Walrasian Equilibrium: Hardness, Approximations and Tractable Instances

Ning Chen and Atri Rudra

Department of Computer Science & Engineering, University of Washington, Seattle, USA
{ning, atri}@cs.washington.edu

Abstract. We study the complexity issues for Walrasian equilibrium in a special case of combinatorial auction, called single-minded auction, in which every participant is interested in only one subset of commodities. Chen et al. [5] showed that it is NP-hard to decide the existence of a Walrasian equilibrium for a single-minded auction and proposed a notion of approximate Walrasian equilibrium called relaxed Walrasian equilibrium. We show that every single-minded auction has a $\frac{2}{3}$-relaxed Walrasian equilibrium proving a conjecture posed in [5]. Motivated by practical considerations, we introduce another concept of approximate Walrasian equilibrium called weak Walrasian equilibrium. We show it is strongly NP-complete to determine the existence of δ-weak Walrasian equilibrium, for any $0 < \delta \leq 1$.

In search of positive results, we restrict our attention to the tollbooth problem [15], where every participant is interested in a single path in some underlying graph. We give a polynomial time algorithm to determine the existence of a Walrasian equilibrium and compute one (if it exists), when the graph is a tree. However, the problem is still hard for general graphs.

1 Introduction

Imagine a scenario where a movie rental store is going out of business and wants to clear out its current inventory. Further suppose that based on their rental records, the store has an estimate of which combinations (or bundles) of items would interest a member and how much they would be willing to pay for those bundles. The store then sets prices for each individual item and allocates bundles to the members (or buyers) with the following "fairness" criterion— given the prices on the items, no buyer would prefer to be allocated any bundle other than those currently allocated to her. Further, it is the last day of the sale and it is imperative that no item remain after the sale is completed. A natural way to satisfy this constraint is to give away any unallocated item for free. This paper looks at the complexity of computing such allocations and prices.

The concept of "fair" pricing and allocation in the above example is similar to the concept of market equilibrium which has been studied in economics literature for more than a hundred years starting with the work of Walras [27]. In this

X. Deng and Y. Ye (Eds.): WINE 2005, LNCS 3828, pp. 141–150, 2005.

model, buyers arrive with an initial endowment of some item and are looking to buy and sell items. A market equilibrium is a vector of prices of the items such that the market clears, that is, no item remains in the market. Arrow and Debreu showed more than half a century ago that when the buyers' utility functions are concave, a market equilibrium always exists [3] though the proof was not constructive. A polynomial time algorithm to compute a market equilibrium for linear utility functions was given last year by Jain [18]. We note that in this setup the items are divisible while we will consider the case of indivisible items.

Equilibrium concepts [22] are an integral part of classical economic theory. However, only in the last decade or so these have been studied from the computational complexity perspective. This perspective is important as it sheds some light on the feasibility of an equilibrium concept. As Kamal Jain remarks in [18]— "If a Turing machine cannot efficiently compute it, then neither can a market". A small sample of the recent work which investigates the efficient computation of equilibrium concepts appears in the following papers: [8, 9, 18, 5, 24].

In this paper, we study the complexity of computing a Walrasian equilibrium [27], one of the most fundamental economic concepts in market conditions. Walrasian equilibrium formalizes the concept of "fair" pricing and allocation mentioned in the clearance sale example. In particular, given an input representing the valuations of bundles for each buyer, a Walrasian equilibrium specifies an allocation and price vector such that any unallocated item is priced at zero and all buyers are satisfied with their corresponding allocations under the given price vector. In other words, if any bundle is allocated to a buyer, the profit[1] made by the buyer from her allocated bundle is no less than the profit she would have made if she were allocated any other bundle for the same price vector.

In the clearance sale example, we assumed that every buyer is interested in some bundles of items— this is the setup for combinatorial auctions which have attracted a lot of attention in recent years due to their wide applicability [26, 7]. In the preceding discussions, we have disregarded one inherent difficulty in this auction model— the number of possible bundles are exponential and thus, even specifying the input makes the problem intractable. In this paper, we focus on a more tractable instance called *single-minded auction* [20] with the linear pricing scheme[2]. In this special case, every buyer is interested in a single bundle. Note that the inputs for these auctions can be specified quite easily. The reader is referred to [2, 1, 23, 5, 13] for more detailed discussions on single-minded auctions.

