The Price of Anarchy of Cournot Oligopoly

Xiaolei Guo and Hai Yang

Department of Civil Engineering, The Hong Kong University of Science and Technology,
Clear Water Bay, Kowloon, Hong Kong, China
guoxl@ust.hk
cehyang@ust.hk

Abstract. Cournot oligopoly is typically inefficient in maximizing social welfare which is total surplus of consumer and producer. This paper quantifies the inefficiency of Cournot oligopoly with the term "price of anarchy", i.e. the worst-case ratio of the maximum possible social welfare to the social welfare at equilibrium. With a parameterization of the equilibrium market share distribution, the inefficiency bounds are dependent on equilibrium market shares as well as market demand and number of firms. Equilibrium market share parameters are practically observable and analytically manageable. As a result, the price of anarchy of Cournot oligopoly established in this paper is applicable to both practical estimation and theoretical analysis.

1 Introduction

It is well-known that Cournot oligopolistic market equilibrium generally does not maximize social welfare (also referred as social surplus, aggregate surplus), which means that Cournot oligopoly is typically inefficient. This paper quantifies the inefficiency of Cournot oligopoly by looking into the worst-case ratio of the social welfare at social optimum (SO) to the social welfare at equilibrium. The philosophy is the same as the term "price of anarchy" which was firstly introduced to congestion games and selfish routing in networks ([9], [11]).

Earlier studies on the inefficiency properties of oligopoly or monopoly focused mainly on empirical analysis (e.g. [6], [4]), while recent papers began to quantify the inefficiency. Anderson and Renault [1] parameterized the curvature of market demand, and derived bounds on the ratios of deadweight loss and consumer surplus to producer surplus. Their results require marginal costs of producers to be constant. Following a series of papers focusing on the quantification of inefficiency for various games (e.g. [2], [3], [12], [13]), Johari and Tsitsiklis [8] studied the efficiency loss in Cournot games. Their discussion on Cournot oligopoly requires the inverse demand curve to be concave, which does not hold for the widely used constant elasticity demand. This paper gives most general results regarding both cost function and market demand. For cost function, all the results of this paper allow arbitrarily convex cost function (nondecreasing marginal cost function). For market demand, this paper begins with general inverse demand function without concavity or convexity assumption, and then studies concave and convex inverse demand functions separately.

In this paper, we introduce to each firm a parameter demoting its market share at equilibrium. The price of anarchy, or inefficiency bounds of Cournot oligopoly are dependent on three terms, market demand, number of firms and equilibrium market

X. Deng and Y. Ye (Eds.): WINE 2005, LNCS 3828, pp. 246 – 257, 2005.

shares. The parameterization of equilibrium market shares has at least three advantages. First, in contrast to cost functions typically unknown, market shares at equilibrium are observable in practice. With observed equilibrium market shares, the results of this paper can be used for practical estimation of the efficiency loss of Cournot oligopoly. Second, equilibrium market share parameters are analytically manageable, and thereby facilitate theoretical analysis. Finally, equilibrium market share parameters are effective in describing various market structures, which naturally makes the results of this paper comprehensive in terms of market structure.

This paper is organized as follows. Section 2 defines the problem under study and the necessary terms. In Section 3, we consider general inverse demand function without concavity or convexity assumption. We present several important lemmas as the fundamentals of the whole paper, and a theorem bounding the inefficiency of general Cournot oligopoly. Section 4 and Section 5 discuss concave and convex inverse demand functions, respectively. In Section 4, the concavity of the inverse demand function leads to good properties. As a result, we have refined inefficiency bounds that give rise to corollaries regarding several special cases intensively studied by previous papers. In Section 5, the convexity of the inverse demand function also leads to tightened inefficiency bounds, and the application to constant elasticity demand has meaningful analytical results. Finally, our conclusions are contained in Section 6.

2 Cournot Oligopoly: Equilibrium, Social Optimum and the Price of Anarchy

Let there be N firms, $i = 1, 2, ..., N$, which supply a homogeneous product in a noncooperative fashion. Let $p(Q)$, $Q \geq 0$ denote the inverse demand curve, where Q is the total supply in the market. Let $q_i \geq 0$ denote the ith firm's supply, then $Q = \sum_{i=1}^{N} q_i$. Let $f_i(q_i)$ denote the ith firm's total cost of supplying q_i units. A Cournot-Nash equilibrium solution is a set of nonnegative output levels q_1^*, q_2^*,…, q_N^* such that q_i^* is an optimal solution to the following problem for all $i = 1, 2, ..., N$:

$$\underset{q_i \geq 0}{\text{maximize}} \; q_i p\left(q_i + Q_i^*\right) - f_i\left(q_i\right)$$

