On the Minimum Size of a Point Set Containing Two Non-intersecting Empty Convex Polygons

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Abstract. Let $n(k,l)$ be the smallest integer such that any set of $n(k,l)$ points in the plane, no three collinear, contains both an empty convex k-gon and an empty convex l-gon, which do not intersect. We show that $n(3,5) = 10$, $12 \leq n(4,5) \leq 14$, $16 \leq n(5,5) \leq 20$.

1 Introduction

Erdős [1] asked the following combinatorial geometry problem in 1979: Find the smallest integer $n(k)$ such that any set of $n(k)$ points in the plane, in general position, i.e., no three points are collinear, contains the vertices of a convex k-gon, whose interior contains no points of the set. Such a subset is called an *empty convex k-gon* or a *k-hole*. Klein [2] found $n(4) = 5$, and $n(5) = 10$ was determined by Harborth [3]. Horton [4] showed that $n(k)$ does not exist for $k \geq 7$. The value of $n(6)$ is still open.

We consider a related problem. Let H_1 and H_2 be a pair of holes in a given point set. We say that H_1 and H_2 are *vertex disjoint* if $H_1 \cap H_2 = \emptyset$. If, moreover, their convex hulls are disjoint, we simply say that the two holes are *disjoint*; $ch(H_1) \cap ch(H_2) = \emptyset$ where ch stands for the convex hull. We study the following function: Let $n(k,l)$ denote the smallest integer such that any set of $n(k,l)$ points in general position in the plane contains a pair of disjoint holes with a k-hole and a l-hole. Clearly, $n(3,3) = 6$ and $n(k,7) = +\infty$. In [5] and [6], we studied the maximum number of pairwise disjoint k-holes and we found $n(3,4) = 7$ and $n(4,4) = 9$, where Figure 1 gives $n(4,4) \geq 9$. In this paper, we estimate the other values; $n(3,5) = 10$, $12 \leq n(4,5) \leq 14$, $16 \leq n(5,5) \leq 20$.

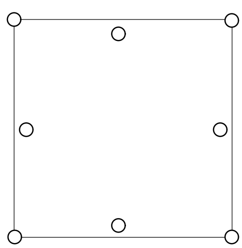

Fig. 1

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2 Definitions and Notations

In this paper, we consider only planar point sets in general position. For such a set P, we denote $P = V(P) \cup I(P)$ such that $V(P)$ is the boundary vertices on $ch(P)$. We denote a k-hole by $(v_1 v_2 \cdots v_k)$ if the vertices are located with order anti-clockwise. When indexing a set of t points, we identify indices modulo t.

Let a, b and c be any three points in the plane. We denote the *convex cone* between $r(a;b)$ and $r(a;c)$ by $C(a;b,c)$, where $r(a;b)$ is the ray emanating from a and passing through b. For $\delta = b$ or c of $C(a;b,c)$, let δ' be a point collinear with a and δ so that a lies on the segment $\delta\delta'$. A convex region is also said to be *empty* if its interior contains no elements of P. Namely, if $C(a;b,c)$ is not empty, we can consider an element p of P in the interior of $C(a;b,c)$ such that $C(a;b,p)$ is empty. We call such p the *attack point* from $r(a;b)$ to $r(a;c)$.

Let R be a convex region and consider $C(a;b,c)$ such that $\{a,b,c\}$ is contained in R. For $C_R(a;b,c) = C(a;b,c) \cap R$, we define $\alpha_R(a;b,c)$ as the element of P in the interior of $C_R(a;b,c)$ such that $C_R(a;b,\alpha_R(a;b,c))$ is empty. Finally, $H(ab;c)$ or $H(ab;\overline{c})$ denotes the open half-plane bounded by the line ab, containing c or not, respectively.

3 Results

In this section, we estimate the values $n(k,5)$ for $k = 3,4,5$. We first show the following result.

Theorem 1. *$n(3,5) = 10$.*

Before showing Theorem 1, we propose the next lemma.

