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Structural Properties of Linear Systems – Part II: Structure at Infinity

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7.1 Introduction

This chapter is the sequel of [7]. Its topic is the structure at infinity of discrete and continuous linear time-varying systems in a unified approach.

In the time-invariant case, the linear systems in [7] are implicitly assumed to be perpetually existing and the smoothness of their behavior is not studied. In practice, however, that behavior must be sufficiently smooth (to avoid undesirable saturations of the variables, or even the destruction of the system), and the system has a limited useful life. These constraints can be taken into account by studying the *structure at infinity* of the system under consideration. As this system is existing during a limited period, it is called a *temporal system* [8].

A list of *errata* and *addenda* for [7] is given at the end of the chapter.

7.2 Differential Polynomials and Non-commutative Formal Series

7.2.1 Differential Polynomials: A Short Review

The notation is the same as in [7], except for the derivation which is now denoted by γ (instead of δ, to avoid confusing with the Dirac distribution). In what follows, the "coefficient field" $\mathbf{K}$ is a differential field, equipped with an α-derivation γ such that α is an automorphism of $\mathbf{K}$, $a^{\gamma} = 0$ implies $a^{\alpha} = a$ (for any $a \in \mathbf{K}$) and $\alpha\,\delta = \delta\,\alpha$. The subfield of constants of $\mathbf{K}$ is denoted by $\mathbf{k}$. The ring of differential polynomials $\mathbf{K}\,[\partial; \alpha, \gamma]$ is denoted by $\mathbf{R}$. This ring is equipped with the commutation rule

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$$\partial a = a^{\alpha}\partial + a^{\gamma}. \tag{7.1}$$

As shown in ([7], Section 6.2.3, Proposition 6.4), $\mathbf{R}$ is an euclidean domain, thus it is a two-sided Ore domain, and it has a field of left fractions and a field of right fractions which coincide ([7], Section 6.2.3); this field is denoted by $\mathbf{Q} = \mathbf{K}(\partial; \alpha, \gamma)$.

7.2.2 Local Complete Rings

Let $\mathbf{A} \neq 0$ be a ring. The set of all left ideals of $\mathbf{A}$ is inductively ordered by inclusion, thus (by Zorn's lemma), any left ideal of $\mathbf{A}$ is included in a maximal left ideal ("Krull's theorem"). The intersection of all maximal left ideals of $\mathbf{A}$ is called the *Jacobson radical* of $\mathbf{A}$; it turns out that this notion is left/right symmetric. A left ideal $\mathbf{a}$ of $\mathbf{A}$ is maximal if, and only if $\mathbf{A}/\mathbf{a}$ is a field ([1], Section I.9.1)[1]. The following result is proved in ([22], Sections 4 and 19):

Proposition 7.1. *(i) If* $\mathbf{A}$ *has a* unique *maximal left ideal* $\mathbf{m}$*, then* $\mathbf{m}$ *is a two-sided ideal, it is also the unique maximal right ideal of* $\mathbf{A}$ *and it consists of all noninvertible elements of* $\mathbf{A}$*. (ii) Conversely, if the set of all noninvertible elements of* $\mathbf{A}$ *is an additive group* $\mathbf{m}$*, then* $\mathbf{m}$ *is the unique maximal left ideal of* $\mathbf{A}$ *(and is the Jacobson radical of that ring).*

Definition 7.1. *A ring* $\mathbf{A}$ *which has a unique maximal left ideal* $\mathbf{m}$ *is called a* local ring *(and is also denoted by* $(\mathbf{A}, \mathbf{m})$*, to emphasize the role of* $\mathbf{m}$*);* $\mathbf{A}/\mathbf{m}$ *is called the* residue field *of* $\mathbf{A}$*.*

Let $(\mathbf{A}, \mathbf{m})$ be a local ring. The set $\left\{\mathbf{m}^i, i \geq 0\right\}$ is a filter base, and $\mathbf{A}$ is a topological ring with $\left\{\mathbf{m}^i, i \geq 0\right\}$ as a neighborhood base of 0 ([3], Section III.6.3). This topology of $\mathbf{A}$ is called the $\mathbf{m}$-adic topology. Assuming that

$$\bigcap_{i \geq 0} \mathbf{m}^i = 0, \tag{7.2}$$

the $\mathbf{m}$-adic topology is Hausdorff, and it is metrizable since the basis $\left\{\mathbf{m}^i, i \geq 0\right\}$ is countable ([4], Section IX.2.4).

Definition 7.2. *The local ring* $(\mathbf{A}, \mathbf{m})$ *is said to be* complete *if it is complete in the* $\mathbf{m}$-adic topology*.*

7.2.3 Formal Power Series

Set $\sigma = 1/\partial$ (σ can be viewed as the "integration operator": see Section 7.4.2) and $\beta = \alpha^{-1}$; $\mathbf{S} := \mathbf{K}[[\sigma; \beta, \gamma]]$ denotes the ring of *formal power series in* σ, equipped with the commutation rule ([12], Section 8.7)

[1] In this chapter, as in [7], *field* means *skew field*.

$$\sigma a = a^{\beta} \sigma - \sigma a^{\beta \gamma} \sigma, \tag{7.3}$$

deduced from (7.1). An element a of $\mathbf{S}$ is of the form

$$a = \sum_{i \geq 0} a_i \, \sigma^i, \quad a_i \in \mathbf{K}.$$

Definition 7.3. *Let a be a nonzero element of $\mathbf{S}$ and set $\omega(a) = \min\{i : a_i \neq 0\}$; the natural integer $\omega(a)$ is called the* order *of a.*

Proposition 7.2. *(i) The ring $\mathbf{S}$ is a principal ideal domain and is local with maximal left ideal $\mathbf{S}\,\sigma = \sigma\,\mathbf{S} = (\sigma)$. (ii) The units of $\mathbf{S}$ are the power series of order zero; any nonzero element $a \in \mathbf{S}$ can be written in the form $a = \upsilon\,\sigma^{\omega(a)} = \sigma^{\omega(a)}\,\upsilon'$, where υ and υ' are units of $\mathbf{S}$. (iii) Let a and b be nonzero elements of $\mathbf{S}$; then $b \parallel a$ (i.e. b is a* total divisor *of a) if, and only if $\omega(b) \leq \omega(a)$; every nonzero element of $\mathbf{S}$ is invariant (see [7], Section 6.2.3). (iv) The local ring $(\mathbf{S}, (\sigma))$ is complete.*

