, 17, 21, 22, 26].

Consider a special case of the the Arrow-Debreu problem, the Fisher exchange market model, where players are divided into two sets: producer and consumer. Consumers have money to buy goods and maximize their individual utility functions; producers sell their goods for money. The price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all the money is spent and all the goods are sold. Eisenberg and Gale [12, 13] gave a convex optimization setting to formulate Fisher's problem with linear utilities. They constructed an aggregated concave objective function that is maximized at the equilibrium. Thus, finding an equilibrium became solving a convex optimization problem, and it could be obtained by using the Ellipsoid method or interior-point algorithms in polynomial time. Here, polynomial time means that one can compute an ϵ-approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$.

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It has turned out that the general Arrow-Debreu problem with linear utilities is also equivalent to a convex optimization setting (see, e.g., Nenakhov and Primak [20] and Jain [15]). The best arithmetic operation bound for solving the Arrow-Debreu problem with linear utilities is $O(n^4 \log \frac{1}{\epsilon})$; see [26]. Moreover, if the input data are rational, then an exact solution can be obtained by solving a system of linear equations and inequalities when $\epsilon < 2^{-L}$, where L is the bit length of the input data. Thus, the arithmetic operation bound becomes $O(n^4 L)$, which is in line with the best complexity bound for linear programming of the same dimension and size.

In this paper we deal more general utility functions: piece-wise linear functions, which include Leontief's utility functions. We show that the Fisher model again reduces to the general analytic center model discussed in [26]. Thus, the same linear programming complexity bound applies to approximating the Fisher equilibrium with these utilities. We also show that the solution to a (pairing) class of Arrow-Debreu problems with Leontief's utility can be decomposed to solutions of two systems of linear equalities and inequalities, and the price vector is the Perron-Frobenius eigen-vector of a scaled Leontief utility matrix. Consequently, if all input data are rational, then there always exists a rational Arrow-Debreu equilibrium, that is, the entries of the equilibrium vector are rational numbers. Additionally, the size (bit-length) of the equilibrium solution is bounded by the size of the input data. This result is interesting since rationality does not hold for Leontief's utility in general. Perhaps more importantly, it also implies, for the first time, that solving this class of Leontief's exchange market problems is equivalent to solving a known linear complementarity problem where its algorithmic complexity status remains open.

2 The Fisher Equilibrium Problem

Without loss of generality, assume that there is 1 unit good from each producer $j \in P$ with $|P| = n$. Let consumer $i \in C$ (with $|C| = m$) has an initial endowment w_i to spend and buy goods to maximize his or her individual linear substitution utility:

$$u_i(x_i) = \min_k \{u_i^k(x_i)\}, \tag{1}$$

where $u_i^k(x_i)$ is a linear function in x_{ij}—the amount of good bought from producer j by consumer i, $j = 1, ..., n$. More precisely,

$$u_i^k(x_i) = (u_i^k)^T x_i = \sum_{j \in P} u_{ij}^k x_{ij}.$$

In particular, the Leontief utility function is the one with

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \; k = j \in P$$

for a given $a_{ik} > 0$, that is, vector u_i^k is an all zero vector except for the kth entry that equals $1/a_{ik}$.

We make the following assumptions: Every consumer's initial money endowment $w_i > 0$, at least one $u_{ij}^k > 0$ for every k and $i \in C$ and at least one $u_{ij}^k > 0$ for every k and $j \in P$. This is to say that every consumer in the market has money to spend and he or she likes at least one good; and every good is valued by at least one consumer. We will see that, with these assumptions, each consumer can have a positive utility value at equilibria. If a consumer has zero budget or his or her utility has zero value for every good, then buying nothing is an optimal solution for him or her so that he or she can be removed from the market; if a good has zero value to every consumer, then it is a "free" good with zero price in a price equilibrium and can be arbitrarily distributed among the consumers so that it can be removed from the market too.

For given prices p_j on good j, consumer i's maximization problem is

$$\begin{array}{rl} \text{maximize} & u_i(x_{i1}, ..., x_{in}) \\ \text{subject to} & \sum_{j\in P} p_j x_{ij} \le w_i, \\ & x_{ij} \ge 0, \quad \forall j. \end{array} \tag{2}$$

Let x_i^* denote a maximal solution vector of (2). Then, vector p is called a Fisher price equilibrium if there is x_i^* for each consumer such that

$$\sum_{i\in C} x_i^* = e$$

where e is the vector of all ones representing available goods on the exchange market.

