Algorithms for the d-Dimensional Rigidity Matroid of Sparse Graphs

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Abstract. Let $\mathcal{R}_d(G)$ be the d-dimensional rigidity matroid for a graph $G = (V, E)$. Combinatorial characterization of generically rigid graphs is known only for the plane $d = 2$ [11]. Recently Jackson and Jordán [5] derived a min-max formula which determines the rank function in $\mathcal{R}_d(G)$ when G is *sparse*, i.e. has maximum degree at most $d + 2$ and minimum degree at most $d + 1$.

We present three efficient algorithms for sparse graphs G that

(i) detect if E is independent in the rigidity matroid for G, and
(ii) construct G using vertex insertions preserving if G is isostatic, and
(iii) compute the rank of $\mathcal{R}_d(G)$.

The algorithms have linear running time assuming that the dimension d is fixed.

1 Introduction

Techniques from Rigidity Theory [4,11] have been recently applied to problems such as collision free robot arm motion planning [1,8], molecular conformations [6,10] and sensor and network topologies [2]. We introduce some notation first, see [4,5,9,11] for more details.

A *framework* (G, p) in d-space is a graph $G = (V, E), n = |V|, m = |E|$ and an embedding $p : V \to \mathbb{R}^d$. Let $p(V) = \{p_1, \ldots, p_n\}$. The *rigidity matrix* of the framework is the $m \times dn$ matrix for the system of m equations

$$(p_i - p_j) \cdot (p'_i - p'_j) = 0, (p_i, p_j) = p(e), e \in E$$

in unknown velocities p'_i. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by independence of rows of the rigidity matrix. A framework (G, p) is *generic* if the coordinates of the points $p(v), v \in V$ are algebraically independent over the rationals. Any two generic frameworks (G, p) and (G, p') have the same rigidity matroid called d-dimensional *rigidity matroid* $\mathcal{R}_d(G) = (E, r_d)$ of G. The rank of $\mathcal{R}_d(G)$ is denoted by $r_d(G)$.

Lemma 1. [9, Lemma 11.1.3] *For a graph G with n vertices, the rank $r_d(G) \leq S(n, d)$ where*

$$S(n, d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d + 1 \\ \binom{n}{2} & \text{if } n \leq d + 1. \end{cases}$$

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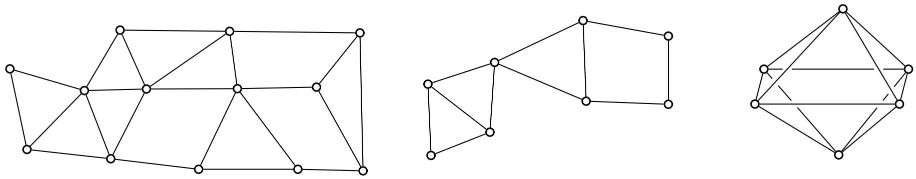

(a) (b) (c)

Fig. 1. (a) Rigid graph in the plane, (b) not rigid graph in the plane, and (c) rigid graph in $\mathbb{R}^3$

We say that a graph $G = (V, E)$ is *rigid* if $r_d(G) = S(n, d)$, see Fig. 1. We say that G is *M-independent*, *M-dependent*, or an *M-circuit* in $\mathbb{R}^d$ if E is independent, dependent, or a circuit, respectively, in $\mathcal{R}_d(G)$. A rigid graph is *minimally rigid* in $\mathbb{R}^d$ (or *generically d-isostatic*) if it is M-independent. The famous Laman Theorem [7] asserts that a graph G with n vertices and m edges is minimally rigid in $\mathbb{R}^2$ if and only if $m = 2n - 3$ and every subgraph induced by k vertices contains at most $2k - 3$ edges for any k.

A combinatorial characterization of rigid graphs is not known for dimensions $d \geq 3$. Recently Jackson and Jordán [5] generalized Laman Theorem to sparse graphs in higher dimensions. Let $G = (V, E)$ be a graph and $d \geq 3$ be a fixed integer. For $X \subseteq V$ let $G[X] = (V(X), E(X))$ be the subgraph of G induced by X. Let $i(X) = |E(X)|$. We say that a graph G is *Laman* if $i(X) \leq S(|X|, d)$ for all $X \subseteq V$. We denote the maximum and minimum degrees of G by $\Delta(G)$ and $\delta(G)$, respectively.