Lemma 1. *If a 10 point set P contains a 6-hole Q, then P has a pair of disjoint holes with a 3-hole and a 5-hole.*

Proof. Let $Q = (q_1 q_2 q_3 q_4 q_5 q_6)$ and consider the convex cone Γ_i determined by $r(q_{i+3};q_i)$ and $r(q_{i-1};q_{i+2})$ for $i = 1,3,5$, where $\Gamma_1 \cup \Gamma_3 \cup \Gamma_5$ covers the area $R^2 \backslash ch(Q)$ and each Γ_i contains exactly one point q_{i+1} of Q in the interior. Since Γ_i contains at least two points of $P \backslash Q$ for some i, we obtain a 5-hole $Q' = Q \backslash \{q_{i+1}\}$ and three points in the convex region $\Gamma_i \backslash ch(Q')$. □

Proof of Theorem 1. The lower bound holds by $n(5) = 10$ as shown in Fig. 2.

For the upper bound we show that every 10 point set P has a disjoint pair of a 3-hole and a 5-hole. By $n(5) = 10$ we obtain a 5-hole $F = (p_1 p_2 p_3 p_4 p_5)$ in P. Denote the finite or infinite region bounded with $r(p_{i-1};p_i)$, $r(p_{i+2};p_{i+1})$ and the line segment $p_i p_{i+1}$ by B_i for $1 \leq i \leq 5$. If some B_i contains a point of P, we are done by Lemma 1 since P has a 6-hole. Suppose that every B_i is empty.

We claim that there are consecutive vertices, say p_3 and p_4 in F such that $\angle p_2 p_3 p_4 + \angle p_3 p_4 p_5 > \pi$. Then we consider three convex regions $R_1 = H(p_4 p_5; \overline{p_1})$, $R_2 = H(p_2 p_3; \overline{p_1}) \backslash R_1$ and $R_3 = C(p_1; p_2', p_5') \backslash (R_1 \cup R_2)$. If some R_i contains at

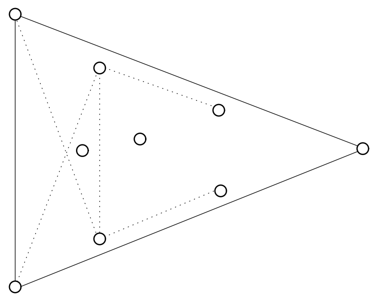

Fig. 2

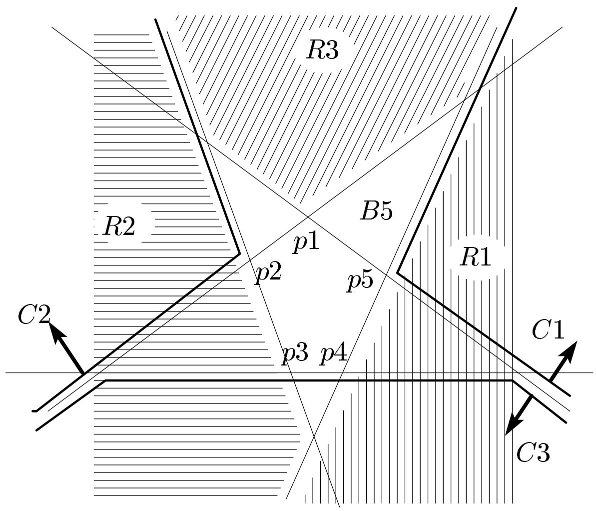

Fig. 3

least three points, we have a triangle in this region and a 5-hole F. We can suppose that each R_i is not empty and contains one or two points.

We first suppose that R_3 contains exactly two points. Then we can assume that $C_1 = C(p_5; p'_1, p'_4)$ is empty since, otherwise, the convex region $R_3 \cup C_1 \cup B_5$ contains at least three points. Similarly, $C_2 = C(p_2; p'_1, p'_3)$ is also empty, which implies that $C_3 = H(p_3p_4; \overline{p_1}) \backslash (C_1 \cup C_2)$ contains exactly three points. Then we have a triangle in C_3 and a 5-hole F. See Fig. 3.

Suppose that R_3 contains exactly one point. If C_1 or C_2 contains at least two points, we are immediately done. Then we suppose that C_3 has exactly two points and that each of C_1 and C_2 contains exactly one point. Moreover, $H(p_3p_4; \overline{p_1}) \cap (C_1 \cup C_2)$ can be assumed to be empty. Then if $C(p_4; p_3, p'_5)$ is empty, R_1 contains exactly three points. Therefore, we can assume that each of $C(p_4; p_3, p'_5)$ and $C(p_3; p_4, p'_2)$ contains exactly one point by symmetry.