where

$$Q_i^* = \sum_{j \neq i} q_j^*$$

It is well-known (e.g. [7], [10]) that if $f_i(\cdot)$ is convex and continuously differentiable for $i = 1, 2, ..., N$, the inverse demand function $p(\cdot)$ is strictly decreasing and continuously differentiable, and the revenue curve $Qp(Q)$ is concave, $Q \geq 0$, then (q_1^* , q_2^* , … , q_N^*) is a Cournot-Nash equilibrium solution if and only if

$$\left[p\left(Q^*\right) + q_i^* p'\left(Q^*\right) - f_i'\left(q_i^*\right)\right] q_i^* = 0 \text{ for each } i = 1, 2, ..., N \tag{1}$$

$$p\left(Q^*\right) + q_i^* p'\left(Q^*\right) - f_i'\left(q_i^*\right) \leq 0 \text{ for each } i = 1, 2, ..., N \tag{2}$$

where

$$Q^* = \sum_{i=1}^{N} q_i^*$$

Assumption 1.

(a) The cost function $f_i(\cdot)$ is convex, strictly increasing and continuously differentiable for $i = 1, 2, ..., N$. In addition, $f_i(0) = 0$.

(b) The inverse demand function $p(\cdot)$ is strictly decreasing and continuously differentiable, and the revenue curve $Qp(Q)$ is concave for $Q \geq 0$.

(c) At equilibrium, all firms are active, namely $q_i^* > 0$ for $i = 1, 2, ..., N$.

Note that in Assumption 1, no assumption is made on the concavity or convexity of the inverse demand function $p(\cdot)$. Part (a) and (b) of Assumption 1 ensure the existence and uniqueness of a Cournot-Nash equilibrium solution. $f_i(0) = 0$, implying that fixed cost is not considered, makes this paper consistent with previous literature ([1], [8]). Part (c) is a reasonable and weak assumption. With Part (c) of Assumption 1, the Cournot-Nash equilibrium condition (1)-(2) is simplified to be

$$p(Q^*) + q_i^* p'(Q^*) - f_i'(q_i^*) = 0 \tag{3}$$

Let s_i denote the ith firm's equilibrium market share

$$s_i = q_i^* / Q^*, \ i = 1, 2, ..., N \tag{4}$$

Without loss of generality, let the first firm have the largest equilibrium market share

$$q_1^* = \max \ \{q_i^*, i = 1, 2, ..., N\}$$

$$s_1 = \max \ \{s_i, i = 1, 2, ..., N\} \tag{5}$$

By definition, we have

$$1/N \leq s_1 \leq 1 \tag{6}$$

s_1 plays a fundamental role in both market structure description and theoretical analysis. For market structure description, we have the following observations

(a) For $N = 1$, "$s_1 = 1$" naturally represents the case of monopoly; for $N \geq 2$, "$s_1 = 1$" also represents the case of monopoly, but is a limit case.

(b) "$s_1 \to 0$ ($N \to \infty$)" represents the case of perfect competition.

(c) "$s_1 = 1/N$" represents the symmetric cost case.

Social welfare is defined to be total surplus of consumer and producer, or total benefits minus total costs, which is mathematically formulated as

$$S = \int_0^Q p(x)\,dx - \sum_{i=1}^{N} f_i(q_i) \tag{7}$$

Then the social optimization (SO) problem is given by

$$\underset{q_i \geq 0}{\text{maximize}} \ \int_0^Q p(x)\,dx - \sum_{i=1}^{N} f_i(q_i) \tag{8}$$

Let ($\overline{q}_1, \overline{q}_2, \ldots, \overline{q}_N$) be an optimal solution to problem (8), then the following first-order conditions hold

$$\left[p\left(\overline{Q}\right)-f_i'(\overline{q}_i)\right]\overline{q}_i=0,\ i=1,2,\ldots,N \tag{9}$$

$$p\left(\overline{Q}\right)-f_i'(\overline{q}_i)\leq 0,\ i=1,2,\ldots,N \tag{10}$$

where

$$\overline{Q}=\sum_{i=1}^{N}\overline{q}_i$$

Let $\overline{S}$ and S^* denote the social welfare at optimum and equilibrium, respectively

$$\overline{S}=\int_0^{\overline{Q}}p(x)\,dx-\sum_{i=1}^{N}f_i(\overline{q}_i) \tag{11}$$

$$S^*=\int_0^{Q^*}p(x)\,dx-\sum_{i=1}^{N}f_i\left(q_i^*\right) \tag{12}$$

Define the following ratio

$$\rho=\overline{S}/S^* \tag{13}$$

Clearly, $\rho\geq 1$. This ratio is called the *inefficiency*, or *price of anarchy* of Cournot oligopoly. We will give upper bounds on ρ under different conditions.

