Incentives in Some Coalition Formation Games

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Abstract. The idea of using the core as a model for predicting the formation of coalitions and the sharing of benefits to coordinated activities has been studied extensively. Basic to the concept of the core is the idea of group rationality as embodied by the blocking condition. The predictions given by the core may run into difficulties if some individuals or coalitions may benefit from not blocking "truthfully". This paper investigates this question in games in some games that generalize assignment games. Some positive results are given, and relationships with Vickrey-Clarke-Groves mechanisms are drawn.

1 Introduction

The idea of using the core as a model for assessing the stability of arrangements made within a society has been proved quite fruitful in various contexts. In particular, as in the pioneering analysis of an assignment game by Shapley and Shubik [13], the core may predict which coalitions form and how benefits are shared within each coalition. This occurs in the situations in which diversity in individual tastes or institutional or organizational constraints do not call for the whole society to coordinate.

Underlying the concept of the core is the idea of group rationality as embodied by the blocking condition. There are however two difficulties: existence and manipulability. No stable outcome exists for a large class of games. Also, the predictions given by the core may run into difficulties if some individuals or coalitions benefit from not blocking "truthfully". It turns out that in some coalitional games, both the problems of existence and manipulability can be resolved in a manner to be made precise.

The paper is organized as follows. Section 2 introduces the model and gives some examples. Section 3 and 4 discusses the manipulability of core correspondences, and Section 5 investigates strategy-proof selections of the core.

2 The Coalitional Formation Model

Before introducing the model, it is worth recalling the basic features of the assignment game.

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The assignment game. There are two types of agents, the "buyers" and the "sellers". Each buyer is interested in buying only one item, say a house. The i-th seller values his own house to c_i while the j-th buyer values the same house at h_{ij} dollars. The important data are the "essential" coalitions, the pairs of buyer-seller, and the total value they derive by forming, here $h_{i,j} - c_i$. The possible outcomes of the market specify which pairs of buyer-seller end up making a transaction and at what price. One seeks a stable outcome, meaning that no pair consisting of a buyer and a seller can make an arrangement that is more satisfactory to both than the given one. As shown in [13], a stable outcome always exists, and each one is supported by an equilibrium price. Furthermore, there is a minimum equilibrium price vector and a maximal one. The strategic properties are the following ones:

(a) selecting the minimum equilibrium price vector gives the incentives to all buyers to reveal their true valuations. This holds true because each buyer reaches his incremental value, or Vickrey payment, at the corresponding stable arrangement ([10] and [3]; incremental values are defined later on). The same result holds for the sellers by selecting the maximal price.

(b) a buyer and a seller cannot each one achieve more than his incremental value by misrepresenting jointly their preferences (Demange [5]).

Our purpose is to explore the extent to which properties (a) and (b) generalize to the following situation.

The coalitional game. A finite set of players, the "society", $N = \{1, ..., n\}$, may organize themselves into pairwise disjoint coalitions, where as usual a coalition is a non empty subset of N. Not all coalitions may form say for organizational or institutional reasons : A collection $\mathcal{C}$ will describe the set of admissible coalitions. Throughout the paper, singletons are allowed to form, hence are members of $\mathcal{C}$.

Players only care about the coalition they join and the amount of money they receive or give. Player i's preferences are represented by a utility function u_i defined over the coalitions that include i : $u_i(S)$ gives in term of money the utility for i to be a member of coalition S.

The set of admissible i's utility functions is denoted by $\mathcal{U}_i$. I shall assume that any utility function is admissible (there is no reason for instance to assume utility functions to be increasing: a member of a coalition may dislike a newcomer or may not enjoy too large coalitions). The n-tuple $u = (u_i)_{i \in N}$ is a preference profile and $\mathcal{U} = \bigotimes_{i=1,..,n} \mathcal{U}_i$ the set of admissible profiles.

