Three Equivalent Partial Orders on Graphs with Real Edge-Weights Drawn on a Convex Polygon

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Abstract. Three partial orders, cut-size order, length order, and operation order, defined between labeled multigraphs with the same order are known to be equivalent. This paper extends the result on edge-capacitated graphs, where the capacities are real numbers, and it presents a proof of the equivalence of the three relations. From this proof, it is also shown that we can determine whether or not a given graph precedes another given graph in polynomial time.

1 Introduction

Let $N = \{x_0, x_1, \ldots, x_{n-1}\}$ be the set of vertices of a convex polygon P in the plane, where the vertices are arranged in this order counter-clockwisely, and hence (x_i, x_{i+1}) is an edge of P for $i = 0, 1, \ldots, n-1$ (We adopt the residue class on n for treating integers in N, i.e., $i \pm j$ is $i' \in N$ such that $i' \equiv i \pm j$ (mod n)). An internal angle of P may be π. We consider graphs whose node set corresponds to N, i.e., the node set is $\{0, 1, \ldots, n-1\}$ and each node i is assigned to x_i, and each edge $e = (i, j)$ of the graph is represented by a line segment $x_i x_j$.

We adopt the cyclic order for treating integers (or numbered vertices) in N. Thus for $i, j \in N$,

$$[i, j] = \begin{cases} \{i,\ i+1,\ \ldots,\ j\}, & \text{if } i \le j, \\ \{i,\ i+1,\ \ldots,\ n-1,\ 0,\ 1,\ \ldots,\ j\}, & \text{if } i > j; \end{cases}$$

for $i, j, k \in N$, $i \le j \le k$ means $j \in [i, k]$; for $i, j, k, h \in N$, $i \le j \le k \le h$ means that i, j, k, h appear in this order when we traverse the nodes of $[i, h]$ from i to h. For notational simplicity, $\{i\}$ may be written as i. For a graph G, $E(G)$ means the edge set of G.

In this paper all graphs are regarded as weighted graphs, i.e., we introduce a weight function $w_G :\ E(G) \to \boldsymbol{R}$ and a weighted graph G always has a weight function w_G in this paper.

Three relations, cut-size order, length order, and operation order, were introduced between vertex-labeled graphs in Reference [5] and shown that they are equivalent [4,5]. However, the proof in Reference [5] is for only multigraphs with the same number of edges and without edge weights. The proof for the general case have been appeared in only Technical Notes [4]. This paper shows a new proof, which is more simple than the previous one, for the general case.

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2 Definitions

We introduce some terms as follows.

Linear Cuts. For a graph G and a pair of distinct nodes $i, j \in N$, a *linear cut* $C_G(i,j)$ is an edge set:

$$C_G(i,j) = \{(k,h) \in E(G) \mid k \in [i, j-1],\ h \in [j, i-1]\}.$$

Fig. 1 show examples of linear cuts. The capacity of a linear cut $C_G(i,j)$ is defined as

$$c_G(i,j) = \sum_{e \in C_G(i,j)} w_G(e).$$

For two subsets N' and N'' of nodes,

$$w_G(N', N'') = \sum_{i \in N', j \in N''} w_G(i,j).$$

The *degree* of a node $i \in N$ of a graph G is defined as $c_G(i, i+1) = w_G(i, [i+1, i-1])$ and may be simply denoted by $d_G(i)$. As a generalization of degree, $d_G(N')$ denotes $w_G(N', N - N')$ for $N' \subset N$. From them, $c_G(i,j) = d_G([i, j-1])$, since they means the same thing.

We introduce a relation based on sizes of linear cuts as follows. For two weighted graph G and G', $G \preceq_c G'$ means that $c_G(i,j) \leq c_{G'}(i,j)$ for all $i, j \in N$, $i \neq j$. This relation is known to be a partial order, since it is easily obtained from the following result presented by Skiena [7].