With Assumption 1, $f_i(0)=0$, then social welfare defined by (7) becomes

$$S=\int_0^{Q}p(x)\,dx-\sum_{i=1}^{N}\int_0^{q_i}f_i'(x)\,dx \tag{14}$$

In some economics textbook [5], social welfare is directly defined by (14) instead of (7) regardless of the value of fixed cost, which justifies the assumption $f_i(0)=0$.

3 General Inverse Demand Function

In this section, we bound the inefficiency ratio ρ with Assumption 1 only.

Define function $\gamma(\cdot)$ as

$$\gamma(x):\ p(x)+s_1xp'(x)=p\left(\gamma(x)x\right),\ x>0 \tag{15}$$

and assume $\gamma(x)$ to be *bounded* for $x>0$. Since $p(\cdot)$ is strictly decreasing, $\gamma(\cdot)$ is generally well-defined. For example, if $p(\cdot)$ takes the form of $p(Q)=\alpha Q^{-\beta}$, $\alpha>0$, $0<\beta<1$, we have

$$\gamma(x)=\left(1-\beta s_1\right)^{-1/\beta},\ x>0 \tag{16}$$

Lemma 1. *With Assumption 1, let* $k=\overline{Q}/Q^*$*, then it holds* $1<k\leq\gamma\left(Q^*\right)$.

Proof: (a) $k>1$. If it holds $\overline{q}_i>q_i^*$ for each $i=1,2,\ldots,N$, then $\overline{Q}>Q^*$ and $k>1$. Otherwise, without loss of generality, suppose $q_j^*\geq\overline{q}_j$, then we have

$$p\left(\overline{Q}\right)\leq f_j'\left(\overline{q}_j\right) \tag{17}$$

$$\leq f_j'\left(q_j^*\right) \tag{18}$$

$$= p\left(Q^*\right) + q_j^* p'\left(Q^*\right) \tag{19}$$

$$< p\left(Q^*\right) \tag{20}$$

where (17) follows from condition (10), (18) follows from that $f_j(\cdot)$ is convex and thus $f_j'(\cdot)$ is nondecreasing, (19) follows from condition (3), and (20) follows from that $p(\cdot)$ is strictly decreasing and thus $p'\left(Q^*\right) < 0$. $p(\cdot)$ is strictly decreasing, then $p\left(\bar{Q}\right) < p\left(Q^*\right)$ leads to $\bar{Q} > Q^*$ and $k > 1$.

(b) $k \leq \gamma\left(Q^*\right)$. Since $\bar{Q} > Q^*$, thus without loss of generality, suppose $\bar{q}_j > q_j^*$, then

$$p\left(\bar{Q}\right) = f_j'\left(\bar{q}_j\right) \tag{21}$$

$$\geq f_j'\left(q_j^*\right) \tag{22}$$

$$= p\left(Q^*\right) + q_j^* p'\left(Q^*\right) \tag{23}$$

$$\geq p\left(Q^*\right) + s_1 Q^* p'\left(Q^*\right) \tag{24}$$

$$= p\left(\gamma\left(Q^*\right) Q^*\right) \tag{25}$$

where (21) follows from condition (9), (22) follows from $f_j'(\cdot)$ nondecreasing, (23) follows from condition (3), (24) follows from $p'\left(Q^*\right) < 0$, and (25) follows from (15), the definition of $\gamma(\cdot)$. $p(\cdot)$ is strictly decreasing, then $p\left(\bar{Q}\right) \geq p\left(\gamma\left(Q^*\right) Q^*\right)$ leads to $\bar{Q} \leq \gamma\left(Q^*\right) Q^*$ and $k \leq \gamma\left(Q^*\right)$, which completes the proof. ♦

Define function $\theta(\cdot)$ as

$$\theta(x) = \frac{1}{2} x^2 \left(-p'(x)\right) \Big/ \left(\int_0^x p(w)\, dw - x p(x) \right), \quad x > 0 \tag{26}$$

and assume $\theta(x)$ to be *bounded* and *nonzero* for $x > 0$. $\theta(\cdot)$ is a well-defined function. For example, if $p(\cdot)$ takes the form of $p(Q) = \alpha Q^{-\beta}$, $\alpha > 0$, $0 < \beta < 1$, then

$$\theta(x) = (1 - \beta)/2, \quad x > 0 \tag{27}$$

If $p(\cdot)$ takes the form of $p(Q) = p_0 - \alpha Q^{\beta}$, $\alpha > 0$, $\beta > 0$, we have

$$\theta(x) = (1 + \beta)/2, \quad x > 0 \tag{28}$$

where $\beta = 1$ gives a linear $p(\cdot)$ with $\theta(x) = 1$ for $x > 0$.