Proof. 1) It is easy to check that the only nonzero elements of $\mathbf{S}$ are the powers $\sigma^i, i \geq 0$, and their associates, thus (i), (ii) and (iii) are obvious; note that the residue field of $\mathbf{S}$ is $\mathbf{S}/(\sigma) \cong \mathbf{K}$. 2) Condition (7.2) is satisfied with $\mathbf{m} = (\sigma)$ (see Exercise 7.1). Let (a_n) be a Cauchy sequence of $\mathbf{S}$, $a_n = \sum_{i \geq 0} a_{n,i}\,\sigma^i$. For any integer $k \geq 1$, there exists a natural integer N such that for all $n, m \geq N$, $a_n - a_m \in \left(\sigma^k\right)$, *i.e.* for any integer i such that $0 \leq i \leq k-1$ we have $a_{n,i} = a_{m,i}$. Let b_i be the latter quantity and $b = \sum_{i \geq 0} b_i\,\sigma^i \in \mathbf{S}$. The sequence (a_n) converges to b in the (σ)-adic topology. ∎

Exercise 7.1. Prove that for any $i \geq 0$, $\left(\sigma^i\right) = (\sigma)^i$ and that $\bigcap_{i \geq 0} \left(\sigma^i\right) = 0$.

7.2.4 A Canonical Cogenerator

For any $\mu \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural integers), set $\tilde{C}_\mu = \frac{\mathbf{S}}{(\sigma^\mu)}$ and let $\tilde{\delta}^{\mu-1}$ be the canonical image of $1 \in \mathbf{S}$ in $\tilde{C}_\mu$; $\tilde{C}_\mu$ is isomorphic to a submodule of $\tilde{C}_{\mu+1}$ under right multiplication by σ and $\tilde{\delta}^{(\mu)}\sigma = \sigma + \left(\sigma^{\mu+1}\right) = \sigma\tilde{\delta}^{(\mu)}$; identifying $\tilde{\delta}^{(\mu-1)}$ with $\sigma\tilde{\delta}^{(\mu)}$, $\tilde{C}_\mu$ is embedded in $\tilde{C}_{\mu+1}$, and

$$\tilde{C}_\mu = \oplus_{i=1}^{\mu} \mathbf{K}\tilde{\delta}^{(i-1)}. \tag{7.4}$$

Set

$$\tilde{\Delta} := \varinjlim_{\mu} \tilde{C}_\mu = \oplus_{\mu \geq 0} \mathbf{K}\tilde{\delta}^{(\mu)}. \tag{7.5}$$

The left $\mathbf{S}$-module $\tilde{\Delta}$ becomes a left $\mathbf{L}$-vector space, setting $\sigma^{-1}\tilde{\delta}^{(\mu)} = \tilde{\delta}^{(\mu+1)}$, thus $\tilde{\Delta}$ becomes also a left $\mathbf{R}$-module by restriction of the ring of scalars. Considering σ and ∂ as operators on $\tilde{\Delta}$, σ is a left inverse of ∂, but σ has no left inverse since $\sigma\tilde{\delta} = 0$.

Exercise 7.2. (i) Prove that $\tilde{\Delta} = E\left(\tilde{C}_\mu\right)$ for any $\mu \geq 1$ (where $E\left(.\right)$ is the injective hull of the module in parentheses). (ii) Prove that $\tilde{C}_1$ is the only simple $\mathbf{S}$-module. (iii) Prove that $\tilde{\Delta}$ is the *canonical cogenerator* of ${}_{\mathbf{S}}\mathbf{Mod}$. (Hint: for (i), first show that $\tilde{\Delta}$ is divisible and then proceed as in the proof of ([7], Section 6.5.1, Proposition 6.26, parts (3) and (4)). For (ii), use ([7], Section 6.3.1, Excercise 6.8(ii)). For (iii), use ([7], Section 6.5.1, Theorem 6.9(i)).)

Remark 7.1. Let $\mathbf{E}$ be the endomorphism ring of $\tilde{\Delta}$ ([7], Section 6.5.1). If $\mathbf{S}$ is commutative (*i.e.* $\mathbf{K} = \mathbf{k}$), then $\mathbf{E} \cong \mathbf{S}$, since $\mathbf{S}$ is complete ([23], Section 3.H), thus these two rings can be identified (this is the main result of the "Matlis theory").

7.2.5 Matrices over S

Unimodular Matrices

Let us study the general linear group $\mathbf{GL}_n\left(\mathbf{S}\right)$, *i.e.* the set of unimodular matrices belonging to $\mathbf{S}^{n\times n}$ [11]:

Proposition 7.3. *(i) Let* $U = \sum_{i\geq 0} \Upsilon_i\,\sigma^i \in \mathbf{S}^{n\times n}$*, where* $\Upsilon_i \in \mathbf{K}^{n\times n}, i \geq 0$. *The matrix* U *belongs to* $\mathbf{GL}_n\left(\mathbf{S}\right)$ *if, and only if* Υ_0 *is invertible, i.e.* $\left|\Upsilon_0\right| \neq 0$. *(ii) Let* $U \in \mathbf{GL}_n\left(\mathbf{S}\right)$ *and* $k \in \mathbb{N}$. *There exist two matrices* $U_k, U'_k \in \mathbf{GL}_n\left(\mathbf{S}\right)$ *such that* $\sigma^k\,U = U_k\,\sigma^k$ *and* $U\,\sigma^k = \sigma^k\,U'_k$.

Proof. (i) If Υ_0 is invertible, U can be written in the form $\Upsilon_0\left(I_n - X\right), X \in \sigma\,\mathbf{S}^{n\times n}$. The matrix $I_n - X$ is invertible with inverse $\sum_{i\geq 0} X^i$. Conversely, if U is invertible, there exists $L = \sum_{i\geq 0} \Lambda_i\,\sigma^i \in \mathbf{S}^{n\times n}$ such that $U\,L = I_n$. This implies $\Upsilon_0\,\Lambda_0 = I_n$, thus Υ_0 is invertible. (ii) Let $U = \sum_{i\geq 0} \Upsilon_i\,\sigma^i \in \mathbf{GL}_n\left(\mathbf{S}\right)$. By (7.3), $\sigma\,U = \left(\sum_{i\geq 0} \Theta_i\,\sigma^i\right)\sigma$ with $\Theta_0 = \Upsilon_0^\beta$. The matrix Υ_0^β (whose entries are the images of the entries of Υ_0 by the automorphism β) is invertible, therefore $\sum_{i\geq 0} \Theta_i\,\sigma^i$ is unimodular by (i). Finally, (ii) is obtained by induction. ∎

Smith Canonical Form over S.