A coalition S that forms can decide to implement transfers between its members, $(t_i)_{i \in S}$. Player i receives t_i (positive or negative), hence achieves a utility level or payoff of $u_i(S) + t_i$. Feasibility requires transfers to be overall balanced: $\sum_{i \in S} t_i \leq 0$. Thus, if S forms, any payoff $(x_i)_{i \in S}$ that satisfies $\sum_{i \in S} x_i \leq \sum_{i \in S} u_i(S)$ can be achieved by S alone through adequate transfers. This leads us to define the value of S by

$$V_u(S) = \sum_{i \in S} u_i(S). \tag{1}$$

The society may split into several self-sufficient groups owing to individuals' preferences (when players dislike large coalitions for instance) or because of the constraints on coalitions as specified by the set $\mathcal{C}$.[1] As defined in [1], a coalition structure describes how players organize themselves into admissible coalitions that are pairwise disjoint (hence membership to a coalition is exclusive) and self-sufficient (which excludes transfers across coalitions).

Definition 1. *A* $\mathcal{C}$**-(coalition) structure** *of* N *is given by* $a = (\pi, t)$ *where*
$\pi = (S_\ell)_{\ell=1,\ldots,L}$ *is a partition of* N *made of elements in* $\mathcal{C}$*:* $S_\ell \in \mathcal{C}$ *and*
$t = (t_i)_{i\in N}$ *specifies transfers that are balanced within each element of* π*:* $\sum_{i\in S_\ell} t_i \le 0$ *for each* S_ℓ *in* π*.*

The payoff reached by i*, denoted by* $\tilde{u}_i(a)$*, is* $\tilde{u}_i(a) = u_i(S_{\ell(i)}) + t_i$ *where* $S_\ell(i)$ *is the unique coalition of which* i *is a member.*

To analyze which structure will emerge, we rely on the standard stability notion as embodied by the blocking condition.

Definition 2. *Given a profile* u*, the coalition structure* $a = (\pi, t)$ *is said to be* **blocked** *by* T *if*

$$\sum_{i\in T} \tilde{u}_i(a) < V_u(T). \tag{2}$$

A $\mathcal{C}$**-stable structure** *is a* $\mathcal{C}$*-structure that is not blocked by any coalition in* $\mathcal{C}$*. Its payoff vector* $(\tilde{u}_i(a))_{i=1,\ldots,n}$ *is called* $\mathcal{C}$**-stable payoff***. The set of* $\mathcal{C}$*-stable structures is called the* $\mathcal{C}$**-core***.*

The blocking condition (2) is justified as usual: since coalition T can achieve to its members any payoff that sums to $V_u(T)$, if (2) is met, then each individual in T could be made better off than under the structure a. Accounting for the set of admissible coalitions, the stability notion follows.

The collections that guarantee the existence of a $\mathcal{C}$-stable structure for any profile are of particular interest (as introduced by Kaneko and Wooders [8]). The guarantee imposes quite severe restrictions. For example, the collection must not contain a "Condorcet triple" that is three coalitions that intersect each other but whose overall intersection is empty (for instance, no $\mathcal{C}$-stable structure exists for a profile that gives value 1 to each of the three coalitions and zero otherwise). The absence of Condorcet triples is however not sufficient to guarantee a non empty core. A sufficient condition can be stated in terms of the balanced families. Recall that a family $\mathcal{B}$ of subsets of N is said to be *balanced* if there are nonnegative weights on the elements in the family $(\gamma_S)_{S\in\mathcal{B}}$ such that $\sum_{S,i\in S} \gamma_S = 1$ for each i. A partition is a balanced family (take weights equal to 1).

Definition 3. *A collection* $\mathcal{C}$ *satisfies the* **partition property** *if any balanced family composed with coalitions in* $\mathcal{C}$ *contains a partition.*

[1] In technical terms, the game is not super-additive. Recall that super-additivity writes as $V_u(S \cup T) \ge V_u(S) + V_u(T)$, for every $S, T\ s.t. S \cap T = \emptyset$.