Lemma 2. *With Assumption 1, the equilibrium social welfare* S^* *satisfies*

$$S^* \geq \left(-p'\left(Q^*\right)\right) \left[\frac{1}{2\theta\left(Q^*\right)} \left(Q^*\right)^2 + \sum_{i=1}^{N} \left(q_i^*\right)^2 \right] \tag{29}$$

Proof: We have

$$\int_0^{q_i^*} f_i'(x)\,dx \le q_i^* f_i'(q_i^*) \tag{30}$$

$$= q_i^*\left(p(Q^*) + q_i^* p'(Q^*)\right) \tag{31}$$

where (30) follows from $f_j'(\cdot)$ nondecreasing, and (31) follows from condition (3). From (14), it holds

$$S^* = \int_0^{Q^*} p(x)\,dx - \sum_{i=1}^N \int_0^{q_i^*} f_i'(x)\,dx \tag{32}$$

Substitute (30)-(31) into (32), we have

$$S^* \ge \int_0^{Q^*} p(x)\,dx - \sum_{i=1}^N q_i^*\left(p(Q^*) + q_i^* p'(Q^*)\right)$$

$$= \int_0^{Q^*} p(x)\,dx - Q^* p(Q^*) + \left(-p'(Q^*)\right)\sum_{i=1}^N (q_i^*)^2$$

$$= \left(-p'(Q^*)\right)\left[\frac{1}{2\theta(Q^*)}(Q^*)^2 + \sum_{i=1}^N (q_i^*)^2\right] \tag{33}$$

where (33) follows from (26), the definition of $\theta(\cdot)$. This completes the proof. ♦

Lemma 3. *With Assumption 1, the following inequality holds*

$$\rho = \frac{\bar{S}}{S^*} \le 1 + \frac{\left(-p'(Q^*)\right)\sum_{i=1}^N (\bar{q}_i - q_i^*)q_i^*}{S^*} \tag{34}$$

Proof: From (11) and (12), we have

$$\bar{S} - S^* = \int_{Q^*}^{\bar{Q}} p(x)\,dx - \sum_{i=1}^N \left(f_i(\bar{q}_i) - f_i(q_i^*)\right) \tag{35}$$

Because $p(\cdot)$ is strictly decreasing, it holds

$$\int_{Q^*}^{\bar{Q}} p(x)\,dx \le (\bar{Q} - Q^*)\,p(Q^*) \tag{36}$$

where "=" may hold only as a limit case. Because $f_i(\cdot)$ is convex, we have

$$f_i(\bar{q}_i) - f_i(q_i^*) \ge (\bar{q}_i - q_i^*) f_i'(q_i^*),\ i = 1, 2, \ldots, N \tag{37}$$

Substitute (36) and (37) into (35), we obtain

$$\bar{S} - S^* \le (\bar{Q} - Q^*)\,p(Q^*) - \sum_{i=1}^N (\bar{q}_i - q_i^*) f_i'(q_i^*)$$

$$= \sum_{i=1}^N (\bar{q}_i - q_i^*)\left(p(Q^*) - f_i'(q_i^*)\right)$$

$$= \left(-p'(Q^*)\right)\sum_{i=1}^N (\bar{q}_i - q_i^*)q_i^* \tag{38}$$

where (38) follows from condition (3). It follows immediately (34) from $\bar{S} - S^* \le \left(-p'(Q^*)\right)\sum_{i=1}^N (\bar{q}_i - q_i^*)q_i^*$, which completes the proof. ♦

Combine Lemma 2 and Lemma 3, substitute (29) into (34), we obtain

$$\rho \le \left[(Q^*)^2 + 2\theta(Q^*)\sum_{i=1}^N \bar{q}_i q_i^*\right] \Big/ \left[(Q^*)^2 + 2\theta(Q^*)\sum_{i=1}^N (q_i^*)^2\right] \tag{39}$$

With $\sum_{i=1}^{N}\bar{q}_i q_i^* \le \sum_{i=1}^{N}\bar{q}_i q_1^* = \bar{Q}q_1^* = ks_1\left(Q^*\right)^2$, $\sum_{i=1}^{N}\left(q_i^*\right)^2 = \left(Q^*\right)^2\sum_{i=1}^{N}s_i^2$, it comes

$$\rho \le \left(1+2\theta\left(Q^*\right)ks_1\right)\Big/\left(1+2\theta\left(Q^*\right)\sum_{i=1}^{N}s_i^2\right) \tag{40}$$