Let $B^{+} \in \mathbf{S}^{q\times k}$. By Proposition 7.2 and ([7], Section 6.3.3, Theorem 6.4), there exist matrices $U \in \mathbf{GL}_q(\mathbf{S})$ and $V \in \mathbf{GL}_k(\mathbf{S})$ such that $U\,B^{+}\,V^{-1} = \Sigma$ where

$$\Sigma = \mathrm{diag}\left(\sigma^{\mu_1}, ..., \sigma^{\mu_r}, 0, ..., 0\right), \quad 0 \leq \mu_1 \leq ... \leq \mu_r. \tag{7.6}$$

Let $\mu_i, 1 \leq i \leq s$ be the zero elements in the list $\{\mu_i, 1 \leq i \leq r\}$ (if any). The following proposition is obvious:

Proposition 7.4. *Σ is the Smith canonical form of B^{+} over* $\mathbf{S}$. *The noninvertible invariant factors σ^{μ_i} $(s+1 \leq i \leq r)$ of B^{+} coincide with its elementary divisors.*

Exercise 7.3. Prove that the Smith canonical form of a matrix $B^{+} \in \mathbf{S}^{q\times k}$ can be obtained using only elementary operations (*i.e.* secondary operations are not necessary).

Exercise 7.4. *Dieudonné determinant over* $\mathbf{S}$. Let $\mathbf{F}$ be a skew field. The "Dieudonné determinant" $|.|$ of a square matrix over $\mathbf{F}$ has the following properties [13]: (a) $|A| = 0$ if, and only if A is singular; (b) if $|A| \neq 0$, $|A| \in \mathbf{F}^{\times}/\mathbf{C}(\mathbf{F}^{\times})$, where $\mathbf{F}^{\times}$ is the multiplicative group consisting of all nonzero elements of $\mathbf{F}$ and $\mathbf{C}(\mathbf{F}^{\times})$ is the *derived group* of $\mathbf{F}^{\times}$, *i.e.* the subgroup generated by all elements $x^{-1}y^{-1}xy$, $x, y \in \mathbf{F}^{\times}$ ([1], Section I.6.2); (c) for any $\lambda \in \mathbf{F}^{\times}$, $|\lambda|$ is the canonical image of λ in $\mathbf{F}^{\times}/\mathbf{C}(\mathbf{F}^{\times})$; (d) $|.|$ is multiplicative, *i.e.* $|A\,B| = |A|\,|B|$; (e) if X is square and nonsingular, $\left|\begin{bmatrix} X & Y \\ Z & T \end{bmatrix}\right| = |X|\,\left|T - Z\,X^{-1}\,Y\right|$. (i) Let $\mathbf{U}$ be the multiplicative group consisting of all units of $\mathbf{S}$; prove that $\mathbf{C}(\mathbf{L}^{\times}) \subset \mathbf{U}$. (ii) Set $\mathbf{1} = \mathbf{U}/\mathbf{C}(\mathbf{L}^{\times})$ and let U be an elementary matrix ([7], Section 6.3.3); show that $|U| \in \mathbf{1}$. (iii) Prove that for any matrix $A \in \mathbf{GL}_n(\mathbf{S})$, $|A| \in \mathbf{1}$. (iv) For any nonsingular matrix $A \in \mathbf{S}^{n\times n}$, prove that there exists $|v| \in \mathbf{1}$ such that $|A| = |\sigma^{\mu}|\,|v|$, where $\mu = \sum_{s+1 \leq i \leq n} \mu_i$ and the σ^{μ_i} $(s+1 \leq i \leq n)$ are the elementary divisors of A. (Hint: for (iii) and (iv), reduce A to its Smith canonical form, and for (iii) show that A is a product of elementary matrices using the result to be proved in Exercise 7.3.)

Canonical Decomposition of an S-Module

Let M^{+} be a finitely generated (f.g.) $\mathbf{S}$-module. The following theorem is an immediate consequence of (7.6) (see [7], Section 6.3.3, Theorem 6.5).

Theorem 7.1. *(i) The following relations hold:*

$$(a) \quad M^{+} = \mathcal{T}\left(M^{+}\right) \oplus \Phi^{+}, \quad (b) \quad \mathcal{T}\left(M^{+}\right) \cong \bigoplus_{s+1 \leq i \leq r} \frac{\mathbf{S}}{\left(\sigma^{\mu_i}\right)}$$

$$(c) \quad \Phi^{+} \cong M^{+}/\mathcal{T}\left(M^{+}\right)$$

where $\mathcal{T}\left(M^{+}\right)$ is the torsion submodule of M^{+}, the module Φ^{+} is free, and $1 \leq \mu_{s+1} \leq ... \leq \mu_r$; the elements σ^{μ_i} $(s+1 \leq i \leq r)$ are uniquely determined from M^{+}. (ii) The module M^{+} can be presented by a short exact sequence

$$0 \longrightarrow \mathbf{S}^r \xrightarrow{\bullet B^{+}} \mathbf{S}^k \longrightarrow M^{+} \longrightarrow 0. \tag{7.7}$$

The terminology in the first part of the definition below is taken from ([2], Section VII.4.8).

Definition 7.4. *(i) The elements σ^{μ_i} $(s+1 \leq i \leq r)$ –or the ideals generated by them– are the* nonzero invariant factors *of M^{+}, and they coincide with its* nonzero elementary divisors*; the number of times a same element σ^{μ_i} is encountered in the list $\{\sigma^{\mu_i}, s+1 \leq i \leq r\}$ is the* multiplicity of that elementary divisor*; $rk\Phi^{+} = k - r$ is the* multiplicity of the elementary divisor 0. *(ii) The integer $\#\left(M^{+}\right) = \sum_{s+1 \leq i \leq r} \mu_i$ is called the* degree *of M^{+}.*

7.2.6 Formal Laurent Series

The Quotient Field L

The quotient field of $\mathbf{S}$ is $\mathbf{L} = \mathbf{K}\left((\sigma; \beta, \gamma)\right)$, the field of *formal Laurent series in* σ, equipped with the commutation rule (7.3). An element a of $\mathbf{L}$ is of the form

$$a = \sum_{i \geq \nu} a_i \, \sigma^i, \quad a_i \in \mathbf{K}, \quad a_\nu \neq 0,$$

where ν belongs to the ring $\mathbb{Z}$ of integers.

The rings $\mathbf{R}$, $\mathbf{Q}$ and $\mathbf{S}$ can be embedded in $\mathbf{L} = \mathbf{K}\left((\sigma; \beta, \gamma)\right)$; all these rings are integral domains, and are noncommutative except if $\mathbf{K} = \mathbf{k}$.

Smith-MacMillan Canonical Form over L

Let $G \in \mathbf{L}^{p \times m}$ be a matrix of rank r.