Theorem 1. *For two weighted graphs G and G', if $c_G(i,j) = c_{G'}(i,j)$ for all $i, j \in N$, $i \neq j$, then $G = G'$.* □

Sum of Edge Lengths. For an edge (i,j) of a weighted graph G and a convex n-gon P, let $\mathrm{dist}(i,j)$ be a length of the line segment $x_i x_j$. We define a sum of weighted edge length of G with respect to P as

$$s_P(G) = \sum_{(i,j) \in E(G)} w(i,j) \cdot \mathrm{dist}(i,j).$$

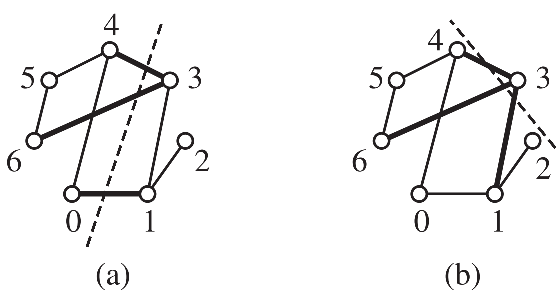

Fig. 1. Linear cuts: (a) $C_G(1,4)$, (b) $C_G(3,4)$

We introduce a relation based on the measure as follows. For two weighted graph G and G', $G \preceq_l G'$ means that $s_P(G) \leq s_P(G')$ for all convex n-gons P. Graph drawing is a very important research aea and the sum of edge lengths is a crucial criterion for evaluating drawing methods [1].

Cross-Operations. We introduce an operation transforming a graph to another one. For a weighted graph G, two distinct $i, j \in N$ and a real value Δ, $\mathrm{ADD}_G(i, j; \Delta)$ means adding Δ to $w(i, j)$ (if $(i, j) \notin E(G)$, adding an edge (i, j) to $E(G)$ previously). The reverse operation of ADD can be defined, i.e., $\mathrm{REMOVE}_G(i, j; \Delta)$ means $\mathrm{ADD}_G(i, j; -\Delta)$. We extend these operations in the case $i = j$, i.e., both $\mathrm{ADD}_G(i, i; \Delta)$ and $\mathrm{REMOVE}_G(i, i; \Delta)$ mean doing nothing. For nodes $i, j, k, h \in N$ with $i \leq j \leq k \leq h$ and a positive $\Delta > 0$ (see, Fig. 2), a *cross-operation* $X_G(i, j, k, h; \Delta)$ is applying.

$$\mathrm{REMOVE}_G(i, j; \Delta),\ \mathrm{REMOVE}_G(k, h; \Delta),\ \mathrm{ADD}_G(i, k; \Delta),\ \text{and}\ \mathrm{ADD}_G(j, h; \Delta).$$

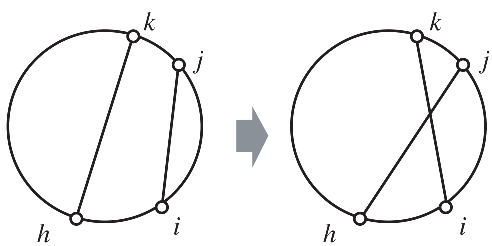

Fig. 2. Cross-operation $X(i, j, k, h; 1)$

If some of $\{i, j, k, h\}$ are equal, a cross-operation may increase edges. In fact, if $i = j < k < h < i$ or $i = j < k = h < i$ (or the cases symmetric with respect to one of them), then the total edge weights increases (see, (a) and (b) of Fig. 3). If $j = k$ or $i = h$, the edge set is not changed (see, (c) and (d) of Fig. 3).

We introduce a relation based on cross-operations as follows. For two weighted graph G and G', $G \preceq_o G'$ means that G' can be obtained from G by applying finite number (including zero) of cross-operations. Cross-operations are very similar to 2-switches, presented by Hakimi [2,3] and developed by West [8]. The only deference between them is that the order of i, j, k, h is not a matter in 2-switches.

3 Equivalence of the Three Relations

We have the following theorem.