Define parameter γ and θ as

$$\gamma = \max_{x>0}\ \gamma(x) \tag{41}$$

$$\theta = \max_{x>0}\ \theta(x) \tag{42}$$

Theorem 1. *With Assumption 1, the inefficacy ratio* ρ *is bounded as*

$$\rho \le \left(1+2\theta\gamma s_1\right)\Big/\left(1+2\theta\sum_{i=1}^{N}s_i^2\right) \tag{43}$$

$$\le \left(1+2\theta\gamma s_1\right)/\left(1+2\theta m\right) \tag{44}$$

where

$$m = \begin{cases} 1, & N=1 \\ s_1^2+\left(1-s_1\right)^2/\left(N-1\right), & N\ge 2 \end{cases} \tag{45}$$

Proof: The right-hand side of (40) increases with k and $\theta\left(Q^*\right)$, and we have $k\le\gamma$ from Lemma 1 and definition (41) and $\theta\left(Q^*\right)\le\theta$ by definition (42), thus we readily obtain (43) by setting $k=\gamma$ and $\theta\left(Q^*\right)=\theta$ in (40). Then (44) follows from $\sum_{i=1}^{N}s_i^2\ge m$ ($\sum_{i=1}^{N}s_i^2 = s_1^2+\sum_{i=2}^{N}s_i^2 \ge s_1^2+\left(1-s_1\right)^2/\left(N-1\right)=m$, for $N\ge 2$).

This completes that proof. ♦

Theorem 1 gives the price of anarchy for general Cournot oligopoly. The two inefficiency bounds given by (43) and (44) are both determined by three terms, the market demand function (represented by θ and γ), the number of firms N and the market share distribution at equilibrium. While (43) requires the equilibrium market share of each individual firm to be known, (44) needs s_1 only because parameter m captures the worst-case market structure for any s_1, i.e. the first firm have the largest market share s_1 and each other firm have market share $\left(1-s_1\right)/\left(N-1\right)$. In general, (43) applies to practical estimation for which equilibrium market shares are observed in practice, while (44) is applicable to theoretical analysis for which it is impossible and unnecessary to know the equilibrium market shares of individual firms.

Apply (44) to the case of perfect competition, we have the following corollary which states that perfect competition, as expected, is fully efficient in terms of the maximization of social welfare.

Corollary 1. *With Assumption 1, for the case of perfect competition, namely* $s_1\to 0$ *(* $N\to\infty$ *), it holds*

$$\rho\to 1$$

Proof: When $N\to\infty$ and $s_1\to 0$, it follows $m\to 0$ from (45), thus the right-hand side of (44) approaches 1 given that θ and γ are bounded. Since $\rho\ge 1$, it follows immediately $\rho\to 1$, which completes that proof. ♦

4 Concave Inverse Demand Function

In this section, we study the case of concave inverse demand function, and apply our general results to the special cases studied by Johari and Tsitsiklis [8] for comparison.

Lemma 4. *With Assumption 1, if $p(\cdot)$ is concave, then it holds*

(a) $\gamma(x) \le 1+s_1$, *for* $x>0$;

(b)
$$\rho = \frac{\bar{S}}{S^*} \le 1 + \frac{\left(-p'(Q^*)\right)\left[\sum_{i=1}^{N}(\bar{q}_i - q_i^*)q_i^* - \frac{1}{2}(\bar{Q}-Q^*)^2\right]}{S^*} \tag{46}$$

Proof: (a) With Assumption 1, if $p(\cdot)$ is concave, we have

$$p(x) + s_1 x p'(x) \ge p(x+s_1 x),\ x>0 \tag{47}$$

From (15), the definition of $\gamma(\cdot)$, (47) gives

$$p(\gamma(x)x) \ge p((1+s_1)x),\ x>0 \tag{48}$$

Since $p(\cdot)$ is strictly decreasing, (48) simply gives $\gamma(x) \le 1+s_1$ for $x>0$.

(b) With Assumption 1, if $p(\cdot)$ is concave, we have

$$\int_{Q^*}^{\bar{Q}} p(x)d \le (\bar{Q}-Q^*)p(Q^*) - \frac{1}{2}\left(-p'(Q^*)\right)(\bar{Q}-Q^*)^2 \tag{49}$$

Let (49) take the place of (36) in Lemma 3, then (34) of Lemma 3 simply becomes (46), which completes the proof. ♦

Part (b) of Lemma 4 in this section takes the place of Lemma 3 in last section. Combine Lemma 2 and Part (b) of Lemma 4, substitute (29) into (46), we obtain

$$\rho \le \frac{(Q^*)^2 + 2\theta(Q^*)\left[\sum_{i=1}^{N}\bar{q}q_i^* - \frac{1}{2}(\bar{Q}-Q^*)^2\right]}{(Q^*)^2 + 2\theta(Q^*)\sum_{i=1}^{N}(q_i^*)^2} \tag{50}$$