If bundles can be priced arbitrarily, a Walrasian equilibrium always exists for combinatorial auctions [6, 21]. However, if the pricing scheme is restricted to be linear, a Walrasian equilibrium may not exist. Several sufficient conditions for the existence of a Walrasian equilibrium with the linear pricing scheme have been studied. For example, if the valuations of buyers satisfy the *gross substitutes* condition [19], the *single improvement* condition [14], or the *no complementarities* condition [14] then a Walrasian equilibrium is guaranteed to exist.

[1] The profit is the difference between how much the buyer values a bundle and the price she pays for it.

[2] The price of a bundle is the sum of prices of items in the bundle.

Bikhchandani et al. [4] and Chen et al. [5] gave an exact characterization for the existence of a Walrasian equilibrium— the total value of the optimal allocation of bundles to buyers, obtained from a suitably defined integer program, is equal to the value of the corresponding relaxed linear program. These conditions are, however, not easy to verify in polynomial time even for the single-minded auction. In fact, checking the existence of a Walrasian equilibrium for single-minded auctions is NP-hard [5].

The intractability result raises a couple of natural questions: (1) Instead of trying to satisfy all the buyers, what fraction of the buyers can be satisfied ? (2) Can we identify some structure in the demands of the buyers for which a Walrasian equilibrium can be computed in polynomial time? In this paper we study these questions and have the following results:

- *Conjecture on relaxed Walrasian equilibrium.* Chen et al. [5] proposed an approximation of Walrasian equilibrium, called *relaxed Walrasian equilibrium.* This approximate Walrasian equilibrium is a natural approximation where instead of trying to satisfy all buyers, we try to satisfy as many buyers as we can. [5] gave a simple single-minded auction for which no more than 2/3 of the buyers can be satisfied. They also conjecture that this ratio is tight— that is, for any single-minded auction one can come up with an allocation and price vector such that at least 2/3 of the buyers are satisfied. In our first result we prove this conjecture. In fact we show that such an allocation and price vector can be computed in polynomial time.
- *Weak Walrasian equilibrium.* There is a problem with the notion of relaxed Walrasian equilibrium— it does not require all the winners to be satisfied. This is infeasible in practice. For instance in the clearance sale example, the store cannot sell a bundle to a buyer at a price greater than her valuation. In other words, at a bare minimum all the winners have to be satisfied. With this motivation, we define a stronger notion of approximation called *weak Walrasian equilibrium* where we try to maximize the number of satisfied buyers subject to the constraint that all winners are satisfied. In our second result we show that computing a weak Walrasian equilibrium is strongly NP-complete. We achieve this via a reduction from the Independent Set problem. Independently, Huang and Li [17] showed the NP-hardness result (but not strongly NP-hard) by a reduction from the problem of checking the existence of a Walrasian equilibrium.
- *Tollbooth problem.* With the plethora of negative results, we shift our attention to special cases of single-minded auctions where computing Walrasian equilibrium is tractable. With this goal in mind, we study the tollbooth problem introduced by Guruswami et al. [15] where the bundle every buyer is interested in is a path in some underlying graph. We show that in a general graph, it is still NP-hard to compute a Walrasian equilibrium given that one exists. We then concentrate on the special case of a tree, and show that we can determine the existence of a Walrasian equilibrium and compute one (if it exists) in polynomial time. Essentially, we prove this result by first showing that the optimal allocation can be computed by a polynomial time

Dynamic Program. In fact, the optimal allocation is equivalent to the maximum weighted edge-disjoint paths in a tree. A polynomial time divide and conquer algorithm for the latter was given by Tarjan [25]. Another Dynamic Program algorithm for computing edge-disjoint paths with maximum cardinality in a tree was shown by Garg et al. [12]. However, their algorithm crucially depends on the fact that all paths have the same valuation while our setup is the more general weighted case.

The paper is organized as follows. In Section 2, we review the basic properties of Walrasian equilibrium for single-minded auctions. In Section 3, we study two different type of approximations— relaxed Walrasian equilibrium and weak Walrasian equilibrium. In Section 4, we study the tollbooth problem— for the different structure of the underlying graphs, we give different hardness or tractability results. We conclude our work in Section 5. Due to space limitations, we omit the proofs in this paper. The proofs will appear in the full version of the paper.

2 Preliminaries

In a *single-minded auction* [20], an auctioneer sells m heterogeneous commodities $\Omega = \{\omega_1, \ldots, \omega_m\}$, with unit quantity each, to n potential buyers. Each buyer i desires a fixed subset of commodities of Ω, called *demand* and denoted by $d_i \subseteq \Omega$, with *valuation* (or *cost*) $c_i \in \mathbb{R}^+$. That is, c_i is the maximum amount of money that i is willing to pay in order to win d_i.