Theorem 2. *Three relations $\preceq_c$, $\preceq_l$, and $\preceq_o$ are equivalent.* $\square$

This theorem was shown in [5] for graphs with the same size (number of edges), but for the general case a proof is shown only in Technical Notes [4]. Moreover,

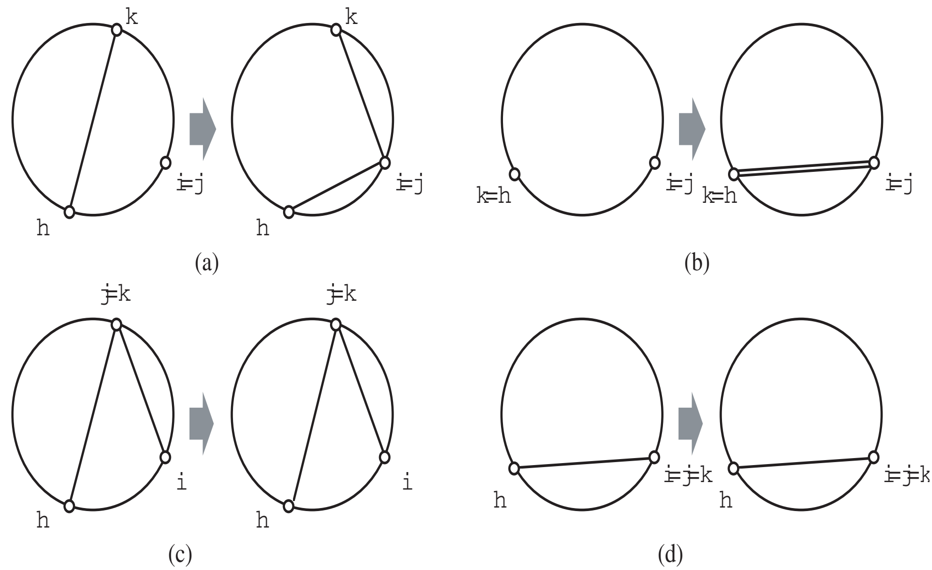

Fig. 3. These cross-operations $X(i, j, k, h; 1)$ when some of nodes are the same

these proofs were a bit long and complicated. We show a more simple proof of this theorem in this section.

In the remaining part of this section, we consider that all weighted graphs are complete graphs without loss of generality, since $(i, j) \notin E$ is equivalent to $w_G(i, j) = 0$. Hence a weighted graph can be represented by a pair of a node set N and a weight function w: $G = (N, w)$. Define a zero weighted graph $G_\emptyset = (N, w_\emptyset)$ as $w_\emptyset(i, j) = 0$ for all $i, j \in N$.

Note that $c_{G_\emptyset}(i, j) = 0$ for any $i, j \in N$ $(i \neq j)$, and $S_P(G_\emptyset) = 0$ for any polygon P. For any pair of $G = (N, w)$ and $G' = (N, w')$, we define $G - G' = (N, w'')$ as $c''(i, j) := c(i, j) - c'(i, j)$ for every $i, j \in N$. $G \preceq G'$ ($\preceq$ is any one of $\preceq_l$, $\preceq_c$, and $\preceq_o$) is equivalent to $G - G' \preceq G_\emptyset$. Therefore, it is enough to consider $G' = G_\emptyset$ for proving Theorem 2, as a result of this fact, the proof of Theorem 2 consists of three parts:

(1) $G \preceq_o G_\emptyset \Rightarrow G \preceq_l G_\emptyset$, (Lemma 1)
(2) $G \preceq_l G_\emptyset \Rightarrow G \preceq_c G_\emptyset$, (Lemma 2) and
(3) $G \preceq_c G_\emptyset \Rightarrow G \preceq_o G_\emptyset$. (Lemma 3)

Lemma 1 ([5]). *If $G \preceq_o G_\emptyset$, then $G \preceq_l G_\emptyset$.*

Proof: It is clear from the triangle inequality. □

Lemma 2 ([5]). *If $G \preceq_l G_\emptyset$, then $G \preceq_c G_\emptyset$.*

Proof. Suppose that $G \preceq_c G_\emptyset$ does not hold, i.e., there are $i, j \in N$ such that $c_G(i, j) > 0$. We construct a polygon P satisfying $S_P(G) > 0$ as follows.