Thanks to Scarf theorem [11], the partition property is sufficient for $\mathcal{C}$ to guarantee stability on any set of utility profiles. As shown in [8], the partition property is also necessary if the set $\mathcal{U}$ is rich enough, so that all super-additive transferable utility games are obtained, which is obviously the case with the whole set of possible profile.

Illustrative examples. **1.** Two-sided society (i.e. divided into two subgroups).

In the assignment game, admissible coalitions are singletons and pairs of buyer-seller and the partition property holds.

In a job market as considered by Kelso and Crawford [9], entities are firms on one side and workers on the other side (or buyers and sellers. with buyers who may be interested in buying several objects). Firms may hire several workers but an employed worker works with a single firm : Apart from singletons, a coalition is admissible if it contains a single firm. Stability is not guaranteed if there are at least two firms and three workers : $\{f_1, w_1, w_2\}$, $\{f_2, w_2, w_3\}$, $\{f_1, w_3, w_1\}$ is a Condorcet triple. Some conditions on preferences are needed to ensure the existence of a stable structure (gross substitutes condition). Also various auction mechanisms and their relationships with Vickrey-Clarke-Groves mechanism have been investigated, see for example Bikhchandani and Ostroy [2]).

2. Networks games. Individuals are linked through a network and only the coalitions that are connected can form. If the network is a tree, individuals are partially ordered, then stability is guaranteed (Demange [6]). Stability fails whenever the network contains a cycle since a Condorcet triple is easily found.

It is worth noting that dropping some coalitions from a collection $\mathcal{C}$ has two effects. On one hand less coalitions can block and on the other hand less can form, hence less structures are feasible. As a result, the cores associated with nested collections cannot be compared. According to this remark, selecting some connected sets of a tree gives a collection that satisfies the partition property and may generate interesting cores.

Incremental values. To account for the possibility of a coalition S to split up into elements of $\mathcal{C}$ let us define the superadditive function $\overline{V}_u(S)$ as follows. Denote by $\Pi_{\mathcal{C}}(S)$ the set of partitions of S made of elements in $\mathcal{C}$ (it is nonempty since it contains the partition of singletons). By choosing to partition into π, each element T in the partition can achieve to its members any payoff that sum to $V_u(T)$, which leads to a total of $V_u(\pi) = \sum_{T\in\pi} V_u(T)$. The value $\overline{V}_u(S)$ is obtained by picking out a partition of S that gives the maximal value. Formally,

$$\overline{V}_u(S) = \max_{\pi \in \Pi_{\mathcal{C}}(S)} V_u(\pi).$$

Observe that a coalition S needs to implement transfers across the distinct elements of an optimal partition so as to reach *any* share of $\overline{V}_u(S)$. Hence the super-additive characteristic function $\overline{V}_u$ does not exactly represent our coalitional game. It is nevertheless a useful tool for describing stability: if a coalition does not achieve $\overline{V}_u(T)$, then surely one of its subset in $\mathcal{C}$ can block. Two remarks follow. First, the set of $\mathcal{C}$-stable payoffs is described by the set of linear inequalities (see [8])

$$\sum_{i\in N} x_i \le \overline{V}_u(N) \text{ and } \sum_{i\in S} x_i \ge V_u(S),\ S \in \mathcal{C} \tag{3}$$

Thus, if the $\mathcal{C}$-core is non empty, which holds true under the partition property, the set described by (3) is nonempty. Furthermore, a partition at a stable structure achieves $\overline{V}_u(N)$, that is is *optimal* for N (typically such a partition is unique).

Second, defining the *incremental value* of a coalition T (to the set of all remaining players) by

$$\overline{V}_u(N) - \overline{V}_u(N-T). \tag{4}$$

yields an upper bound on the sum of the payoffs that players in T can achieve at a $\mathcal{C}$-stable payoff. To see this, note simply that if players in T get strictly more than their incremental value, then, by feasibility, $N-T$ gets strictly less than $\overline{V}_u(N-T)$, hence an admissible subset can block.