Theorem 7.2. *There exist matrices $W \in \mathbf{GL}_p\left(\mathbf{S}\right)$ and $V \in \mathbf{GL}_m\left(\mathbf{S}\right)$ such that*

$$W\,G\,V^{-1} = diag\left(\sigma^{\nu_1}, ..., \sigma^{\nu_r}, 0, ..., 0\right), \quad \nu_1 \leq ... \leq \nu_r, \tag{7.8}$$

and the integers $\nu_i \in \mathbb{Z}$ $(1 \leq i \leq r)$ are uniquely determined from G.

Proof. Let σ^k be a least common denominator of all entries of G and $A^+ = \sigma^k G \in \mathbf{S}^{p\times m}$. According to Proposition 7.4, there exist matrices $U \in \mathbf{GL}_p(\mathbf{S})$ and $V \in \mathbf{GL}_m(\mathbf{S})$ such that $U A^+ V^{-1} = \Sigma$, where Σ is given by (7.6). Therefore, $U \sigma^k G V^{-1} = \Sigma$, and by Proposition 7.3, $U \sigma^k = \sigma^k W$ where $W = U'_k \in \mathbf{GL}_p(\mathbf{S})$. Thus, the equality in (7.8) holds with $\nu_i = \mu_i - k$ $(1 \leq i \leq r)$. ■

Definition 7.5. *The matrix in the right-hand side of the equality in* (7.8) *is called the* Smith-MacMillan canonical form *of* $G = G(\partial)$ *over* $\mathbf{L}$.

7.3 Transmission Poles and Zeros at Infinity

7.3.1 Transfer Matrix of an Input-Output System

Let M be an input-output system with input u and output y ([7], Section 6.4.1); M is a f.g. left $\mathbf{R}$-module, the input $u = (u_i)_{1\leq i\leq m}$ (such that the module $M/[u]_{\mathbf{R}}$ is torsion) is assumed to be *independent*, and $y = (y_i)_{1\leq i\leq p}$.

Since $\mathbf{R}$ is a two-sided Ore domain, the functor $\mathbf{Q}\bigotimes_{\mathbf{R}} -$ is well-defined, it is covariant from the category of left $\mathbf{R}$-modules to the category of left $\mathbf{Q}$-vector spaces, and it is exact (*i.e.* $\mathbf{Q}$ is a flat $\mathbf{R}$-module: see [12], Sections 0.9 and Appendix 2). Let $\hat{M} = \mathbf{Q}\bigotimes_{\mathbf{R}} M$ and $\hat{\varphi} : M \to \hat{M}$ be the canonical map defined by $\hat{\varphi}(w) = \hat{w} = 1_{\mathbf{Q}} \otimes_{\mathbf{R}} w$, where $1_{\mathbf{Q}}$ is the unit-element of $\mathbf{Q}$; then $\ker \hat{\varphi} = \mathcal{T}(M)$ ([7], Section 6.3.1, Excercise 6.6).

Definition 7.6. *[15]* $\mathbf{Q}\bigotimes_{\mathbf{R}} -$ *is called the* Laplace functor.

The following theorem is due to Fliess ([14], [15]).

Theorem 7.3. *(i) $\hat{u}$ is a basis of $\hat{M}$. (ii) There exists a unique matrix $G(\partial) \in \mathbf{Q}^{p\times m}$ such that $\hat{y} = G(\partial)\,\hat{u}$.*

Proof. There exists a short exact sequence

$$0 \to [u]_{\mathbf{R}} \to M \to M/[u]_{\mathbf{R}} \to 0;$$

by exactness of the functor $\mathbf{Q}\bigotimes_{\mathbf{R}} -$, this yields the short exact sequence

$$0 \to [\hat{u}]_{\mathbf{Q}} \to \hat{M} \to 0 \to 0$$

since the module $M/[u]_{\mathbf{R}}$ is torsion; therefore, $\hat{M} = [\hat{u}]_{\mathbf{Q}}$. In addition, $\dim [\hat{u}]_{\mathbf{Q}} = \mathrm{rk}\,[u]_{\mathbf{R}} = m$ ([7], Section 6.4.1), thus $\hat{u}$ is a basis of $\hat{M}$. (ii) is an obvious consequence of (i). ■

Definition 7.7. *[15] The matrix $G(\partial) \in \mathbf{Q}^{p\times m}$ is called the* transfer matrix *of the input-output system.*

7.3.2 Structure at Infinity of a Transfer Matrix

Embedding $\mathbf{Q}$ into $\mathbf{L}$, a transfer matrix $G(\partial) \in \mathbf{Q}^{p\times m}$ can be considered as an element of $\mathbf{L}^{p\times m}$, thus its Smith-MacMillan canonical form over $\mathbf{L}$ can be determined. The following definition, taken from ([11], [8]), generalizes notions which are classical in the context of time-invariant linear systems ([19], [28], [27]).

Definition 7.8. *(i) The Smith-MacMillan canonical form* (7.8) *of* $G(\partial)$ *over* $\mathbf{L}$ *is called its* Smith-MacMillan canonical form at infinity. *(ii) Define the finite sequences* $(\bar{\varsigma}_i)_{1\leq i\leq r}$ *and* $(\bar{\pi}_i)_{1\leq i\leq r}$ *as:* $\bar{\varsigma}_i = \max(0,\nu_i)$ *and* $\bar{\pi}_i = \max(0,-\nu_i)$. *Among the natural integers* $\bar{\varsigma}_i$ *(resp.* $\bar{\pi}_i$*), those which are nonzero (if any) are called the* structural indexes *of the* zeros at infinity *(resp. of the* poles at infinity*) of the matrix* $G(\partial)$*; they are put in increasing (resp. decreasing) order and denoted by* $\varsigma_i, 1\leq i\leq \rho$ *(resp.* $\pi_i, 1\leq i\leq s$*). (iii) If* $\rho \geq 1$ *(resp.* $s\geq 1$*),* $G(\partial)$ *is said to have* ρ *zeros (resp.* s *poles) at infinity, the* i*-th one of* order ς_i *(resp.* π_i*). (iii) If* $\nu_1 > 0$, $G(\partial)$ *is said to have a* blocking zero at infinity of order ν_1. *The natural integer* $\#(TP_\infty) = \sum_{1\leq i\leq s} \pi_i$ *(resp.* $\#(TZ_\infty) = \sum_{1\leq i\leq \rho} \varsigma_i$*) is called the* degree *of the poles (resp. the zeros) at infinity of* $G(\partial)$.