With $\sum_{i=1}^{N}\bar{q}_i q_i^* \le \sum_{i=1}^{N}\bar{q}_i q_1^* = \bar{Q}q_1^* = ks_1(Q^*)^2$, $\sum_{i=1}^{N}(q_i^*)^2 = (Q^*)^2\sum_{i=1}^{N}s_i^2$, it comes

$$\rho \le \left[1+\theta(Q^*)\left(2ks_1-(k-1)^2\right)\right] \Big/ \left(1+2\theta(Q^*)\sum_{i=1}^{N}s_i^2\right) \tag{51}$$

Theorem 2. *With Assumption 1, if $p(\cdot)$ is concave, then it holds*

$$\rho \le \left[1+\theta(2+s_1)s_1\right] \Big/ \left(1+2\theta\sum_{i=1}^{N}s_i^2\right) \tag{52}$$

$$\le \left[1+\theta(2+s_1)s_1\right] \big/ \left[1+2\theta m\right] \tag{53}$$

Proof: The right-hand side of (51) increases with $\theta(Q^*)$ and k, and we have $\theta(Q^*) \le \theta$ by definition (42) and $k \le 1+s_1$ from Lemma 1 and Part (a) of Lemma 4, thus we readily obtain (52) by setting $\theta(Q^*)=\theta$ and $k=1+s_1$ in (51). Then (53) follows immediately from $\sum_{i=1}^{N}s_i^2 \ge m$, which completes the proof. ♦

Theorem 2 gives the price of anarchy of Cournot oligopoly for concave inverse demand function, which is a tightened and refined counterpart of Theorem 1. Like Theorem 1, the two inefficiency bounds given by (52) and (53) are applicable to practical estimation and theoretical analysis, respectively. The bounds do not use parameter γ thanks to Part (a) of Lemma 4, a good property brought about by the concavity of $p(\cdot)$. Without γ in its formulation, (53) gives rise to the following corollaries regarding the worst-case inefficiency for several special cases.

Corollary 2. *With Assumption 1, if* $p(\cdot)$ *is concave, then it holds*

$$\rho \le (2+s_1)s_1/2m \tag{54}$$

where "=" may hold only if $\theta \to \infty$.

Corollary 2 gives the worst-case inefficiency for given N and s_1, and has two important applications, the case of monopoly and the symmetric cost case, both studied intensively by previous papers. For the case of monopoly, namely $N=1$ and $s_1=1$, it follows $m=1$ from (45), thus (54) gives $\rho \le 3/2$. For the symmetric cost case, namely all firms share the same cost function and thereby $s_1 = 1/N$, it follows $m=1/N$ from (45), thus (54) gives $\rho \le 1+1/2N$. These two results are exactly the same as those of Johari and Tsitsiklis (Corollary 17 and 18 of [8]).

Corollary 3. *With Assumption 1, if* $p(\cdot)$ *is concave, then it holds*

$$\rho \le \left(\sqrt{4N+5}+3\right)\Big/4 \tag{55}$$

where "=" may hold only if $\theta \to \infty$ *and* $s_1 = \left(\sqrt{4N+5}+1\right)\Big/2(N+1)$.

Corollary 3 gives the worst-case inefficiency for given N. Particularly, if $N=1$, (55) gives $\rho \le 3/2$, the same as the result of Corollary 2.

Corollary 4. *With Assumption 1, if* $p(\cdot)$ *is concave, then it holds*

$$\rho \le \left(3+\sqrt{1+8\theta}\right)\Big/4 \tag{56}$$

where "=" may hold only if $N \to \infty$ *and* $s_1 = \left(\sqrt{1+8\theta}-1\right)\Big/4\theta$.

Corollary 4 gives the worst-case inefficiency for given θ. The most important application of Corollary 4 is the case of linear (affine) market demand function, which appears in a lot of papers. For linear market demand function, namely $\theta=1$, (56) gives $\rho \le 3/2$ (where "=" holds if and only if $N \to \infty$ and $s_1 = 1/2$). This result (including the condition for "=" to hold) is again the same as that of Johari and Tsitsiklis (Theorem 19 of [8]).

5 Convex Inverse Demand Function

In this section, we study the case of convex inverse demand function including the constant elasticity demand.