Problem (2) can be rewritten as an linear program, after introducing a scalar variable u_i, as

$$\begin{array}{rc} \text{maximize} & u_i \\ \text{subject to} & \sum_{j\in P} p_j x_{ij} \le w_i, \\ & u_i - \sum_{j\in P} u_{ij}^k x_{ij} \le 0, \ \forall k, \\ & u_i, \ x_{ij} \ge 0, \quad \forall j. \end{array} \tag{3}$$

Besides (u_i, x_i) being feasible, the optimality conditions of (3) are

$$\begin{array}{c} \lambda_i p_j - \sum_k \pi_i^k u_{ij}^k \ge 0, \ \forall j \in P \\ \sum_k \pi_i^k = 1 \\ \lambda_i w_i = u_i. \end{array} \tag{4}$$

for some $\lambda_i, \ \pi_i^k \ge 0$.

It has been shown by Eisenberg and Gale [12, 11, 13] (independently later by Codenotti et al. [3]) that a Fisher price equilibrium is an optimal Largrange multiplier vector of an aggregated convex optimization problem:

$$\begin{array}{rc} \text{maximize} & \sum_{i\in C} w_i \log u_i \\ \text{subject to} & \sum_{i\in C} x_{ij} = 1, \quad \forall j \in P, \\ & u_i - \sum_{j\in P} u_{ij}^k x_{ij} \le 0, \quad \forall k, i \in C, \\ & u_i, \ x_{ij} \ge 0, \quad \forall i, j. \end{array} \tag{5}$$

Conversely, an optimal Largrange multiplier vector is also a Fisher price equilibrium, which can be seen from the optimality conditions of (5):

$$\begin{array}{rl} p_j - \sum_k \pi_i^k u_{ij}^k \geq 0, & \forall i, j \\ \pi_i^k (\sum_{j \in P} u_{ij}^k x_{ij} - u_i) = 0, & \forall i, k \\ x_{ij}(p_j - \sum_k \pi_i^k u_{ij}^k) = 0, & \forall i, j \\ u_i \sum_k \pi_i^k = w_i, & \forall i. \end{array} \tag{6}$$

for some p_j, the Largarange multiplier of equality constraint of $j \in P$, and some $\pi_i^k \geq 0$, the Largarange multiplier of inequality constraint of $i \in C$ and k. Summing the second constraint over k we have

$$w_i = \sum_k \pi_i^k u_i = \sum_k \pi_i^k \sum_{j \in P} u_{ij}^k x_{ij} = \sum_{j \in P} \left(x_{ij} \sum_k \pi_i^k u_{ij}^k \right), \ \forall i;$$

then summing the third constraint over j we have

$$\sum_{j \in P} p_j x_{ij} = \sum_{j \in P} \left(x_{ij} \sum_k \pi_i^k u_{ij}^k \right) = w_i.$$

This implies that x_i from the aggregate problem is feasible for (3). Moreover, note that π_i^k in (6) equals π_i^k / λ_i in (4). Thus, finding a Fisher price equilibrium is equivalent to finding an optimal Largrange multiplier of (5).

In particular, if each $u_i^k(x_i)$ has the Leontief utility form, i.e.,

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \ \forall k = j \in P$$

for a given $a_{ik} > 0$. Then, upon using u_i to replace variable x_{ij}, the aggregated convex optimization problem can be simplified to

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log u_i \\ \text{subject to} & A^T u \leq e, \\ & u \geq 0; \end{array} \tag{7}$$

where the Leontief matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and variable vector} \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_m \end{pmatrix}. \tag{8}$$

3 The Weighted Analytic Center Problem

In [26] the Eisenberg-Gale aggregated problem was related to the (linear) analytic center problem studied in interior-point algorithms

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^n w_j \log(x_j) \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \tag{9}$$

where the given A is an $m \times n$-dimensional matrix with full row rank, b is an m-dimensional vector, and w_j is the nonnegative weight on the jth variable. Any x who satisfies the constraints is called a primal feasible solution, while any optimal solution to the problem is called a weighted analytic center.

If the weighted analytic center problem has an optimal solution, the optimality conditions are

$$\begin{aligned} Sx &= w, \\ Ax &= b, \;\; x \geq 0, \\ -A^T y + s &= 0, \;\; s \geq 0, \end{aligned} \tag{10}$$

where y and s are the Largrange or KKT multipliers or dual variable and slacks of the dual linear program:

$$\min \;\; b^T y \quad \text{subject to} \quad s = A^T y \geq 0,$$

and S is the diagonal matrix with slack vector s on its diagonals.