Definition 7.9. *The poles (resp. the zeros) at infinity of* $G(\partial)$ *are called the* transmission poles *(resp. the* transmission zeros*)* at infinity *of the input-output system with transfer matrix* $G(\partial)$. *See also Exercise 7.17, Section 7.5.6.*

The matrix $G(\partial)$ can be expanded as:

$$G(\partial) = \sum_{i\geq \nu_1} \Theta_i\, \sigma^i, \quad \Theta_{\nu_1} \neq 0.$$

Definition 7.10. *The transfer matrix* $G(\partial)$ *is said to be* proper *(resp.* strictly proper*) if* $\nu_1 \geq 0$ *(resp.* $\nu_1 \geq 1$*). It is said to be* biproper *if it is invertible, proper and with a proper inverse [16].*

The following notion, introduced in [30] in the time-invariant case, was generalized in [25] to time-varying systems.

Definition 7.11. *The integer* $c_\infty(G) = \#(TP_\infty) - \#(TZ_\infty)$ *is called the* content at infinity *of* $G(\partial)$.

Note that $c_\infty(G) = -\sum_{1\leq i\leq r} \nu_i$, where the integers $\nu_i \in \mathbb{Z}$ are defined according to (7.8).

Exercise 7.5. Let $G \in \mathbf{Q}^{p \times m}$ and denote by g_{ij} the entries of G ($1 \le i \le p$, $1 \le j \le m$). Write $g_{ij} = a_{ij}^{-1} b_{ij}$, $0 \neq a_{ij} \in \mathbf{R}$, $b_{ij} \in \mathbf{R}$. Let $a \in \mathbf{R}$ be an l.c.l.m. of all elements a_{ij} (*i.e.* a least common left denominator of all elements g_{ij}); this means that

$$\mathbf{R}\, a = \bigcap_{\substack{1 \le i \le p \\ 1 \le j \le m}} \mathbf{R}\, a_{ij}$$

([7], Section 6.2.3). Let $aG = C$, $L \in \mathbf{R}^{p \times p}$ be a g.c.l.d. of aI_p and C ([7], Section 6.3.3, Excercise 6.13), set $D_l = L^{-1} a I_p$ and $N_l = L^{-1} C$. Prove that $(D_l(\partial), N_l(\partial))$ is a left-coprime factorization of $G(\partial)$ over $\mathbf{R}$, *i.e.* $G = D_l^{-1} N_l$ and $\{D_l, N_l\}$ are left-coprime over $\mathbf{R}$.

Let $(D_l(\partial), N_l(\partial))$ be a left-coprime factorization of $G(\partial)$ over $\mathbf{R}$. The input-output system $M = [y, u]_{\mathbf{R}}$ defined by the equation $D_l(\partial)\, y = N_l(\partial)\, u$ is observable and controllable ([7], Section 6.4.1, Excercise 6.20). Let $\dim_{\mathbf{K}}(M / [u]_{\mathbf{R}})$ be the dimension of the $\mathbf{K}$-vector space $M/[u]_{\mathbf{R}}$ (*i.e.* the "order" of the above input-output system).

Definition 7.12. *The natural integer* $\dim_{\mathbf{K}}(M/[u]_{\mathbf{R}}) + \#(TP_\infty)$ *is called the* MacMillan degree *of* $G(\partial)$, *and is denoted by* $\delta_M(G)$.

Remark 7.2. If $G(\partial)$ is a *polynomial matrix*, $\delta_M(G) = \#(TP_\infty) = c_\infty(G) + \#(TZ_\infty)$.

Exercise 7.6. [25] Let $G_1(\partial) \in \mathbf{Q}^{p \times r}$ and $G_2(\partial) \in \mathbf{Q}^{r \times m}$ be two matrices of rank r. Prove that $c_\infty(G_1 G_2) = c_\infty(G_1) + c_\infty(G_2)$. (Hint: using the Dieudonné determinant and its properties established in Exercise 7.4, first show that $c_\infty(\bar{G}_1 U \bar{G}_2) = c_\infty(\bar{G}_1) + c_\infty(\bar{G}_2)$ when $\bar{G}_1(\partial) = \operatorname{diag}\{\sigma^{\mu_i}\}_{1 \le i \le r}$, $\bar{G}_2(\partial) = \operatorname{diag}\{\sigma^{\nu_i}\}_{1 \le i \le r}$ and $U \in \mathbf{GL}_r(\mathbf{S})$.)

Exercise 7.7. Let $A(\partial) \in \mathbf{Q}^{p \times m}$ and $B(\partial) \in \mathbf{Q}^{q \times m}$ be two matrices of rank r and set $F(\partial) = \begin{bmatrix} A(\partial) \\ B(\partial) \end{bmatrix}$. (i) Assuming that the Smith-MacMillan form at infinity of $A(\partial)$ and of $B(\partial)$ are $\operatorname{diag}(\sigma^{\nu_1}, ..., \sigma^{\nu_r}, 0, ..., 0)$ and $\operatorname{diag}(\sigma^{\lambda_1}, ..., \sigma^{\lambda_r}, 0, ..., 0)$, respectively, show that the Smith-MacMillan form at infinity of $F(\partial)$ is $\operatorname{diag}(\sigma^{\varepsilon_1}, ..., \sigma^{\varepsilon_r}, 0, ..., 0)$ with $\varepsilon_i = \min\{\nu_i, \lambda_i\}, 1 \le i \le r$. (ii) Deduce from (i) that if $F(\partial)$ has no pole at infinity, then $A(\partial)$ and $B(\partial)$ have the same property and that $c_\infty(F) \ge \max\{c_\infty(A), c_\infty(B)\}$. (iii) Show that $\begin{bmatrix} A(\partial) \\ I_m \end{bmatrix}$ and $A(\partial)$ have the same poles at infinity (with the same orders). (Hint: for (i), $B(\partial)$ and $B(\partial)\, V$ have the same Smith-MacMillan form over $\mathbf{S}$ if $V \in \mathbf{GL}_m(\mathbf{S})$; use row elementary operations once $A(\partial)$ and $B(\partial)\,V$ have been reduced to their Smith-MacMillan form over $\mathbf{S}$, where V is a suitable element of $\mathbf{GL}_m(\mathbf{S})$.)