$X = \{x_k \mid k \in [i, j-1]\}$ and $Y = \{x_k \mid k \in [j, i-1]\}$. Let $p, r > 0$ be real numbers. Put all vertices $x_i \in X$ in a circle whose center is $(0, 0)$ and radius is r. Put all vertices $x_i \in Y$ in a circle whose center is $(p, 0)$ and radius is r. We can locate all vertices satisfying the above conditions and convexity for any r and p. By letting p be far larger than r, $S_P(G) > 0$. □

Lemma 3. *If $G \preceq_c G_\emptyset$, then $G \preceq_o G_\emptyset$.*

In this paper we show a new proof, which is more simple than the previous one, for this lemma. The following proposition is well-known. Since the proof is easy, it is omitted.

Proposition 1. *Let $A, B, C \subset N$ be three mutually disjoint subsets and G be a weighted graph, then*

$$d_G(A \cup B) + d_G(B \cup C) = d_G(B) + d_G(A \cup B \cup C) + 2w_G(A, C).$$

□

Proof of Lemma 3. Assume that $G \preceq_c G_\emptyset$, i.e.,

$$d_G([i, j]) = c_G(i, j+1) \leq 0 \text{ for all } i, j \in N. \tag{1}$$

We use an example shown in Fig. 4 (a) for a help of understanding. Let k_0 be the largest integer such that

$$d_G([i, j]) = 0 \text{ for all } (i, j) \in \{(i, j) \mid i, j \in N, j - i < k_0\}. \tag{2}$$

Note that the residue class is used for the difference. (For an example for Fig. 4 (a), $k_0 = 2$ since $d_G(0) = d_G(1) = \cdots = d_G(11) = 0$, $d_G([0, 1]) =$

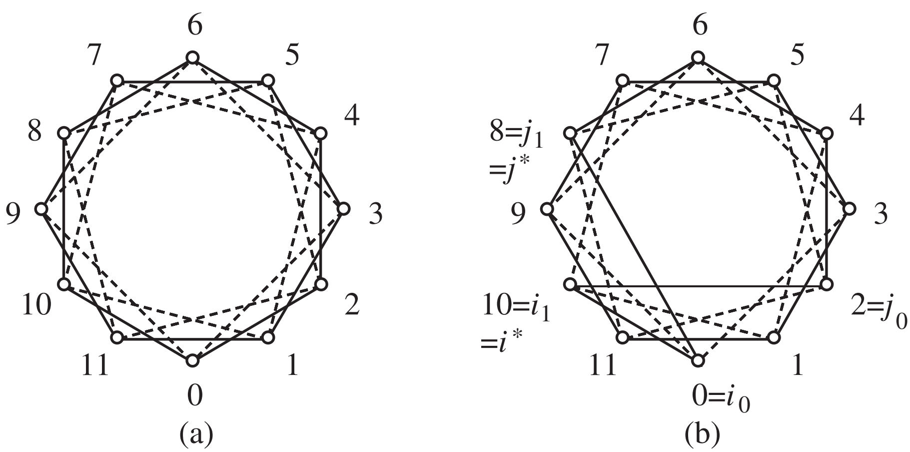

Fig. 4. An example of G: $w(e) = 1$ for solid edges and $w(e) = -1$ for broken edges

$d_G([1,2]) = \cdots = d_G([11,0]) = 0$, and $d_G([0,2]) = -2 < 0$.) If $k_0 \geq \lfloor n/2 \rfloor$, $G = G_\emptyset$. Hence, we assume $k_0 < \lfloor n/2 \rfloor$. Then their exists (i_0, j_0) such that $j_0 - i_0 = k_0$ and

$$d_G([i_0, j_0]) < 0. \tag{3}$$

(For an example for Fig. 4 (a), $i_0 = 0$ and $j_0 = 2$.) By considering Proposition 1 with $A = \{i_0\}$, $B = [i_0 + 1, j_0 - 1]$, and $C = \{j_0\}$, we obtain

$$\begin{aligned} & d_G([i_0, j_0 - 1]) + d_G([i_0 + 1, j_0]) \\ = & \, d_G([i_0 + 1, j_0 - 1]) + d_G([i_0, j_0]) + 2w_G(i_0, j_0). \end{aligned}$$