Let the feasible set of (9) be bounded and has a (relative) interior, i.e., has a strictly feasible point $x > 0$ with $Ax = b$ (clearly holds for problem (5) and (7)). Then, there is a strictly feasible dual solution $s > 0$ with $s = A^T y$ for some y. Furthermore, [26], based on the literature of interior-point algorithms (e.g., Megiddo and Kojima et al. [19, 18] and Güler [14]), has shown that

Theorem 1. *Let A, b be fixed and consider a solution $(x(w), y(w), s(w))$ of (10) as a mapping of $w \geq 0$ with $\sum_j w_j = 1$. Then,*

- *The mapping of $S^n_{++} = \{x > 0 \in R^n : \; e^T x = 1\}$ to $F_{++} = \{(x > 0, y, s > 0) : \; Ax = b, \; s = A^T y\}$ is one-to-one, continuously and differentiable.*
- *The mapping of $S^n_{+} = \{x \geq 0 \in R^n : \; e^T x = 1\}$ to $F_{+} = \{(x \geq 0, y, s \geq 0) : Ax = b, \; s = A^T y\}$ is upper semi-continuous.*
- *The pair $(x_j(w), s_j(w))$ is unique for any $j \in W = \{j : \; w_j > 0\}$, and*

  $$x'_j(w) s''_j(w) = x''_j(w) s'_j(w) = 0, \;\; \forall j \notin W$$

  and for any two solutions $(x'(w), y'(w), s'(w))$ and $(x''(w), y''(w), s''(w))$ of (10).

From this theorem, we see that, in the Fisher equilibrium problem (5) or (7), $u_i(w)$, the utility value of each consumer, is unique; but the price vector $p(w)$ can be non-unique.

In addition, a modified primal-dual path-following algorithm was developed in [26], for computing an ϵ-solution for any $\epsilon > 0$:

$$\begin{aligned} \|Sx - w\| &\leq \epsilon, \\ Ax &= b, \;\; x \geq 0, \\ -A^T y + s &= 0, \;\; s \geq 0. \end{aligned} \tag{11}$$

Theorem 2. *The primal-dual interior-point algorithm solves the weight analytic center problem (9) in $O(\sqrt{n} \log(n \max(w)/\epsilon))$ iterations and each iteration solves a system of linear equations in $O(nm^2 + m^3)$ arithmetic operations. If Karmarkar's rank-one update technique is used, the average arithmetic operations per iteration can be reduced to $O(n^{1.5} m)$.*

A rounding algorithm is also developed for certain types of problems possessing a rational solution, and the total iteration bound would be $O(\sqrt{n}L)$ and the average arithmetic operation bound would be $O(n^{1.5}m)$ per iteration, where L is the bit-length of the input data A, b, w. These results indicate, for the first time, that the complexity of the Fisher equilibrium problem with linear substitution utility functions is completely in line with linear programming of the same dimension and size.

4 The Arrow-Debreu Equilibrium Problem

The Arrow-Debreu exchange market equilibrium problem which was first formulated by Leon Walras in 1874 [24]. In this problem everyone in a population of m players has an initial endowment of a divisible good and a utility function for consuming all goods—their own and others. Every player sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such equilibrium would exist if the utility functions were concave.

We consider a special class of Arrow-Debreu's problems, where each of the $m = n$ players have exactly one unit of a divisible good for trade (e.g., see [15, 26]), and let player i, $i = 1, ..., m$, bring good $j = i$ and have the linear substitution utility function of (1). We call this class of problems the pairing class. The main difference between Fisher's and Arrow-Debreu' models is that, in the latter, each player is both producer and consumer and the initial endowment w_i of player i is *not* given and will be the price assigned to his or her good i. Nevertheless, we can still write a (parametric) convex optimization model

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log u_i \\ \text{subject to} & \sum_i x_{ij} = 1, \quad \forall j, \\ & u_i \le \sum_j u_{ij}^k x_{ij}, \quad \forall i, k, \\ & u_i,\ x_{ij} \ge 0, \quad \forall i, j, \end{array}$$

where we wish to select weights w_i's such that an optimal Largrange multiplier vector p equals w. It is easily seen that any optimal Largrange multiplier vector p satisfies

$$p \ge 0 \quad \text{and} \quad e^T p = e^T w.$$

For fixed u_{ij}^k, consider p be a map from w. Then, the mapping is from S_+^n to S_+^n, and the mapping is upper semi-continuous from Theorem 1. Thus, there is a $w \in S_+^n$ such that an Largrange multiplier vector $p(w) = w$ from the Kakutani fixed-point theorem (see, e.g., [22, 23, 25]). This may be seen as an alternative, restricted to the case of the linear substitution utility functions, to Arrow-Debreu's general proof of the existence of equilibria.