Exercise 7.8. *Proper model matching [25].* Let $A(\partial) \in \mathbf{Q}^{p\times m}$ and $B(\partial) \in \mathbf{Q}^{q\times m}$. The *model matching problem* is to determine a matrix $H(\partial) \in \mathbf{Q}^{q\times p}$ such that $H(\partial)\,A(\partial) = B(\partial)$. The *proper model matching problem* is to determine a *proper* solution $H(\partial) \in \mathbf{Q}^{q\times p}$ to the model matching problem. (i) Let $F(\partial)$ be defined as in Exercise 7.7. Show that the model matching problem has a solution if, and only if

$$\operatorname{rk} A(\partial) = \operatorname{rk} F(\partial). \tag{7.9}$$

(ii) Let $H(\partial) \in \mathbf{Q}^{q\times p}$; show that $H(\partial)$ is proper if, and only if, $c_\infty\left(\begin{bmatrix} H(\partial) \\ I_p \end{bmatrix}\right) = 0$. (iii) Considering the model matching problem, and assuming that $m = r = \operatorname{rk} A(\partial)$, show that there exists an invertible matrix $Q(\partial) \in \mathbf{Q}^{r\times r}$ such that $F(\partial) = \bar{F}(\partial)\,Q(\partial)$ where $\bar{F}(\partial)$ has no pole and no zero at infinity and $Q(\partial)$ has the same structural indexes at infinity as $F(\partial)$. (iv) Let $\bar{F}(\partial) = \begin{bmatrix} \bar{A}(\partial) \\ \bar{B}(\partial) \end{bmatrix}$; the model matching problem can be written in the form $\begin{bmatrix} H(\partial) \\ I_p \end{bmatrix} \bar{A}(\partial) = \bar{F}(\partial)$. Prove that $H(\partial)$ is proper if and only if

$$c_\infty(A) = c_\infty(F). \tag{7.10}$$

(v) To summarize: (7.9) and (7.10) are a *necessary and sufficient condition* for the *proper model matching problem to have a solution* when $A(\partial)$ is full column rank. (Hint: use the results to be proved in exercises 7.6 and 7.7.)

7.4 Impulsive Systems and Behaviors

7.4.1 Temporal Systems

Definition of a Module by Generators and Relations

Let $M = \operatorname{coker} \bullet B(\partial)$ be a system, where $B(\partial) \in \mathbf{R}^{q\times k}$. The system equations can be written

$$\begin{cases} B(\partial)\,w = e, \\ e = 0. \end{cases} \tag{7.11}$$

The above module M is said to be defined by generators and relations ([7], Section 6.3.1, Definition 6.1). Equations (7.11) correspond to the exact sequence

$$\mathbf{R}^q \overset{\bullet B}{\to} \mathbf{R}^k \overset{\varphi}{\to} M \to 0. \tag{7.12}$$

The module of generators is $\mathbf{R}^k = [\mathring{w}]_{\mathbf{R}}$ where $\mathring{w} = (\mathring{w}_i)_{1 \leq i \leq k}$ is the canonical basis of $\mathbf{R}^k$; the module of relations is $\operatorname{Im} \bullet B = [\mathring{e}]_{\mathbf{R}}$ where $\mathring{e} = B(\partial)\,\mathring{w} = (\mathring{e}_j)_{1 \leq j \leq q}$. Let w_i and e_j be the canonical image of $\mathring{w}_i$ and $\mathring{e}_j$, respectively, in the quotient $M = \mathbf{R}^k / [\mathring{e}]_{\mathbf{R}}$ $(1 \leq i \leq k, 1 \leq j \leq q)$. Equations (7.11) are satisfied, and the second one (*i.e.* $e = 0$) expresses the fact that the relations existing between the system variables are active.

Continuous-time Temporal System

Assuming that $\mathbf{K} = \mathrm{Re}$, set $\mathbb{T} = \mathrm{Re}$ and $\mathbb{T}_0 = [0, +\infty[$. In place of (7.11), consider the equations

$$\begin{cases} B(\partial)\, w(t) = e(t), \; t \in \mathbb{T}, \\ \quad e(t) = 0, \; t \in \mathbb{T}_0. \end{cases} \tag{7.13}$$

The relations between the system variables are now active only during the time period $\mathbb{T}_0$, *i.e.* the system is *formed at initial time zero* (due, *e.g.*, to a failure or a switch). On the complement $\mathbb{T} \setminus \mathbb{T}_0$ of $\mathbb{T}_0$ in $\mathbb{T}$, e can be any C^∞ function. Let us give a provisional definition of a temporal system (the final one is given in Section 7.4.5):

Definition 7.13. *[8] The system of differential equations* (7.13) *is the* temporal system with matrix of definition $B(\partial)$*.*

Once the input and output variables have been chosen, one can assume without loss of generality that the first line of (7.13) corresponds to a polynomial matrix description (PMD) ([7], Section 6.4.1), *i.e.*

$$B(\partial) = \begin{bmatrix} D(\partial) & -N(\partial) & 0 \\ Q(\partial) & W(\partial) & -I_p \end{bmatrix}, \tag{7.14}$$

with module of generators $\mathbf{R}^k = [\mathring{w}]_{\mathbf{R}}$ where $\mathring{w} = \begin{bmatrix} \mathring{\xi}^T & \mathring{u}^T & \mathring{y}^T \end{bmatrix}^T$. In that case, the temporal system under consideration is called an *input-output temporal system* (this definition is consistent with the one in ([7], Section 6.4.1) of an *input-output system.*

Discrete-time Temporal System

Assuming that $\mathbf{K} = \mathrm{Re}$, set $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T}_0 = \{..., -2, -1, 0\}$. Definition 7.13 still holds (since "discrete-time differential equations" are well-defined: see [7], Section 6.1), and it means that the relations between the system variables are active only *up to final time zero.* On $\mathbb{T}\setminus\mathbb{T}_0$, the sequence $(e(t))_{t\in\mathbb{T}}$ can have any values.

Such temporal systems are encountered in various fields, notably in economy; see [8] and the references therein for more details.

7.4.2 A Key Isomorphism

Continuous-time Case

From the analytic point of view, the temporal system $\mathbf{\Sigma}$ defined by (7.13) is formed as follows: take for e in the first line of (7.13) any $\mathcal{C}^\infty$ function; then multiply e by $1-\Upsilon$, where Υ is the Heaviside function (*i.e.* $\Upsilon(t)=1$ for $t>0$ and 0 otherwise). Let $W=\mathcal{C}^\infty(\mathrm{Re};\mathrm{Re})$ and set

$$\varDelta=\oplus_{\mu\geq 0}\mathrm{Re}\;\delta^{(\mu)} \tag{7.15}$$

where δ is the Dirac distribution. The $\mathbf{R}$-module generated by $S_0:=(1-\Upsilon)W$ is (as Re-vector space): $S=S_0\oplus\varDelta$. The operator ∂ is an automorphism of the Re-vector space S, and $\sigma=\partial^{-1}$ is the operator defined on S by: $(\sigma w)(t)=\int_{+\infty}^{t}w(\tau)\,d\tau$. The space S is an $\mathbf{L}$-vector space (and thus an $\mathbf{S}$-module which is an $\mathbf{R}$-module, by restriction of the ring of scalars), and S_0 is an $\mathbf{S}$-submodule of S. The $\mathbf{R}$-module $\varDelta$ is not an $\mathbf{S}$-module, but $\varDelta\cong_{\mathrm{Re}}S/S_0$; the set S/S_0 (denoted by $\bar{\varDelta}$ in the sequel) is clearly an $\mathbf{L}$-vector space (and thus an $\mathbf{R}$-module which is an $\mathbf{S}$-module). The nature of the above isomorphism, denoted by τ, can be further detailed:

Lemma 7.1. *The isomorphism τ, defined as: $\varDelta\ni\lambda\,\delta\overset{\backsim}{\longrightarrow}\lambda\,\bar{\delta}\in\bar{\varDelta}$, is* $\mathbf{R}$-*linear.*

Proof. First, notice that any element of $\varDelta$ (resp. $\bar{\varDelta}$) is uniquely expressible in the form $\lambda\,\delta$ (resp. $\lambda\,\bar{\delta}$) for some $\lambda\in\mathbf{R}$, thus τ is a well-defined $\mathbb{Z}$-isomorphism. In addition, for any $x\in\varDelta$, such that $x=\lambda\,\delta,\lambda\in\mathbf{R}$, and any $\mu\in\mathbf{R}$, $\tau(\mu\,x)=\tau(\mu\lambda\,\delta)=\mu\lambda\,\bar{\delta}=\mu\,\tau(x)$. ■

Therefore,

$$\varDelta\cong_{\mathbf{R}}\bar{\varDelta} \tag{7.16}$$

We have $\sigma\delta=\Upsilon-1$; setting $\bar{\delta}=\tau(\delta)$, we obtain $\sigma\bar{\delta}=0$, thus $\tilde{\delta}$ and $\bar{\delta}$ can be identified, as well as the $\mathbf{S}$-modules $\tilde{\varDelta}$ and $\bar{\varDelta}$. As a result, by (7.15), (7.5), we can write

$$\tilde{\varDelta}=\bar{\varDelta}=\oplus_{\mu\geq 0}\mathrm{Re}\tilde{\delta}^{(\mu)}. \tag{7.17}$$

The canonical epimorphism $S\rightarrow\tilde{\varDelta}$ is denoted by $\tilde{\phi}$. Let θ be the Re-linear projection $S_0\oplus\varDelta\rightarrow\varDelta$; the following diagram is commutative:

$$\begin{array}{ccc} S_0 \oplus \varDelta & \xrightarrow{\tilde{\phi}} & \tilde{\varDelta} \\ \downarrow \theta & \tau \nearrow & \\ \varDelta & & \end{array} \tag{7.18}$$

Discrete-time Case

Let $\varUpsilon$ be the sequence defined by $\varUpsilon(t) = 1$ for $t > 0$ and 0 otherwise. From the analytic point of view, the temporal system $\mathbf{\Sigma}$ defined by (7.13) is formed as follows: take for e in the first line of (7.13) any sequence; then multiply e by $\varUpsilon$. Set $W = \mathrm{Re}^{\mathbb{Z}}$ and $S_0 = \varUpsilon W$. Let $\varDelta$ be defined as in (7.15), but where $\delta := \partial\varUpsilon$ is the "Kronecker sequence", such that $\delta(t) = 1$ for $t = 0$ and 0 otherwise (thus, $\varDelta$ is the $\mathbf{R}$-module consisting of all sequences with left and finite support). The $\mathbf{R}$-module generated by S_0 is (as Re-vector space) $S = S_0 \oplus \varDelta$. The operator ∂ is an automorphism of the Re-vector space S, and $\sigma = \partial^{-1}$ is the operator defined on S by: $(\sigma w)(t) = \sum_{j=-\infty}^{t-1} w(j)$; S is an $\mathbf{L}$-vector space. The $\mathbf{R}$-isomorphism (7.16) still holds; the same identifications as in the continuous-time case can be made and the same notation can be used. Obviously, the continuous- and discrete-time cases are now completely analogous.

7.4.3 Impulsive Behavior

Assuming that $\mathbf{K} = \mathrm{Re}$, consider a temporal system $\mathbf{\Sigma}$ with matrix of definition $B(\partial) \in \mathbf{R}^{q \times k}$.

Proposition 7.5. *The following properties are equivalent: (i) For any $e \in S_0^q$, there exists $w \in S^k$ such that* (7.13) *is satisfied. (ii) The matrix $B(\partial)$ is full row rank.*

Proof. (i) $\Rightarrow$ (ii): If the matrix $B(\partial)$ is not full row rank, $\bullet B(\partial)$ is not injective, *i.e.* there exists a nonzero element $\eta(\partial) \in \mathbf{R}^q$ (*i.e.* a $1 \times q$ matrix with entries in $\mathbf{R}$) such that $\eta(\partial)\, B(\partial) = 0$. Therefore, for $w \in S^k$ and $e \in S_0^q$ to satisfy (7.13), e must satisfy the "compatibility condition" $\eta(\partial)\, e = 0$ (see [21], [20], where this compatibility condition is further detailed). (ii) $\Rightarrow$ (i): By (7.8) with $B = G$, assuming that $q = r$, (7.13) is equivalent to

$$\begin{bmatrix} \mathrm{diag}\,\{\sigma^{\nu_i}\}_{1 \le i \le r} & 0 \end{bmatrix} v = h \tag{7.19}$$

where $v = V(\sigma)\, w$ and $h = W(\sigma)\, e$; (7.19) is equivalent to $\sigma^{\nu_i} v_i = h_i$, $1 \le i \le q$. For any $\nu_i \in \mathbb{Z}$ and any $h_i \in S_0$, $v_i = \partial^{\nu_i} h_i$ belongs to S.

Therefore, (i) holds because h spans S_0^q as e spans the same space (since S_0 is an $\mathbf{S}$-module). ■

In the sequel, the matrix $B(\partial)$ is assumed to be full row rank (*i.e.* $q = r$).