Theorem 1. [5, Theorem 3.5] *Let G be a connected graph with $\Delta(G) \leq d + 2$ and $\delta(G) \leq d + 1$. Then G is M-independent if and only if G is Laman.*

Jackson and Jordán [5] derived a min-max formula for the rank $r_d(G)$ of a sparse graph. A *cover* of G is a collection $\mathcal{X}$ of subsets of V, each of size at lest two, such that $\cup_{X \in \mathcal{X}} E(X) = E$. For $X \subseteq V$ let $f(X) = S(|X|, d)$ and $val(\mathcal{X}) = \sum_{X \in \mathcal{X}} f(X)$. A cover $\mathcal{X}$ is 1-*thin* if $|X \cap X'| \leq 1$ for all distinct $X, X' \in \mathcal{X}$.

Theorem 2. [5, Theorem 3.9] *Let G be a connected graph with $\Delta(G) \leq d + 2$ and $\delta(G) \leq d+1$. Then $r_d(G) = min_{\mathcal{X}} val(\mathcal{X})$ where the minimum is taken over all 1-thin covers $\mathcal{X}$ of G.*

A direct computation of the rank $r_d(G)$ by Theorem 2 leads to an exponential algorithm since the number of 1-thin covers can be exponential. Thus, it would be interesting to design an efficient algorithm (with polynomial running time) for computing the rank $r_d(G)$.

Isostatic Graphs. By Theorem 60.1.2 [11], a graph $G = (V, E)$ is generically d-isostatic if and only if it is rigid and $|E| = S(|V|, d)$. Inductive constructions are useful for isostatic graphs.

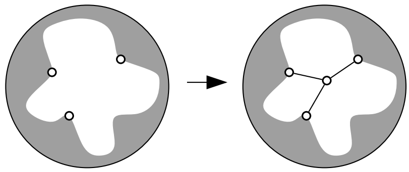

Fig. 2. Vertex addition for $d = 3$

Theorem 3. [11, Theorem 60.1.6] *Vertex Addition.*
Let G be a graph with a vertex v of degree d. Let G' denote the graph obtained by deleting v and the edges incident to it. Then G is generically d-isostatic if and only if G' is generically d-isostatic.

Theorem 4. [11, Theorem 60.1.7] *Edge Split.*
Let G be a graph with a vertex v of degree $d+1$. Let G' denote the graph obtained by deleting v and its $d+1$ incident edges. Then G is generically d-isostatic if and only if there is a pair u, w of vertices of G adjacent to v such that (u, w) is not an edge of G and the graph $G' + (u, w)$ is generically d-isostatic.

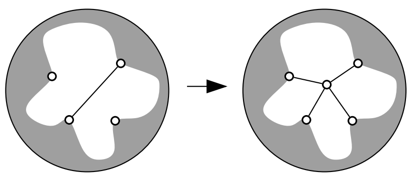

Fig. 3. Edge split for $d = 3$

1.1 Our Results

We present three efficient algorithms.

Theorem 5. *Let G be a graph with $\Delta(G) \leq d + 2$ and $\delta(G) \leq d + 1$. The following problems can be solved in linear time.*

(i) *Determine whether G is M-independent.*
(ii) *If G is generically d-isostatic then compute a sequence of vertex additions and edge splits that yield the graph G.*
(iii) *Compute the rank $r_d(G)$ and a basis of the rigidity matroid $\mathcal{R}_d(G)$.*

2 Detecting M-Independence of a Sparse Graph

Lemma 2. *Let G be a graph with $\Delta(G) \leq d + 2$ and $\delta(G) \leq d + 1$. Let $E' \subseteq E$ be a minimal M-dependent set and let X be the set of endvertices of the edges of E'. Then X contains at most M_d vertices where*

$$M_d = \left\lfloor \frac{(d-1)(d+1)}{d-2} \right\rfloor .$$

Proof. For every vertex $v \in X$, its degree in $G[X]$ is bounded by $d+2$, $d_X(v) \leq d+2$. Therefore

$$2i(X) = \sum_{v \in X} d_X(v) \leq (d+2)|X|.$$

The graph $G[X]$ is connected since E' is a minimal M-dependent set. By Theorem 1, $G[X]$ is not a Laman graph since $G[X]$ is M-dependent. Therefore $i(X) \geq S(|X|, d) + 1$. We assume that $|X| > d+1$ (the lemma follows otherwise). Therefore $S(|X|, d) = d|X| - \binom{d+1}{2}$ and

$$\begin{aligned} 2d|X| - d(d+1) + 2 &\leq 2i(X) \leq (d+2)|X|, \\ (d-2)|X| &\leq (d-1)(d+2), \\ |X| &\leq \frac{(d-1)(d+2)}{d-2}, \\ |X| &\leq M_d. \end{aligned}$$