Similarly the incremental value of a player i to a coalition S possibly smaller than $N-i$, is simply defined as $\overline{V}_u(S+i) - \overline{V}_u(S)$ (to simplify notation, $\{,\}$ is dropped when there is no possible confusion. Also $S+i$ denotes the set $S \cup \{i\}$.)

3 Optimistic Manipulability and Incremental Values

The set of stable structures, if non empty, is typically multi-valued. Therefore, when a player contemplates misrepresenting his preferences he compares two subsets of coalition structures. Various notions of manipulability are possible, depending on how preferences over coalition structures are extended over subsets. We choose here a concept that answers to the following main objection. Although it seems quite surprising at first sight, it may happen that all members of a coalition prefer an alternative that they can block to *any* alternative that is stable. Why, then, should these individuals agree to block? (for a discussion see Demange [5] or [7]). As usual, given a profile u, (v_T, u_{N-T}) denote the profile with functions u_i for individuals not in T and v_i for those in T.

Definition 4. *Optimistic T can manipulate correspondence $\mathcal{S}$ at u if there is v_T and b in $\mathcal{S}(v_T, u_{N-T})$ for which*

$$\tilde{u}_i(b) > \tilde{u}_i(a), \forall\ a \in \mathcal{S}(u), \forall i \in T. \tag{5}$$

Applying the definition to the $\mathcal{C}$-core correspondence, the members of T can manipulate if by misrepresenting their preferences, a structure b that they all prefer to each C-stable structure becomes stable. When applied to a single individual, the definition amounts to assume that the individual evaluates a subset by considering the best element in the set, hence the qualification of "optimistic" manipulation.

The link between strategy-proofness and incremental values is known since the work of Vickrey [14]. The argument extends to correspondences under optimistic manipulability.

Proposition 1. *Let u be a profile. A coalition that achieves at a stable structure its incremental value cannot optimistically manipulate the core.*

Proof. Let coalition T achieve its incremental value at u. By contradiction, suppose T can optimistically manipulate. Denote by $x = \tilde{u}(b)$ the payoff vector (under the "true" preferences u) at the preferred structure b. Surely $\sum_{i \in T} x_i > \overline{V}_u(N) - \overline{V}_u(N-T)$. Also, by feasibility, $\sum_{i \in N} x_i \leq \overline{V}_u(N)$ holds. These inequalities imply $\sum_{i \in N-T} x_i < \overline{V}_u(N-T)$, in contradiction with the stability of b at profile (v_T, u_{N-T}). □

It is worth noting that the core may be manipulable. Consider player 1 in the game V_u: $V_u(2,3) = V_u(2,4) = c$, $V_u(1,3,4) = d$, $V_u(1,2,3,4) = 1$, and all other values are nil. Assume $c \leq 1$ and $d \leq 1$ so that the game is super additive. The incremental value of player 1 is $(1-c)$. The only possible stable payoff at which it can be achieved is $(1-c, c, 0, 0)$. However, for $1 - c < d$, this payoff is not stable (it is blocked by $\{1,3,4\}$) and 1's maximum stable payoff is reached at the extreme point $(2-2c-d, 1-d, c+d-1, c+d-1)$. By lowering his utility d for $\{1,3,4\}$, player 1's maximal payoff is increased possibly up to $(1-c)$.