Thus

$$\begin{aligned} & w_G(i_0, j_0) \\ = & \frac{d_G([i_0, j_0 - 1]) + d_G([i_0 + 1, j_0]) - d_G([i_0 + 1, j_0 - 1]) - d_G([i_0, j_0])}{2} \\ > & \, 0, \end{aligned} \tag{4}$$

since $d_G([i_0, j_0 - 1]) = d_G([i_0 + 1, j_0]) = d_G([i_0 + 1, j_0 - 1]) = 0$, and $d_G([i_0, j_0]) < 0$. (In the example, $w_G(0, 2) = 1 > 0$.) Let I be a set of (i, j) $(i, j \in N)$ satisfying the following conditions:

(a) $i < i_0 \leq j_0 < j$, and
(b) $d_G([i', j']) < 0$ for all $i < i' \leq i_0$ and $j_0 \leq j' < j$.

(For an example for Fig. 4 (a), $I = \{(11,3), (11,4), \ldots, (11,9), (10,3), (10,4), \ldots, (10,8), (9,3), (9,4), \ldots, (9,7), (8,3), (8,4), (8,5), (8,6), (7,3), (7,4), (7,5), (6,3), (6,4), (5,3)\}$.) $I \neq \emptyset$ since $(i_0 - 1, j_0 + 1) \in I$. Let (i_1, j_1) be an extremal element of I, i.e., they satisfies (a), (b), and

(c) there are $i_1 < i_2 \leq i_0$ and $j_0 \leq j_2 < j_1$ such that $d_G([i_1, j_2]) = d_G([i_2, j_1]) = 0$

(For an example for Fig. 4 (a), $i_1 = 10$, $j_1 = 8$, $i_2 = 11$, $j_2 = 7$.) Such i_1, j_1 (and i_2, j_2) must exist since $d_G([i, j]) = 0$ for $i, j \in N$ with $j - i > n - k_0$ (note $d_G([i, j]) = d_G([j + 1, i - 1])$ and (2)). By considering Proposition 1 with $A = [i_1, i_2 - 1]$, $B = [i_2, j_2]$, and $C = [j_2 + 1, j_1]$ (see, Fig. 5), we obtain

$$\begin{aligned} & d_G([i_1, j_2]) + d_G([i_2, j_1]) \\ = & \, d_G([i_2, j_2]) + d_G([i_1, j_1]) + 2w_G([i_1, i_2 - 1], [j_2 + 1, j_1]). \end{aligned}$$

Thus

$$\begin{aligned} & w_G([i_1, i_2 - 1], [j_2 + 1, j_1]) \\ = & \frac{d_G([i_1, j_2]) + d_G([i_2, j_1]) - d_G([i_2, j_2]) - d_G([i_1, j_1])}{2} > 0, \end{aligned} \tag{5}$$

since $d_G([i_1, j_2]) = d_G([i_2, j_1]) = 0$ from (c), $d_G([i_2, j_2]) < 0$ from (b), and $d_G([i_1, j_1]) \leq 0$. (For an example for Fig. 4 (a), $w_G(10, 8) = 1 > 0$.) Hence there is a pair $i^* \in [i_1, i_2 - 1]$ and $j^* \in [j_2 + 1, j_1]$ such that

$$w_G(i^*, j^*) > 0. \tag{6}$$

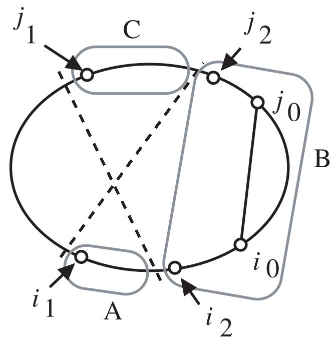

Fig. 5. Relation between nodes and cuts

(For an example for Fig. 4 (a), $i^* = i_1 = 10$ and $j^* = i_1 = 8$.) Since $(i^*, j^*) \in I$, it satisfies (b), i.e.,

$$d_G([i, j]) < 0 \text{ for all } i^* < i \leq i_0 \text{ and } j_0 \leq j < j^*. \tag{7}$$

From (4), (6), and (7) we can apply a cross-operation $X(i_0, j_0, j^*, i^*; \Delta)$ on G with

$$\Delta = \min\{w_G(i_0, j_0), w_G(i^*, j^*), \min_{i^* < i \leq i_0, j_0 \leq j < j^*} \{-d_G([i, j])/2\}\} > 0.$$

(For an example for Fig. 4 (a), $\Delta = 1$ and we obtain a graph of Fig. 4 (b) by the cross-operation.)