After receiving the submitted tuple (d_i, c_i) from each buyer i (the *input*), the auctioneer specifies the tuple (X, p) as the *output* of the auction:

- *Allocation vector* $X = (x_1, \ldots, x_n)$, where $x_i \in \{0, 1\}$ indicates if i wins d_i ($x_i = 1$) or not ($x_i = 0$). Note that we require $\sum_{i:\ \omega_j \in d_i} x_i \leq 1$ for any $\omega_j \in \Omega$. $X^* = (x_1^*, \ldots, x_n^*)$ is said to be an *optimal allocation* if for any allocation X, we have
$$\sum_{i=1}^{n} c_i \cdot x_i^* \geq \sum_{i=1}^{n} c_i \cdot x_i.$$
That is, X^* maximizes total valuations of the winning buyers.
- *Price vector* $p = (p(\omega_1), \ldots, p(\omega_m))$ (or simply, $(p_1, \ldots, p_m)$) such that $p(\omega_j) \geq 0$ for all $\omega_j \in \Omega$. In this paper, we consider linear pricing scheme, *i.e.*, $p(\Omega') = \sum_{\omega_j \in \Omega'} p(\omega_j)$, for any $\Omega' \subseteq \Omega$.

If buyer i is a winner (*i.e.*, $x_i = 1$), her (quasi-linear) *utility* is $u_i(p) = c_i - p(d_i)$; otherwise (*i.e.*, i is a loser), her *utility* is zero.

We now define the concept of Walrasian equilibrium.

Definition 1. (Walrasian equilibrium) [14] *A* Walrasian equilibrium *of a single-minded auction is a tuple (X, p), where $X = (x_1, \ldots, x_n)$ is an allocation vector and $p \geq 0$ is a price vector, such that*

1. *$p(X_0) = 0$, where $X_0 = \Omega \setminus \left(\bigcup_{i:\ x_i=1} d_i\right)$ is the set of commodities that are not allocated to any buyer.*
2. *The utility of each buyer is maximized. That is, for any winner i, $c_i \geq p(d_i)$, whereas for any loser i, $c_i \leq p(d_i)$*

Chen et al. [5] showed that checking the existence of a Walrasian equilibrium in general is hard.

Theorem 1. [5] *Determining the existence of a Walrasian equilibrium in a single-minded auction is NP-complete.*

Lehmann et al. [20] showed that in a single-minded auction, the computation of the optimal allocation is also NP-hard. Indeed, we have the following relation between Walrasian equilibrium and optimal allocation.

Theorem 2. [5] *For any single-minded auction, if there exists a Walrasian equilibrium (X, p), then X must be an optimal allocation.*

Due to the above theorem, we know that computing a Walrasian equilibrium is at least as hard as computing the optimal allocation. However, if we know the optimal allocation X^*, we can compute the price vector by the following linear program:

$$\begin{aligned}
&\sum_{j:\ \omega_j \in d_i} p_j \leq c_i, \quad \forall\ x_i^* = 1 \\
&\sum_{j:\ \omega_j \in d_i} p_j \geq c_i, \quad \forall\ x_i^* = 0 \\
&p_j \geq 0, \quad \forall\ \omega_j \in \Omega \\
&p_j = 0, \quad \forall\ \omega_j \in \Omega \setminus \left(\bigcup_{i:\ x_i^*=1} d_i \right)
\end{aligned}$$

Any feasible solution p of the above linear program defines a Walrasian equilibrium (X^*, p). In fact, note that if a Walrasian equilibrium exists, we can compute one which maximizes the revenue by adding the following objective function to the above linear program

$$\max \sum_{j:\ \omega_j \in \Omega} p_j$$

Thus, we have the following conclusion:

Corollary 1. *For any single-minded auction, if the optimal allocation can be computed in polynomial time, then we can determine if a Walrasian equilibrium exists or not and compute one (if it exists) which maximizes the revenue in polynomial time.*

3 Approximate Walrasian Equilibrium

Theorem 1 says that in general computing a Walrasian equilibrium is hard. In this section, we consider two notions of approximation of a Walrasian equilibrium. The first one called *relaxed Walrasian equilibrium*, due to Chen et al. [5], tries to maximize the number of buyers for which the second condition of Definition 1 is satisfied. We also introduce a stronger notion of approximation called *weak Walrasian equilibrium* that in addition to being a relaxed Walrasian equilibrium has the extra constraint that the second condition of Definition 1 is satisfied for *all* winners. We now define these notions and record some of their properties.