4 Non Manipulability Result

Proposition 2. *Let collection $\mathcal{C}$ satisfy the partition property. Then, for each coalition T in $\mathcal{C}$ there is a $\mathcal{C}$-stable structure at which that coalition reaches its incremental value. Therefore, no admissible coalition can optimistically manipulate the $\mathcal{C}$-core. This applies in particular to each singleton.*

Proof. Given profile u, let (π, t) be a $\mathcal{C}$-stable structure at which T achieves its maximal payoff, denoted by M_T. Of course, $M_T \geq \overline{V}_u(T)$. We have to show that for T admissible

$$M_T = \overline{V}_u(N) - \overline{V}_u(N-T). \tag{6}$$

If T belongs to π, T gets exactly its value at the structure: $M_T = V_u(T)$, and furthermore $V_u(\pi) = \overline{V}_u(N) = \overline{V}_u(T) + V_u(N-T)$: (6) holds. Suppose that T does not belong to π. Change u_i into v_i for each i in T by increasing $u_i(T)$ everything else equal. Denote $v = (v_T, u_{N-T})$. As long as $M_T \geq V_v(T)$, the structure (π, t) remains stable for profle v. For $M_T < V_v(T)$ (π, t) is no longer stable. Furthermore, T is a member of a partition at any stable structure for v, say (π', t'): otherwise (π', t') would also be stable at profile u and the payoffs to T (computed at u) would be strictly larger than M_T, a contradiction. Thus the value $\overline{V}_v(N)$ is given by:

$$\overline{V}_u(N) \text{ for } M_T \geq V_v(T) \text{ and by } V_v(T) + \overline{V}_u(N-T) \text{ for } M_T < V_v(T). \tag{7}$$

The continuity of the value $\overline{V}_v$ with respect to v at $M_T = V_v(T)$ yields

$$M_T + \overline{V}_u(N-T) = \overline{V}_u(N),$$

the desired result. □

The result can be proved through linear programming methods, by computing some extreme points of the set of stable payoffs (see [7]). The proof provided here is somewhat more intuitive. It makes clear that the crucial property is that the $\mathcal{C}$-core is not empty for any profile. Hence the result may extend to the case with money but without transferable utility (for an extension in an assignment game see Demange and Gale [4]).

Finally, the non manipulability result does not apply to the coalitions that are not admissible, as illustrated by the following example. There are three individuals on a line with 1 in between and $\mathcal{C}$ is the collection of all connected sets (thus only $\{2,3\}$ is not admissible) and $V_u(1,2,3) = 1$, $V_u(i) = 0$, $V_u(1,2) = c_2$ and $V_u(1,3) = c_3$. Assume c_2 and c_3 between 0 and 1 and $c_2 + c_3 > 1$. Consider players 2 and 3. Each one gets his incremental value at the same (extreme) stable payoff $(c_2 + c_3 - 1, 1 - c_3, 1 - c_2\}$. As for the non admissible coalition $\{2,3\}$, its incremental value, equal to 1, is not reached. (check that the maximal payoff is $2-(c_2+c_3) < 1$. Players 2 and 3 can be better off by falsifying their preferences as follows: 2 announces a lower utility for $\{1,2\}$, thereby lowering the value of $\{1,2\}$ hence increasing the incremental payoff of player 3, and similarly 3 makes 2 better off by lowering her utility for $\{1,3\}$.

5 Strategy-Proof Selection

We consider here a collection that satisfies the partition property. A selection of the $\mathcal{C}$-core is a function that assigns a $\mathcal{C}$-stable structure at each profile. Recall that individual i can manipulate f at u if for some v_i in $\mathcal{U}_i$

$$\tilde{u}_i(f(v_i, u_{N-i})\} > \tilde{u}_i(f(u)).$$

Function f is strategy-proof for an individual if this individual cannot manipulate f at any profile.

From the previous result, one easily derives that selecting a preferred core structure for a given player is strategy-proof for that player. Is it possible to get strategy-proofness for more than one player ? The answer is positive in an assignment game as recalled above (property (a)). This section aims at understanding under which conditions on the collection $\mathcal{C}$ a selection of the $\mathcal{C}$-core is strategy-proof for a given subset of players. As a preliminary, note that such a selection has to give to each of these players his incremental value.