Algorithm 1.
// Determine whether G is M-independent.
1. For each vertex v of G do
2. Compute $A = \{u \mid d(u,v) < M_d)\}$.
3. For each subset X of A such that $v \in X$ and $|X| \leq M_d$ and $G[X]$ is connected
4. Compute $i(X)$.
5. If $i(X) > S(|X|, d)$ then return "G is M-dependent"
6. return "G is M-independent"

□

Theorem 6. *Let G be a graph with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$. The above algorithm detects in linear time whether G is M-independent or not, in $O(n)$ time assuming that d is fixed.*

Proof. The algorithm checks all subsets of V of size at most M_d that induce connected graphs. By Lemma 2 the graph G is M-dependent if and only if at least one of these sets induces the connected and M-dependent graph. By Theorem 1 it is necessary and sufficient to test if the induced graph is Laman.

We analyze the running time in terms of both n and d, and show later that the dependence on n is linear. The degree of each vertex is bounded by $d+2$. Therefore the size of A is at most

$$|A| \leq 1 + (d+2) + (d+2)^2 + \cdots + (d+2)^{M_d - 1} = \frac{(d+2)^{M_d} - 1}{d+1}.$$

Let A_d be the number of the subsets of A of size at most M_d. Then

$$A_d = \binom{|A|}{1} + \binom{|A|}{2} + \cdots + \binom{|A|}{M_d} \leq |A|^{M_d}.$$

The running time is $O(A_d(d+2)^{M_d - 1} n)$ since we need $O((d+2)^{M_d - 1})$ time to compute $i(X)$ for each subset. The theorem follows since d is a constant. □

3 Isostatic and M-Independent Graphs

A set $X \subseteq V$ is *critical* if $|X| \geq 2$ and $i(X) = S(|X|, d)$.

We show that M-independent graph with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$ can be constructed using (i) the operations of vertex addition and edge split as in Theorems 3 and 4, and (ii) addition of a vertex of degree less than d. We need the following bound on the size of a critical set.

Lemma 3. *Let G be a graph with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$. Any critical set in G contains at most N_d vertices where*

$$N_d = \left\lfloor \frac{d(d+1)}{d-2} \right\rfloor .$$

Proof. Let X be a critical set in G. For every vertex $v \in X$, its degree in $G[X]$ is bounded by $d+2$, $d_X(v) \leq d+2$. Therefore

$$2i(X) = \sum_{v \in X} d_X(v) \leq (d+2)|X|.$$

On the other hand, $i(X) = S(|X|, d)$ since X is critical. We assume that $|X| > d+1$ (the lemma follows otherwise). Therefore $S(|X|, d) = |X|d - \binom{d+1}{2}$ and

$$\begin{aligned} 2i(X) = 2d|X| - d(d+1) &\leq (d+2)|X|, \\ (d-2)|X| &\leq d(d+1), \\ |X| &\leq \frac{d(d+1)}{d-2}, \\ |X| &\leq N_d. \end{aligned}$$

Algorithm 2.
// Given a M-independent graph G with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$,
// find a sequence of vertex additions and edge splits that creates G.
1. Partition V into sets V_d, V_{d+1} and V_{d+2} of vertices of degree $\leq d, d$ and $d+2$, respectively.
2. **while** $E \neq \emptyset$ **do**
3.   **if** $V_d \neq \emptyset$ **then**
4.   Remove a vertex v from V_d.
5.   Update E, V_d, V_{d+1} and V_{d+2}.
6.   **else**
7.   Let v be a vertex of V_{d+1}.
8.   Compute $N(v) = \{u \mid (u,v) \in E\}$.
9.   Compute $A = \{u \mid d(u,v) \leq N_d)\}$.
10.   Compute the set $\mathcal{C}$ of all maximal critical sets $C \subseteq A$.
11.   **for each** $u \in N(v)$
12.   Find $C(u) \in \mathcal{C}$ such that $u \in C(u)$; if $C(u)$ does not exist then $C(u) = \{u\}$.
13.   Find a pair $u, w \in N(v)$ such that $(u,w) \notin E$ and $C(u) \neq C(w)$.
14.   Remove v from G and add the edge (u,w) to E.
15.   Update E, V_d, V_{d+1} and V_{d+2}.