We now focus on the Arrow-Debreu equilibrium with the (complete) Leontief utility function:

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}},\ \forall k = j = 1, ..., m$$

for a given $a_{ik} > 0$. Recall the parametric convex optimization model (7) where the Leontief matrix A of (8) is a $m \times m$ positive matrix. Let $p \in R^m$ be an optimal Largrange multiplier vector of the constraints. Then, we have

$$\begin{array}{ccc} u_i \sum_j a_{ij} p_j = w_i \ \forall i, & \text{and} & p_j(1 - \sum_i a_{ij} u_i) = 0 \ \forall j, \\ \sum_i a_{ij} u_i \leq 1 \ \forall j, & \text{and} & u_i,\ p_j \geq 0 \ \forall i, j. \end{array}$$

Thus, the Arrow-Debreu equilibrium $p \in R^m$, together with $u \in R^m$, satisfy

$$\begin{aligned} UAp &= p, \\ P(e - A^T u) &= 0, \\ A^T u &\leq e, \\ u,\ p &\geq 0, \end{aligned} \tag{12}$$

where U and P are diagonal matrices whose diagonal entries are u and p, respectively. The Arrow-Debreu theorem implies that nonzero p and u exist for this system of equalities and inequalities, even in general case where $A \geq 0$, that is, some $a_{ik} = 0$ in the Leontief matrix.

5 Characterization of an Arrow-Debreu Equilibrium

If $u_i > 0$ at a solution $(u, p \neq 0)$ of system (12), we must have $p_i > 0$, that is, player i's good must be priced positively in order to have a positive utility value. On the other hand, $p_i > 0$ implies that $\sum_k^m a_{ki} u_k = 1$, that is, good i must be all consumed and gone. Conversely, if $p_i > 0$, we must have $u_i > 0$, that is, player i's utility value must be positive. Thus, there is a partition of all players (or goods) such that

$$B = \{i : \ p_i > 0\} \quad \text{and} \quad N = \{i : \ p_i = 0\}$$

where the union of B and N is $\{1, 2, ..., m\}$. Then, (u, p) satisfies

$$\begin{aligned} (U_B A_{BB}) p_B &= p_B, \\ A_{BB}^T u_B &= e, \\ A_{BN}^T u_B &\leq e, \\ u_B,\ p_B &> 0. \end{aligned}$$

Here A_{BB} is the principal submatrix of A corresponding to the index set B, A_{BN} is the submatrix of A whose rows in B and columns in N. Similarly, u_B and p_B are subvectors of u and p with entries in B, respectively.

Since the scaled Leontief matrix $U_B A_{BB}$ is a (column) stochastic matrix (i.e., $e^T U_B A_{BB} = e^T$), p_B must be the (right) Perron-Frobenius eigen-vector of $U_B A_{BB}$. Moreover, A_{BB} is irreducible because $U_B A_{BB}$ is irreducible and $u_B > 0$, and $U_B A_{BB}$ is irreducible because $p_B > 0$. To summarize, we have

Theorem 3. *Let $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$, A_{BB} be irreducible, and u_B satisfy the linear system*

$$A_{BB}^T u_B = e, \quad A_{BN}^T u_B \leq e, \quad \text{and} \quad u_B > 0.$$

Then the (right) Perron-Frobenius eigen-vector p_B of $U_B A_{BB}$ together with $p_N = 0$ will be an Arrow-Debreu equilibrium. And the converse is also true. Moreover, there is always a rational Arrow-Debreu equilibrium for every such B, that is, the entries of price vector are rational numbers, if the entries of A are rational. Furthermore, the size (bit-length) of the equilibrium is bounded by the size of A.