Lemma 5. *With Assumption 1, if* $p(\cdot)$ *is convex, then it holds*

$$\rho = \frac{\bar{S}}{S^*} \le 1 + \frac{\left(-p'\left(Q^*\right)\right)\sum_{i=1}^{N}\left(\bar{q}_i - q_i^*\right)q_i^* - \frac{1}{2}\left(\bar{Q} - Q^*\right)\left(p\left(Q^*\right) - p\left(\bar{Q}\right)\right)}{S^*} \tag{57}$$

Proof: With Assumption 1, if $p(\cdot)$ is convex, we have

$$\int_{Q^*}^{\bar{Q}} p(x)\,dx \le \left(\bar{Q} - Q^*\right)p\left(Q^*\right) - \frac{1}{2}\left(\bar{Q} - Q^*\right)\left(p\left(Q^*\right) - p\left(\bar{Q}\right)\right) \tag{58}$$

Then the proof follows the same line as the proof Lemma 3. Let (58) take the place of (36) in Lemma 3, then (34) of Lemma 3 simply becomes (57).

This completes the proof. ♦

In this section, Lemma 5, like Part (b) of Lemma 4 in last section, takes exactly the place of Lemma 3 in Section 3. Combine Lemma 2 and Lemma 5, substitute (29) into (57), we obtain

$$\rho \le \frac{\left(Q^*\right)^2 + 2\theta\left(Q^*\right)\left[\sum_{i=1}^{N}\bar{q}q_i^* - \frac{(k-1)Q^*\left(p\left(Q^*\right) - p\left(\bar{Q}\right)\right)}{2\left(-p'\left(Q^*\right)\right)}\right]}{\left(Q^*\right)^2 + 2\theta\left(Q^*\right)\sum_{i=1}^{N}\left(q_i^*\right)^2} \tag{59}$$

With $\sum_{i=1}^{N}\bar{q}_i q_i^* \le \sum_{i=1}^{N}\bar{q}_i q_1^* = \bar{Q}q_1^* = ks_1\left(Q^*\right)^2$, $\sum_{i=1}^{N}\left(q_i^*\right)^2 = \left(Q^*\right)^2\sum_{i=1}^{N}s_i^2$, it comes

$$\rho \le \left[1 + \theta\left(Q^*\right)\left(2ks_1 - (k-1)h\left(Q^*, k\right)\right)\right] \Big/ \left(1 + 2\theta\left(Q^*\right)\sum_{i=1}^{N}s_i^2\right) \tag{60}$$

where $h: E_2 \to E_1$ is a function defined as

$$h(x,k) = \left(p(x) - p(kx)\right) \big/ \left(-p'(x)x\right),\ x > 0,\ 1 < k \le \gamma(x) \tag{61}$$

Like $\gamma(\cdot)$ and $\theta(\cdot)$, $h(x,k)$ is a well-defined function. For example, if $p(\cdot)$ takes the form of $p(Q) = \alpha Q^{-\beta}$, $\alpha > 0$, $0 < \beta < 1$, we have

$$h(x,k) = \left(1 - k^{-\beta}\right) \big/ \beta,\ x > 0,\ 1 < k \le \gamma(x) \tag{62}$$

Furthermore, $h(x,k)$ have the following relationship with $\gamma(\cdot)$ defined by (15)

$$h\left(x, \gamma(x)\right) = s_1,\ x > 0 \tag{63}$$

Assumption 2. *For any* $1/N \le s_1 \le 1$ *and* $x > 0$, $\left(2ks_1 - (k-1)h(x,k)\right)$ *increases with* k *for* $1 < k \le \gamma(x)$.

With Assumption 2, set $k = \gamma\left(Q^*\right)$ in (60), and make use of (63), we obtain

$$\rho \le \left[1 + \theta\left(Q^*\right)\left(1 + \gamma\left(Q^*\right)\right)s_1\right] \Big/ \left(1 + 2\theta\left(Q^*\right)\sum_{i=1}^{N}s_i^2\right) \tag{64}$$

Theorem 3. *With Assumption 1 and 2, if* $p(\cdot)$ *is convex, then it holds*

$$\rho \le \left[1 + \theta(1+\gamma)s_1\right] \Big/ \left(1 + 2\theta\sum_{i=1}^{N}s_i^2\right) \tag{65}$$

$$\leq \left[1+\theta(1+\gamma)s_1\right] \big/ (1+2\theta m) \tag{66}$$

Proof: The right-hand side of (64) increases with $\gamma(Q^*)$ and $\theta(Q^*)$, and we have $\gamma(Q^*) \leq \gamma$ and $\theta(Q^*) \leq \theta$ by definitions (41)-(42), thus we readily obtain (65) by setting $\gamma(Q^*) = \gamma$ and $\theta(Q^*) = \theta$ in (64). Then, like the proof of Theorem 1 and 2, (66) follows immediately from $\sum_{i=1}^{N} s_i^2 \geq m$, which completes the proof. ♦

Theorem 3 gives the price of anarchy of Cournot oligopoly for convex inverse demand function. Like Theorem 1 and 2, the two inefficiency bounds given by (65) and (66) are applicable to practical estimation and theoretical analysis, respectively.