Proposition 3. *Let collection $\mathcal{C}$ satisfy the partition property and consider a coalition T. A selection of the $\mathcal{C}$-core is strategy-proof for each player in T if and only if each one reaches his incremental value at any profile.*

An immediate consequence is that for a selection of the core to be strategy-proof for each player the core has to be single valued. This occurs only in the uninteresting case where no coalition apart the singletons are admissible.

A second consequence is that the incentives properties of a selection are much related to the properties of substitutes or complements as defined in [13]. We first recall the definition, restricting to two players, α and β. Denote $S + \alpha\beta$ the set $S \cup \{\alpha, \beta\}$ (and similarly $S - \alpha\beta$ for the set $S - \{\alpha, \beta\}$).

Definition 5. *Two players are* **substitutes** *at u if*

$$\overline{V}_u(S+\alpha\beta) - \overline{V}_u(S+\beta) \leq \overline{V}_u(S+\alpha) - \overline{V}_u(S) \text{ all } S, \alpha \notin S, \beta \notin S \quad (8)$$

They are **complements** *if*

$$\overline{V}_u(S+\alpha\beta) - \overline{V}_u(S+\beta) \geq \overline{V}_u(S+\alpha) - \overline{V}_u(S) \text{ all } S, \alpha \notin S, \beta \notin S \quad (9)$$

In other words, players α and β are substitutes (resp. complements) if the incremental value of one of the players to a coalition is not positively (resp. negatively) affected by the arrival of the other player in the coalition. To see the relationship with the incentives properties, it suffices to observe that the two players α and β simultaneously reach their incremental value at a stable structure only if

$$\overline{V}_u(N) - \overline{V}_u(N-\alpha) + \overline{V}_u(N) - \overline{V}_u(N-\beta) \leq \overline{V}_u(N) - \overline{V}_u(N-\alpha\beta) \quad (10)$$

holds. This inequality says that the sum of the players' incremental values is less than the incremental value of $\alpha\beta$, which is an upper bound on the payoffs to $\alpha\beta$ at a stable structure. Condition (10) is surely satisfied if the players are substitutes (apply (8) to $S = N - \alpha\beta$ and rearrange). At the opposite, complements players can reach their incremental values at the same stable outcome in the very special case where (10) holds as an equality. Let us start with this case.

5.1 Complements

Recall that when $\{\alpha, \beta\}$ is admissible, the incremental value of the coalition is reached. Since also each single player gets at most his own incremental value, surely the reverse of (10) holds. This suggests that the players are complements We give here a direct proof.

Proposition 4. *Let collection $\mathcal{C}$ satisfy the partition property and $\{\alpha, \beta\}$ be in $\mathcal{C}$. Then players α and β are complements.*

Proof. For each $j = \alpha, \beta$, take an optimal $\mathcal{C}$-partition π^j of $S + j$. Add to the family $\mathcal{B}$ composed of all the elements of π^j the admissible coalition $\{\alpha, \beta\}$. $\mathcal{B}$ is composed of coalitions in $\mathcal{C}$. Furthermore it is a balanced family of $S + \alpha\beta$: Each i in $S + \alpha\beta$ belongs to 2 sets (counting twice a set that belongs to 2 partitions). Formally $\mathcal{B}$ is balanced with a weight vector γ equal to half $\delta^{\pi^\alpha} + \delta^{\pi^\beta} + 1_{\{\alpha,\beta\}}$. where δ^π is the vector associated with a partition π ($\delta^\pi_C = 1$ for C element of π and 0 otherwise). Consider the set of balanced weights for $\mathcal{B}$. As shown in [12], an extreme point is associated with a balanced family included in $\mathcal{B}$ and minimal (i.e. containing no balanced family). Under the partition property, a minimal balanced family is a partition. Thus, vector γ is a convex combination of some δ^π associated to partitions of $S + \alpha\beta$: there are μ_π such that

$$\mu_\pi \geq 0, \sum_\pi \mu_\pi = 1 \text{ and } \delta^{\pi^\alpha} + \delta^{\pi^\beta} + 1_{\{\alpha,\beta\}} = 2 \sum_\pi \mu_\pi \delta^\pi \quad (11)$$