For any tuple (X, p), we say buyer i is *satisfied* if her utility is maximized. That is, if $x_i = 1$, then $p(d_i) \leq c_i$; if $x_i = 0$, then $p(d_i) \geq c_i$. For any single-minded auction $\mathcal{A}$, let $\delta_{\mathcal{A}}(X, p)$ denote the number of satisfied buyers under (X, p). In the following discussion, unless specified otherwise, we assume all unallocated commodities are priced at zero.

3.1 Relaxed Walrasian Equilibrium

We first define the notion of relaxed Walrasian equilibrium [5].

Definition 2. (relaxed Walrasian equilibrium) *Given any* $0 < \delta \leq 1$*, a* δ-relaxed Walrasian equilibrium *of single-minded auction* $\mathcal{A}$ *is a tuple* (X, p) *that satisfies the following conditions:*

- $p(X_0) = 0$*, where* $X_0 = \Omega \setminus \left(\bigcup_{i:\ x_i=1} d_i\right)$.
- $\frac{\delta_{\mathcal{A}}(X,p)}{n} \geq \delta$*, where* n *is the number of buyers in* $\mathcal{A}$.

Note that a Walrasian equilibrium is a 1-relaxed Walrasian equilibrium. Theorem 1 implies that it hard to maximize $0 < \delta \leq 1$ such that δ-relaxed Walrasian equilibrium exists for a single-minded auction. We now consider the following example which is due to Chen et al. [5].

Example 1. Consider the following single-minded auction $\mathcal{A}$: Three buyers bid for three commodities, where $d_1 = \{\omega_1, \omega_2\}, d_2 = \{\omega_2, \omega_3\}, d_3 = \{\omega_1, \omega_3\}$, and $c_1 = 3, c_2 = 3, c_3 = 3$. Note that there is at most one winner for any allocation, say buyer 1. Thus, under the condition of $p(\omega_3) = 0$ (since ω_3 is an unallocated commodity), at most two inequalities of $p(\omega_1\omega_2) \leq 3$, $p(\omega_2\omega_3) \geq 3$, $p(\omega_1\omega_3) \geq 3$ can hold simultaneously. Therefore at most two buyers can be satisfied.

Chen et al. conjectured that this ratio $\delta = \frac{2}{3}$ is tight [5]. We prove this conjecture in the following theorem.

Theorem 3. *Any single-minded auction has a* $\frac{2}{3}$*-relaxed Walrasian equilibrium, and thus* $\delta = \frac{2}{3}$ *is a tight bound. Further, a* $\frac{2}{3}$*-relaxed Walrasian equilibrium can be computed in polynomial time.*

	w_1	w_2	w_3	valuation
Buyer 1	+	+		3
Buyer 2		+	+	3
Buyer 3	+		+	3

Fig. 1. An example of single-minded auction where at most 2/3 buyers can be satisfied

3.2 Weak Walrasian Equilibrium

The notion of relaxed Walrasian equilibrium does not require all the winners to be satisfied. Indeed, in our proof of Theorem 3, to get a higher value of $\delta_{\mathcal{A}}(X,p)$, we may require some winner to pay a payment higher than her valuation. This is infeasible in practice: the auctioneer cannot expect a winner to pay a price higher than her valuation. Motivated by this observation, we introduce a stronger concept of approximate Walrasian equilibrium.

Definition 3. (Weak Walrasian Equilibrium) *Given any* $0 < \delta \leq 1$*, a* δ-weak Walrasian equilibrium *of single-minded auction* $\mathcal{A}$ *is a tuple* (X,p) *that satisfies the following conditions:*

- $p(X_0) = 0$*, where* $X_0 = \Omega \setminus \left(\bigcup_{i:\ x_i=1} d_i\right)$*.*
- *The utility of each winner is maximized. That is, for any winner* i*,* $c_i \geq p(d_i)$*.*
- $\frac{\delta_{\mathcal{A}}(X,p)}{n} \geq \delta$*, where* n *is the number of buyers in* $\mathcal{A}$*.*

Unfortunately, this extra restriction makes the problem much harder as the following theorem shows.