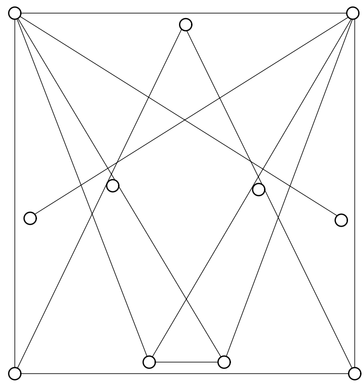

Fig. 4

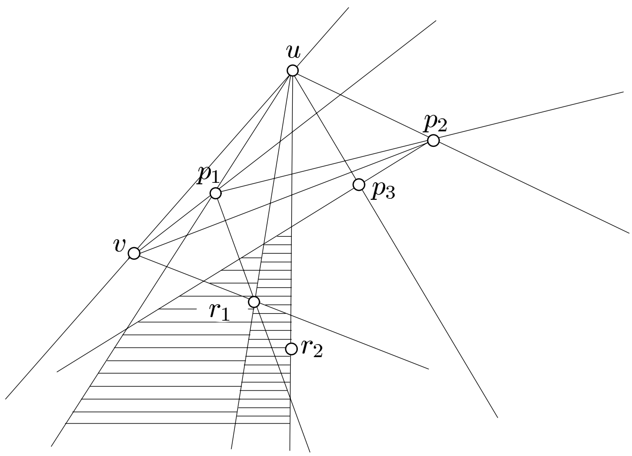

Fig. 5

Let q_1, q_2 or q_3 be in R_3, C_2 or $C(p_4; p_3, p_5')$, respectively. If both q_1 and q_3 are in $H(p_1p_3; p_4)$, a 6-hole $(q_1p_1p_3q_3p_4p_5)$ appears and we are done. If either q_1 or q_3 is in $H(p_1p_3; p_4)$, we obtain a 5-hole and a 3-hole in $H(p_1p_3; p_2)$. Hence, the remaining case is that $H(p_1p_3; \overline{p_4})$ contains both q_1 and q_3. If $\triangle q_1q_2q_3$ contains p_2, we obtain a 5-hole $(q_1p_2q_3p_3p_1)$ and a 3-hole in $H(p_1p_3; p_4)$. If not so, we obtain F itself and $\triangle q_1q_2q_3$. □

Theorem 2. $12 \leq n(4,5) \leq 14$.

Proof. Figure 4 gives an 11 point set which does not contain both a 4-hole and a 5-hole, following that $n(4,5) \geq 12$.

For a 14 point set P, we take any edge uv of $ch(P)$ and consider the point p_1 with $\{u, v, p_1\}$ anti-clockwise order such that $C(u; v, p_1)$ is empty. If $C(p_1; u, v')$ is not empty, we obtain a 4-hole $Q = (uvp_1\alpha(p_1; u, v'))$. Since $C(p_1; u', \alpha(p_1; u, v'))$