Since $\overline{V}_u(S+j) = V_u(\pi^j) = \sum_{C\in\mathcal{C}} \delta_C^{\pi^j} V_u(C)$, $j = \alpha, \beta$, one deduces

$$\overline{V}_u(S+\alpha) + \overline{V}_u(S+\beta) = 2\sum_{\pi} \mu_\pi V_u(\pi) - V_u(\{\alpha,\beta\}) \tag{12}$$

One always has $V_u(\pi) \leq \overline{V}_u(S+\alpha\beta)$ and furthermore for a partition of $S+\alpha\beta$ that contains $\{\alpha,\beta\}$, $V_u(\pi) \leq V_u(\{\alpha,\beta\}) + \overline{V}_u(S)$. Since from (11) surely $\sum_{\pi,\{\alpha,\beta\}.\in\pi} \mu_\pi = 1/2$ and $\sum_{\pi,\{\alpha,\beta\}\notin\pi} \mu_\pi = 1/2$ this gives $2\sum_\pi \mu_\pi V_u(\pi) \leq V_u(S) + \overline{V}_u(S+\alpha\beta)) + \overline{V}_u((\{\alpha,\beta\})$. This inequality together with (12) yields (9), the desired result. □

The lesson to be drawn is the following. Recall that when $\{\alpha,\beta\}$ is admissible, both players cannot be better off than at their preferred outcome by misrepresenting jointly their preferences. However by making precise how coalitions split their benefits, typically at least one of these two players will benefit from misrepresentation (since typically (10) holds as a strict inequality). There is no strategy-proof selection of the core for these two players.

5.2 Incentives for Substitutes Players

Condition (10) is necessary for a strategy-proof selection of the $\mathcal{C}$-core to exist at a given profile. Is it sufficient ? Also, what kind of restrictions on admissible coalitions ensures it is satisfied at all profile ? An answer to this question can be stated in terms of chains, which we now define.

Definition 6. *A* **chain** *between α and β is defined by two families of admissible coalitions, $(S_k,\ k=1,..,\ell+1)$ and $(T_k,\ k=1,..,\ell\ )$ with $\ell \geq 0$, each formed with disjoint elements, that satisfy*
α belongs to S_1 and β to $S_{\ell+1}$, no T_k contains α or β
T_k intersects S_k and S_{k+1}, $k=1,..,\ell$.

For $\ell = 0$, a chain is simply an admissible coalition that contains both α and β. For $\ell = 1$, a chain is given by two disjoint coalitions, S_1 and S_2, one that contains α, the other β, and a third coalition T_1 that intersects both S_1 and S_2 but contains neither α nor β.

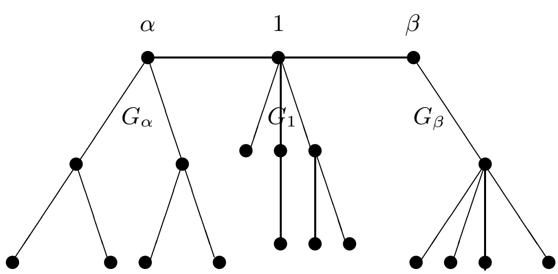

Fig. 1.

Proposition 5. *Let collection $\mathcal{C}$ satisfy the partition property. Consider two players α and β. The following properties are equivalent:*

1. *players α and β are substitutes at any profile*
2. *there is no chain between α and β*
3. *condition (10) is met for α and β at any profile.*

The absence of a chain between two players imply that no admissible coalition contains both. In particular N cannot be admissible. It is easy to check that in an assignment game there is no chain between two sellers or between two buyers. Let us consider a tree and two players α and β who are linked through player 1 as in figure 1. Letting $\mathcal{C}$ be the set of all connected coalitions except those that contain both α and β, one can show that players α and β are substitutes.

Notes and Comments. An extended version and different proofs are in Demange [7].

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