Theorem 4. *For any* $0 < \delta \leq 1$*, checking the existence of a* δ*-weak Walrasian equilibrium problem is strongly NP-complete.*

We use a reduction from the Independent Set problem in the proof of the above result. If we regard Independent Set and weak Walrasian equilibrium as optimization problems, then our reduction is a gap-preserving reduction [16]. Therefore we have the following conclusion:

Corollary 2. *There is an* $\epsilon > 0$ *such that approximation of weak Walrasian equilibrium problem within a factor* n^{ϵ} *is NP-hard, where* n *is the number of buyers.*

4 The Tollbooth Problem

As we have seen in the preceding discussions, in general the computation of a (weak) Walrasian equilibrium in a single-minded auctions is hard. In search of tractable instances, we consider a special case of single-minded auctions where the set of commodities and demands can be represented by a graph. Specifically, we consider the *tollbooth problem* [15], in which we are given a graph, the commodities are edges of the graph, and the demand of each buyer is a path in the graph. For the rest of this section, we will use the phrase tollbooth problem for a single-minded auction where the input is from a tollbooth problem.

4.1 Tollbooth Problem in General Graphs

For the general graph, as the following theorem shows, both the computation of optimal allocation and Walrasian equilibrium (if it exists) are difficult.

Theorem 5. *If a Walrasian equilibrium exists in the tollbooth problem, it is NP-hard to compute one.*

4.2 Tollbooth Problem in a Tree

In this subsection, we consider the tollbooth problem in a tree. Note that even for this simple structure, Walrasian equilibrium may not exist, as shown by Example 1. In this special case, however, we can determine whether Walrasian equilibrium exists or not and compute one (if it exists) in polynomial time. Due to Corollary 1, we only describe how to compute an optimal allocation efficiently. To this end, we give a polynomial time Dynamic Program algorithm (details are deferred to the full version). We note that computing an optimal allocation is the same as computing the maximum weighted edge-disjoint paths in a tree for which a polynomial time algorithm already exists [25]. Therefore, we have the following conclusion:

Theorem 6. *For any tollbooth problem in a tree, it is polynomial time to compute an optimal allocation, determine if a Walrasian equilibrium exists or not and compute one (if it exists).*

5 Conclusion

In the notions of approximate Walrasian equilibrium that we studied in this paper, we are concerned with relaxing the second condition of Walrasian equilibrium (Definition 1). That is, we guarantee the prices of the two approximate Walrasian equilibria clear the market (Definition 2, 3). Relaxing of the first condition is called *envy-free auction* and is well studied, *e.g.*, in [15]. Unlike the general Walrasian equilibrium, an envy-free pricing always exists, and thus, a natural non-trivial goal is to compute an envy-free pricing which maximizes the revenue [15]. Similarly, trying to compute a revenue maximizing Walrasian equilibrium is an important goal. However, even checking if a Walrasian equilibrium exists is NP-hard [5] which makes the task of finding a revenue maximizing Walrasian equilibrium an even more ambitious task. Corollary 1 shows that it is as hard as computing an optimal allocation.

Our polynomial time algorithm for determining the existence of a Walrasian equilibrium and computing one (if it exists) in a tree generalizes the result for the line case, where Walrasian equilibrium always exists [4, 5] and can be computed efficiently.

Our work leaves some open questions. For relaxed Walrasian equilibrium, we showed $\delta_{\mathcal{A}}(X,p) \geq 2/3$ for some (X,p) in any single-minded auction $\mathcal{A}$. An interesting question is how to approximate $\max_{(X,p)} \delta_{\mathcal{A}}(X,p)$ within an approximation ratio better than 2/3. In addition, we showed that for the tollbooth problem on a general graph, it is NP-hard to compute the optimal allocation, which implies that given that a Walrasian equilibrium exists, computing one is also hard. A natural question is to resolve the complexity of determining the existence of a Walrasian equilibrium (as opposed to computing one if it exists) in a general graph.

Acknowledgments

We thank Neva Cherniavsky, Xiaotie Deng, Venkatesan Guruswami and Anna Karlin for helpful discussions and suggestions. We thank Yossi Azar for pointing out the references [25, 12], and thank Xiaotie Deng for pointing out the reference [17].