□

Theorem 7. *Let G be a M-independent graph in $\mathbb{R}^d$ with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$. The above algorithm computes in linear time a sequence of additions of vertices of degree at most $d+1$ and edge splits that yields the graph G.*

Proof. First, we prove the correctness of the algorithm. There are two updates of G in the algorithm: the removal of vertex v in the line 4 and the removal of vertex v with the insertion of edge (u, w) in the line 14. The degree of a vertex $u \neq v$ does not increase after either update. Therefore the graph G preserves the property $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$ after its modification.

The graph G remains M-independent after the deletion in the line 4 since the degree of v is at most d. We show that the update of G in the line 14 preserves M-independence of G. Arguing by contradiction we suppose that G is M-dependent after the update. Then there exists $E' \subseteq E$ that is dependent in $\mathcal{R}_d(G)$. Let V' denote the set of the vertices incident to an edge of E'. The graph (V', E') is M-circuit since $E' - \{(u, w)\}$ is independent. Therefore the graph $G' = (V', E' - \{(u, w)\})$ is critical. This contradicts the choice of (u, w).

The existence of the edge (u, w) (the line 13) follows from Lemma 4. The algorithm finds all critical sets in A since the size of a critical set is bounded by N_d by Lemma 3.

For analysis of the running of the algorithm time we assume that $d = O(1)$. The running time is linear since (i) $|E| = O(n)$, and (ii) $|N(v)| = O(1), |A| = O(1), |\mathcal{C}| = O(1)$, and (iii) the sets E, V_d, V_{d+1} and V_{d+2} can be updated in $O(1)$ time after each modification of G. □

Corollary 1. *Let G be a d-isostatic graph in $\mathbb{R}^d$ with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$. The above algorithm computes in linear time a sequence of additions of vertices of degree at most $d+1$ and edge splits that yields the graph G.*

Proof. The graph G has no vertices of degree less than d. By Theorems 3 and 4 the graph after removal of a vertex of degree d (the line 4) or degree $d+1$ (the line 14) is isostatic. Therefore the new graph does not contain a vertex of degree less than d. The corollary follows. □

Lemma 4. [5, Corollary 3.8] *Let G be a connected M-independent graph with $\Delta(G) = d+2$ and $\delta(G) = d+1$. Let X_1, X_2 be maximal critical subsets of V and suppose that $|X_i| \geq d+2$ for each $i \in \{1, 2\}$. Then $X_1 \cap X_2 = \emptyset$.*

4 Basis of the Rigidity Matroid

An independent set all of whose proper supersets are dependent is called a *basis*. We say that a set of vertices $X \subseteq V$ is *dependent* if the graph induced by X is M-dependent.

The algorithm for finding a basis of G maintains a graph $G' = (V, E')$ by inserting edges of G that are independent in G'. For a set $X \subset V$, we denote by $i'(X)$ the number of edges in the graph $G'[X]$ induced by X.

Algorithm 3.
// Given a graph G with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$,
// compute the rank r of $r_d(G)$ and a basis B of the rigidity matroid $\mathcal{R}_d(G)$.
Initialize $r = 0$ and $B = \emptyset$ and $G' = (V, \emptyset)$.
for each edge (u, v) of G **do**
  flag=TRUE //boolean flag indicates whether (u, v) is independent
  Compute $A = \{w \mid d(u, w) < N_d$ and $d(v, w) < N_d\}$.
  for each subset X of A such that $u, v \in X$ and $|X| \leq N_d$
    and X is connected
    Compute $i'(X)$.
    if $i'(X) = S(|X|, d)$ **then**
      print "(u, v) is dependent" and set flag=FALSE
  if flag **then**
    Add (u, v) to G'.
    $r = r + 1$ and $B = B \cup \{(u, v)\}$
return r and B

Theorem 8. *Let G be a graph with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$. The above algorithm computes the rank $r_d(G)$ and a basis of the rigidity matroid $\mathcal{R}_d(G)$ in linear time.*

Proof. The graph G' has the property that $\Delta(G') \leq d+2$ and $\delta(G') \leq d+1$. By Theorem 1 and Lemma 3 the edges rejected for insertion to G' are dependent and the set B is M-independent. Therefore B is the basis of G and the rank is computed correctly.

The running time follows since $|E| = O(n)$ and the number of subsets of A is $O(1)$. □

5 Conclusion

We presented three efficient algorithms for sparse graphs for (i) detecting M-independent graphs, and (ii) constructing M-independent graphs, and (iii) computing the rank of a graph. All algorithms have linear running time assuming that d is fixed. The hidden constants are exponential in d. In the journal version we show that the algorithms can be improved so that the dependence of the running time on d is polynomial.

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