Now, we have found a cross-operation that makes G be closer to $G_\emptyset$. By applying the preceding discussion iteratively, we can find a sequence of cross-operations that makes G be closer to $G_\emptyset$. For completing the proof, we must show that the length of the sequence is finite. It is shown as follows.

Let G' be a graph obtained by applying $X(i_0, j_0, j^*, i^*; \Delta)$ to G. There are three cases: (I) $\Delta = w_G(i_0, j_0)$, (II) $\Delta = \min_{i^* < i \leq i_0, j_0 \leq j < j^*} \{-d_G([i, j])/2\}$, and (III) $\Delta = w_G(i^*, j^*)$. We consider each case as follows.

(I) $\Delta = w_G(i_0, j_0)$. In this case, $w_{G'}(i_0, j_0)$ becomes zero. Then by applying Proposition 1 with $A = \{i_0\}$, $B = [i_0 + 1, j_0 - 1]$, and $C = \{j_0\}$, we obtain $d_{G'}([i_0, j_0]) = 0$. Thus, the number of zero-linear-cuts of G' is greater than the one of G. Therefore (I) occurs at most $\binom{n}{2} < n^2$ times.

(II) $\Delta = \min_{i^* < i \leq i_0,\ j_0 \leq j < j^*} \{-d_G([i, j])/2\}$. Let i' and j' be nodes satisfying $i^* < i' \leq i_0$, $j_0 \leq j' < j^*$, and $\Delta = -d_G([i, j])/2$. Thus $d_{G'}([i', j'])$ becomes zero. Hence the number of zero-linear-cuts of G' is greater than the one of G. Therefore (II) occurs at most $\binom{n}{2} < n^2$ times.

(III) $\Delta = w_G(i^*, j^*)$. It is enough to consider a case of $\Delta < w_G(i_0, j_0)$, because if $\Delta = w_G(i_0, j_0)$, then case (I) could be applied. Then $w_{G'}(i_0, j_0) > 0$ and $w_{G'}(i^*, j^*) = 0$. In this case, we try again to find another pair of (i^*, j^*) for the same (i_0, j_0) (the same (i^*, j^*) be never found since $w_{G'}(i^*, j^*) = 0$). Thus (III) occurs successively at most $\binom{n}{2} < n^2$ times.

From (I)–(III), the length of the sequence of cross-operations is less than n^4. By using the sequence, G is transformed into $G_\emptyset$, i.e., $G \preceq_o G_\emptyset$. □

Proof of Theorem 2. Follows immediately from Lemmas 1, 2, and 3. □

Corollary 1. *Three relations $\preceq_c$, $\preceq_l$, and $\preceq_o$ are all partial orders.*

Proof. Clear from Theorem 2 and that $\preceq_c$ is a partial order. □

From Theorem 2, these three partial orders can be denoted by $\preceq$ simply. Moreover, we easily get the next.

Corollary 2. *Whether or not $G \preceq G'$ for a given pair of graphs G and G' can be determined in polynomial time.*

Proof. Clear from Theorem 2 and that the number of linear-cuts is $O(n^2)$. □

4 Concluding Remarks and Future Work

This paper extends the three orders, cut-size order, length order, and operation order, onto real capacitated (vertex labeled) graphs, and presents a proof for the equivalence of them.

Theorem 2 guarantees that there is a sequence of graphs $G = G_0, G_1, \ldots, G_p = G'$ such that G_i $(i = 1, \ldots, p)$ can be obtained from G_{i-1} by applying a cross operation if $G \prec G'$. These graphs G_i $(i = 1, \ldots, p)$ may be not simple even if G and G' are both simple. Whether or not there is a sequence consists of simple graphs only in this case is an interesting problem. Some results have been obtained for this problem [6], but our conjecture that such sequence always exists if $d_G(i) = d_{G'}(i)$ for all $i \in N$ remains for future work.

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