Our theorem implies that the players in block B can trade among them self and keep others goods "free." In particular, if one player likes his or her own good more than any other good, that is, $a_{ii} \geq a_{ij}$ for all j. Then, $u_i = 1/a_{ii}$, $p_i = 1$, and $u_j = p_j = 0$ for all $j \neq i$, that is, $B = \{i\}$, makes an Arrow-Debreu equilibrium. The theorem thus establishes, for the first time, a combinatorial algorithm to compute an Arrow-Debreu equilibrium with Leontief's utility by finding a right block $B \neq \emptyset$, which is actually a non-trivial complementarity solution to a *linear complementarity problem* (LCP)

$$A^T u + v = e, \; u^T v = 0, \; 0 \neq u, v \geq 0; \tag{13}$$

see, e.g., [5] for more references on LCP. If $A > 0$, then any complementarity solution $u \neq 0$ and $B = \{j : \; u_j > 0\}$ of (13) induce an equilibrium that is the (right) Perron-Frobenius eigen-vector of $U_B A_{BB}$, and it can be computed in polynomial time by solving a linear equation. If A is not strictly positive, then any complementarity solution $u \neq 0$ and $B = \{j : \; u_j > 0\}$, as long as A_{BB} is irreducible, induces an equilibrium. The equivalence between the pairing Arrow-Debreu model and the LCP also implies that LCP (13) always has a complementarity solution $u \neq 0$ such that A_{BB} is irreducible where $B = \{j : \; u_j > 0\}$.

The pairing class of Arrow-Debreu's problems is a rather restrictive class of problems. Consider a general supply matrix $0 \leq G \in R^{m \times n}$ where row i of G represents the multiple goods brought to the market by player i, $i = 1, ..., m$. The pairing model represents the case that $G = I$, the identity matrix, or $G = P$ where P is any permutation matrix of $m \times m$.

What to do if one player brings two different goods? One solution is to copy the same player's utility function twice and treat the player as two players with an identical Leontief utility function, where each of them brings only one type of good. Then, the problem reduces to the pairing model. Thus, we have

Corollary 1. *If all goods are different from each other in the general Arrow-Debreu problem with Leontief's utility, i.e., each column of $G \in R^{m \times n}$ has exactly one positive entry, then there is always a rational equilibrium, that is, the entries of a price vector are rational numbers.*

Now what to do if two players bring the same type of good? In our present pairing class, they will be treated as two different goods, and one can set the same utility coefficients to them so that they receive an identical appreciation from all the players. Again, the problem reduces to the pairing class, which leads to rationality. The difference is that now these two "same" goods may receive two different prices; for example, one is priced higher and the other is at a discount level. I guess this could happen in the real world since two "same" goods may not be really the same and the market does have "freedom" to price them.

6 An Illustrative Example

The rationality result is interesting since the existence of a rational equilibrium is not true for Leontief's utility in Fisher's model with rational data, see the following example converted in Arrow-Debreu's setting, with three consumers each of whom has 1 unit money (the first good) and two other goods (the second and third) brought by a seller (the fourth player) who is only interested in money, adapted from Codenotti et al. [3] and Eaves [9].

$$A = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{5} \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

There is a unique equilibrium for this example, where the utility values of the three consumers are $u_1^* = \frac{2}{3\sqrt{3}}$, $u_2^* = \frac{1}{3} + \frac{1}{3\sqrt{3}}$, $u_3^* = \frac{10}{3} - \frac{10}{3\sqrt{3}}$, and the utility value of the seller $u_4^* = 3$. The equilibrium price for good 1 (money) is $p_1^* = 1$, and for other two goods are $p_2^* = 3(\sqrt{3} - 1)$, and $p_3^* = 3(2 - \sqrt{3})$.

However, if we treat the money from each consumer differently, that is, let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{5} \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then, there are multiple rational equilibria and here are a few:

1. $B = \{1, 4\}$, with $u_1^* = u_4^* = 1$ and $p_1^* = p_4^* = 1$, and $u_2^* = u_3^* = p_2^* = p_3^* = p_5^* = 0$;
2. $B = \{2, 5\}$, with $u_2^* = u_4^* = 1$ and $p_2^* = p_5^* = 1$, and $u_1^* = u_3^* = p_1^* = p_3^* = p_4^* = 0$;
3. $B = \{3, 4\}$, with $u_3^* = 4$ and $u_4^* = 1$ and $p_3^* = p_4^* = 1$, and $u_1^* = u_2^* = p_1^* = p_2^* = p_5^* = 0$;
4. $B = \{1, 2, 3, 4, 5\}$, with an equilibrium $u_1^* = \frac{11}{30}$, $u_2^* = \frac{31}{60}$, $u_3^* = \frac{3}{2}$, $u_4^* = 1$, $p_1^* = \frac{66}{80}$, $p_2^* = \frac{93}{80}$, $p_3^* = \frac{81}{80}$ and $p_4^* = p_5^* = \frac{3}{2}$, .

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