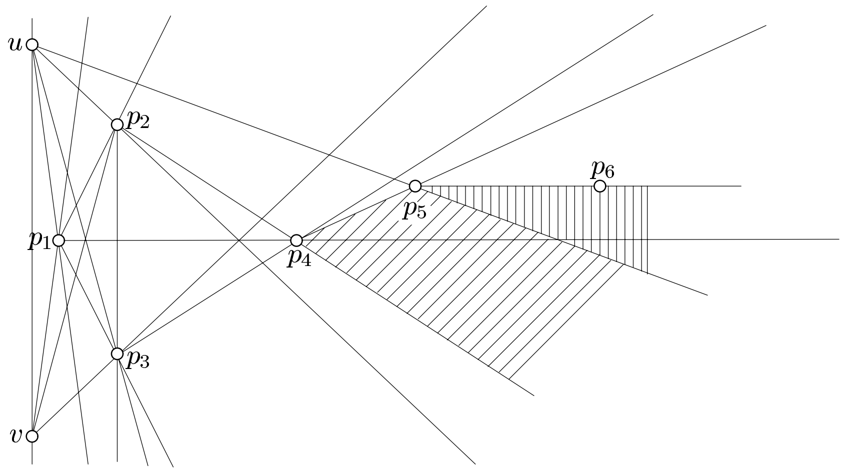

Fig. 6

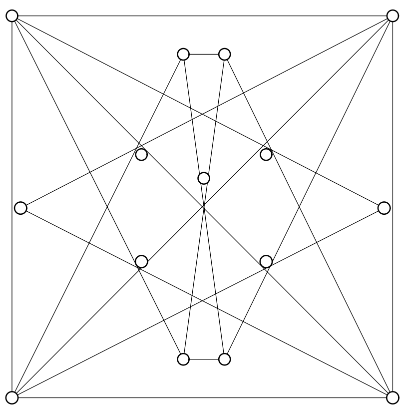

Fig. 7

contains precisely 10 points in the interior, we also obtain a 5-hole in it by $n(5) = 10$.

Suppose that $C(v; u, p_1)$ is empty and let $p_2 = \alpha(p_1; v', u')$. We can assume that $\triangle v p_2 p_1$ is empty since, otherwise, we obtain a 4-hole $(p_2 u p_1 \alpha(p_2; p_1, v))$ and $H(\alpha(p_2; p_1, v) p_2; v)$ contains 10 points. Then since, if $C(p_2; v, u')$ is empty, we obtain a 4-hole $(p_2 p_1 v \alpha(p_2; u', p_1'))$ and $H(p_2 \alpha(p_2; u', p_1'); u)$ contains 10 points, we consider $p_3 = \alpha(p_2; v, u')$.

We can assume that $C(p_3; p_2', u')$ is empty. In fact, if this region contains a unique interior point q, we obtain a 4-hole $(v q p_3 p_1)$ and $H(u p_3; p_2) \cup \{u\}$ with 10 points. If this contains at least two points, we consider two attack points of $r_1 = \alpha(u; p_1, p_3)$ and $r_2 = \alpha(u; r_1, p_3)$. If $\triangle p_1 v r_2$ contains r_1, we have a 5-hole

$(up_1r_1r_2p_3)$ or $(up_1r_1r_2\alpha(r_2;u,p_3))$, respectively if $C(r_2;u,p_3)$ is emty or not. Then since $H(r_2p_3;p_2)$ or $H(r_2\alpha(r_2;u,p_3);p_2)$ contains 8 points, it has a 4-hole by $n(4)=5$. See Fig. 5 where the shaded portions are empty. For otherwise, we have a 4-hole of either $(r_1r_2p_1v)$ or $(r_2r_1p_1v)$ and $H(ur_2;p_3)\cup\{u\}$ with 10 points.

We now suppose that $C(p_3;u',p_1')$ is empty since, otherwise, we obtain a 4-hole $(p_3p_1v\alpha(p_3;u',p_1'))$ and $H(p_3\alpha(p_3;u',p_1');u)$ with 10 points. We can assume that $C(p_2;p_3,u')$ is empty since, otherwise, we have a 5-hole $(p_2up_1p_3\alpha(p_2;p_3,u'))$ and $H(p_2\alpha(p_2;p_3,u');\overline{u})$ with 8 points. By symmetry, $C(p_3;p_2,v')$ is also empty. For $p_4=\alpha(p_3;v',p_1')$, we suppose that $C(p_2;p_4,u')$ is empty since, otherwise, we obtain a 5-hole $(p_4p_2p_1p_3\alpha_R(p_4;p_3,p_2'))$ where $R=C(p_1;p_2,p_3)$ and $H(p_4\alpha_R;\overline{p_1})$ with 7 or 8 points.

We consider $p_5=\alpha(p_4;p_3',p_1')$ by symmetry. If $\alpha_{R'}(p_5;p_4,u')$ exists for $R'=C(p_2;p_4,u)$, we obtain a 5-hole $(p_5up_2p_4\alpha_{R'})$ and $H(p_5\alpha_{R'};\overline{p_4})$ with at least 6 points. If $C_{R'}(p_5;p_4,u')$ is empty, we obtain a 5-hole $(p_6p_5p_4p_3v)$ for $p_6=\alpha(p_5;u',p_4')$ and $C(p_5;p_6,p_4')$ contains 6 points in the interior as shown in Fig. 6. □

Theorem 3. $16\le n(5,5)\le 20$.

Proof. Figure 7 gives a 15 point set which does not contain two disjoint 5-holes, referred to in [5]. For any set of 20 points, there is a line which halves the set. Then the upper bound holds by $n(5)=10$. □

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