Apply (66) to the constant elasticity demand, we have the following corollary.

Corollary 5. *Suppose* $p(\cdot)$ *takes the constant elasticity form,* $p(Q) = \alpha Q^{-\beta}$, $\alpha > 0$, $0 < \beta < 1$. *With Assumption 1, it holds*

$$\rho \leq \frac{1+\frac{1}{2}(1-\beta)\left[1+(1-\beta s_1)^{-1/\beta}\right]s_1}{1+(1-\beta)m} \tag{67}$$

Furthermore, monopoly $s_1 = 1$ *gives the worst-case inefficiency*

$$\rho \leq \left[3-\beta+(1-\beta)^{1-1/\beta}\right] \big/ 2(2-\beta) \tag{68}$$

and $\beta = 0.8670$ *gives the overall worst-case inefficiency* $\rho \leq 1.5427$.

Corollary 5 does not require Assumption 2 explicitly because constant elasticity demand automatically meets Assumption 2. From Corollary 5, for a constant elasticity demand, the worst-case inefficiency can be attained only by monopoly, and the overall worst-case inefficiency is that the optimal social welfare is 1.5427 times of the equilibrium social welfare.

6 Conclusion

This paper studies the price of anarchy of Cournot oligopoly, and gives most general results regarding cost function and market demand. General, concave and convex inverse demand functions are studied separately. The general inefficiency bounds given by Theorem 1-3 are determined by three terms, market demand, number of firms and equilibrium market shares. These general results are applicable to both practical estimation and theoretical analysis thanks to the practical observability and analytical manageability of the equilibrium market share parameters. Furthermore, since equilibrium market share parameters can effectively describe various special market structures, the general results can be readily applied to special cases such as monopoly, perfect competition and the symmetric cost case.

One point worth mentioning is that the general inefficiency bounds given by Theorem 1-3 are *not* independent of the cost functions of firms because the cost characteristics of firms to a large extent determine the equilibrium market shares. However, with these general results, equilibrium market share parameters are enough for practical estimation and theoretical analysis of the price of anarchy of Cournot oligopoly. Thus it is unnecessary to know any information on the cost functions (as long as the marginal costs are nondecreasing with output). In addition, inefficiency bounds independent of cost functions can simply be obtained by searching the worst-case equilibrium market share distribution, like in Corollary 3-4 where the inefficiency bounds are dependent on θ or N only.

References

1. Anderson, S. P. and R. Renault. Efficiency and surplus bounds in Cournot competition. Journal of Economic Theory 113, 2003, pp. 253-264.
2. Correa, J. R., A. S. Schulz, and N. E. Stier-Moses. Selfish routing in capacitated networks. Mathematics of Operations Research 29, 2004, pp. 961-976.
3. Chau, C. K. and K. M. Sim. The price of anarchy for non-atomic congestion games with symmetric cost maps and elastic demands. Operations Research Letters 31, 2003, pp. 327-334.
4. Cowling, K. and D. Mueller. The social costs of monopoly. Econom. J. 88, 1978, 727-748.
5. Geoffrey, A. J. and J. R. Philip. Advanced Microeconomic Theory. Addison Wesley Longman, 2001.
6. Harberger, A. C. Monopoly and resource allocation. American Economic Review 44, 1954, pp. 77-87.
7. Harker, P. T. A variational inequality approach for the determination of oligopolistic market equilibrium. Mathematical Programming 30, 1984, pp. 105-111.
8. Johari, R. and J. N. Tsitsiklis. Efficiency loss in Cournot games. Technical Report LIDS-P-2639, Laboratory for Information and Decision Systems, MIT, January 2005.
9. Koutsoupias, E. and C. H. Papadimitriou. Worst-case equilibria. In Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science, Trier, Germany, Vol. 1563 of Lecture Notes in Computer Science, 404-413. Springer, Berlin, 1999.
10. Murphy, F. H., H. D. Sherali and A. L. Soyster. A mathematical programming approach for the determination of oligopolistic market equilibrium. Mathematical Programming 24, 1982, pp. 92-106.
11. Papadimitriou, C. H. Algorithms, games, and the Internet. In Proceedings of the 33rd Annual ACM Symposium on Theory of Computing (STOC), Heraklion, Greece, pp. 749-753. ACM Press, New York, NY, 2001.
12. Roughgarden, T. and E. Tardos. How bad is selfish routing? Journal of the ACM 49, 2002, pp. 236-259.
13. Roughgarden, T. The price of anarchy is independent of the network topology. Journal of Computer and System Sciences 67, 2003, pp. 341-364.