References

1. A. Archer, C. H. Papadimitriou, K. Talwar, E. Tardos, *An Approximate Truthful Mechanism For Combinatorial Auctions with Single Parameter Agents*, SODA 2003, 205-214.
2. A. Archer, E. Tardos, *Truthful Mechanisms for One-Parameter Agents*, FOCS 2001, 482-491.
3. K. K. Arrow, G. Debreu, *Existence of An Equilibrium for a Competitive Economy*, Econometrica, V. 22, 265-290, 1954.
4. S. Bikhchandani, J. W. Mamer, *Competitive Equilibrium in an Economy with Indivisibilities*, Journal of Economic Theory, V.74, 385-413, 1997.
5. N. Chen, X. Deng, X. Sun, *On Complexity of Single-Minded Auction*, Journal of Computer and System Sciences, V.69(4), 675-687, 2004.
6. W. Conen, T. Sandholm, *Coherent Pricing of Efficient Allocations in Combinatorial Economies*, AAAI 2002, Workshop on Game Theoretic and Decision Theoretic Agents (GTDT).
7. P. Cramton, Y. Shoham, R. Steinberg (editors), *Combinatorial Auctions*, MIT Press, 2005.
8. X. Deng, C. H. Papadimitriou, S. Safra, *On the Complexity of Equilibria*, STOC 2002, 67-71. Full version appeared in Journal of Computer and System Sciences, V.67(2), 311-324, 2003.
9. N. Devanur, C. H. Papadimitriou, A. Saberi, V. V. Vazirani, *Market Equilibrium via a Primal-Dual-Type Algorithm*, FOCS 2002, 389-395.
10. Z. Galil, S. Micali, H. Gabow, *An $O(EV \log V)$ Algorithm for Finding a Maximal Weighted Matching in General Graphs*, SIAM Journal on Computing, V.15, 120-130, 1986.
11. M. R. Garey, D. S. Johnson, *Computers and Intractability: a Guide to the Theory of NP-Completeness*, Freeman, San Francisco, 1979.

12. N. Garg, V. V. Vazirani, M. Yannakakis, *Primal-Dual Approximation Algorithms for Integral Flow and Multicut in Trees*, Algorithmica, V.18, 3-20, 1997.
13. A. V. Goldberg, J. D. Hartline, *Collusion-Resistant Mechanisms for Single-Parameter Agents*, SODA 2005, 620-629.
14. F. Gul, E. Stacchetti, *Walrasian Equilibrium with Gross Substitutes*, Journal of Economic Theory, V.87, 95-124, 1999.
15. V. Guruswami, J. D. Hartline, A. R. Karlin, D. Kempe, C. Kenyon, F. McSherry, *On Profit-Maximizing Envy-Free Pricing*, SODA 2005, 1164-1173.
16. D. S. Hochbaum (editor), *Approximation Algorithms for NP-Hard Problems*, PWS Publishing Company, 1997.
17. S. L. Huang, M. Li, *Approximation of Walrasian Equilibrium in Single-Minded Auctions*, submitted.
18. K. Jain, *A Polynomial Time Algorithm for Computing the Arrow-Debreu Market Equilibrium for Linear Utilities*, FOCS 2004, 286-294.
19. A. S. Kelso, V. P. Crawford, *Job Matching, Coalition Formation, and Gross Substitutes*, Econometrica, V.50, 1483-1504, 1982.
20. D. Lehmann, L. I. O'Callaghan, Y. Shoham, *Truth Revelation in Approximately Efficient Combinatorial Auctions*, ACM Conference on E-Commerce 1999, 96-102. Full version appeared in *JACM*, V.49(5), 577-602, 2002.
21. H. B. Leonard, *Elicitation of Honest Preferences for the Assignment of Individual to Positions*, Journal of Political Economy, V.91(3), 461-479, 1983.
22. A. Mas-Collel, W. Whinston, J. Green, *Microeconomic Theory*, Oxford University Press, 1995.
23. A. Mu'alem, N. Nisan, *Truthful Approximation Mechanisms for Restricted Combinatorial Auctions*, AAAI 2002, 379-384.
24. C. H. Papadimitriou, T. Roughgarden, *Computing Equilibria in Multi-Player Games*, SODA 2005, 82-91.
25. R. E. Tarjan, *Decomposition by clique separators*, Discrete Math, V.55, 221-231, 1985.
26. S. de Vries, R. Vohra, *Combinatorial Auctions: A Survey*, INFORMS Journal on Computing, V.15(3), 284-309, 2003.
27. L. Walras, *Elements d'economie politique pure; ou, Theorie de la richesse sociale*, (Elements of Pure Economics, or the Theory of Social Wealth), Lausanne, Paris, 1874. (Translated by William Jaffé, Irwin, 1954).