Notation 1 *For any scalar operator ω and any integer $l \geq 1$, $\omega_{(l)}$ denotes the operator diag $(\omega, ..., \omega)$, where ω is repeated l times.*

Definition 7.14. *[8] Let $\mathcal{W} \subset S^k$ be the space spanned by the elements w satisfying* (7.13) *as e spans S_0^q. The* impulsive behavior *of $\mathbf{\Sigma}$ is: $\mathcal{B}_\infty = \theta_{(k)}\mathcal{W}$.*

Definition 7.15. *[8] The* pseudo-impulsive behavior *of $\mathbf{\Sigma}$ is: $\mathcal{A}_\infty = \tau_{(k)}\mathcal{B}_\infty$.*

7.4.4 Impulsive System

Assuming that the right-hand member of the equality in (7.8) is the Smith-MacMillan form at infinity of $B(\partial)$ with $q = r$, $m = k$ and $W = U$, set

$$\varPi(\sigma) = \text{diag}\left\{\sigma^{\bar{\pi}_i}\right\}_{1 \leq i \leq r}, \quad \varSigma(\sigma) = \text{diag}\left\{\sigma^{\bar{\varsigma}_i}\right\}_{1 \leq i \leq r} \tag{7.20}$$

so that diag $\{\sigma^{\nu_i}\}_{1 \leq i \leq r} = \varPi^{-1}(\sigma)\,\varSigma(\sigma) = \varSigma(\sigma)\,\varPi^{-1}(\sigma)$. By (7.8),

$$B(\partial) = A^{-1}(\sigma)\,B^+(\sigma) = B'^+(\sigma)\,A'^{-1}(\sigma) \tag{7.21}$$

where

$$A = \varPi\,U, \quad B^+ = \begin{bmatrix} \varSigma & 0 \end{bmatrix} V, \tag{7.22}$$

$$A' = V^{-1}(\varPi \oplus I_{k-r}), \quad B'^+ = U^{-1}\begin{bmatrix} \varSigma & 0 \end{bmatrix}. \tag{7.23}$$

The above expressions, the results to be proved in exercises 7.9 and 7.10, and Definitions 7.16 and 7.17 below, are valid when $\mathbf{K}$ is any differential field.

Exercise 7.9. (i) The above pair $(A(\sigma), B^+(\sigma))$ (resp. $(B'^+(\sigma), A'(\sigma))$) is a left-coprime (resp. right-coprime) factorization of $B(\partial)$ over $\mathbf{S}$. (ii) Let $\left(A_1(\sigma), B_1^+(\sigma)\right)$ and $\left(A_2(\sigma), B_2^+(\sigma)\right)$ be two left-coprime factorizations of $B(\partial)$ over $\mathbf{S}$; then, there exists a unimodular matrix $W(\sigma)$ over $\mathbf{S}$ such that $B_2^+(\sigma) = W(\sigma)\,B_1^+(\sigma)$ and $A_2(\sigma) = W(\sigma)\,A_1(\sigma)$. (iii) A similar result holds for right-coprime factorizations of $B(\partial)$ over $\mathbf{S}$; make it explicit. (Hint: see, *e.g.*, [31], Section 4.1, (43).)

Let $(A(\sigma), B^+(\sigma))$ be any left-coprime factorization of $B(\partial)$ over $\mathbf{S}$. According to the result to be proved in Exercise 7.9(ii), the module $M^+ =$ coker $\bullet\, B^+(\sigma)$ is uniquely defined from $B(\partial)$.

Definition 7.16. *[8] (i) The* $\mathbf{S}$*-module* $M^{+} = coker \bullet B^{+}(\sigma)$ *is called the* impulsive system. *(ii) The torsion submodule of* M^{+}*, written* $\mathcal{T}(M^{+})$*, is called the* module of uncontrollable poles at infinity*.*

Exercise 7.10. Let $B(\partial) = [C(\partial)\ D(\partial)]$, $D(\partial) \in \mathbf{R}^{r\times m_2}$ and $C(\partial) \in \mathbf{R}^{r\times m_1}$. Assuming that $\operatorname{rk} B(\partial) = r$, let $(A(\sigma), B^{+}(\sigma))$ be a left-coprime factorization of $B(\partial)$ over $\mathbf{S}$, and set $B^{+}(\sigma) = \begin{bmatrix} C^{+}(\sigma) & D^{+}(\sigma) \end{bmatrix}$, $C(\sigma) \in \mathbf{S}^{r\times m_1}$, $D(\sigma) \in \mathbf{S}^{r\times m_2}$. (i) Prove that the Smith-MacMillan form of $B^{+}(\sigma)$ over $\mathbf{S}$ is $\begin{bmatrix} I_r & 0 \end{bmatrix}$ if, and only if $B(\partial)$ has no zero at infinity. (ii) Deduce that $\{C^{+}(\sigma), D^{+}(\sigma)\}$ are left-coprime if, and only if $\begin{bmatrix} C(\partial) & D(\partial) \end{bmatrix}$ has no zero at infinity.

Definition 7.17. *(i) Let* $C(\partial) \in \mathbf{R}^{r\times m_1}$ *and* $D(\partial) \in \mathbf{R}^{r\times m_2}$ *be such that* $rk \begin{bmatrix} C(\partial) & D(\partial) \end{bmatrix} = r$*. The matrices* $C(\partial)$ *and* $D(\partial)$ *are said to be* left-coprime at infinity *if* $\begin{bmatrix} C(\partial) & D(\partial) \end{bmatrix}$ *has no zero at infinity. (ii)* Right-coprimeness at infinity *is defined analogously.*

The connection between the pseudo-impulsive behavior $\mathcal{A}_\infty$ and the impulsive system M^{+} is given below, with the notation $(.)^{*} := \operatorname{Hom}_{\mathbf{S}}\left(., \tilde{\Delta}\right)$ ([7], Section 6.5.1). Let us assume that $\mathbf{K} = \mathrm{Re}$.

Theorem 7.4. *[8]* $\mathcal{A}_\infty = (M^{+})^{*}$.

Proof. By Definition 7.15 and the commutativity of the diagram (7.18), $\mathcal{A}_\infty$ is the $\mathbf{E}$-module consisting of all elements $\tilde{w} = \tilde{\phi}_{(k)} w$ for which there exists $h \in S_0^q$ such that (7.19) is satisfied (recall that $q = r$). With the notation in the proof of Proposition 7.5, this equation is equivalent to $\sigma^{\nu_i} v_i = h_i, 1 \leq i \leq q$. For any index i such that $\nu_i \leq 0$, $v_i = \sigma^{-\nu_i} h_i$ belongs to S_0, thus $\tilde{v}_i = 0$ (where $\tilde{v}_i := \tilde{\phi}\, v_i$). Therefore, $\mathcal{A}_\infty$ is the $\mathbf{S}$-module consisting of all elements $\tilde{w} = V^{-1}(\sigma)\, \tilde{v}$ such that $\tilde{v} \in \tilde{\Delta}^k$ satisfies $\begin{bmatrix} \Sigma(\sigma) & 0 \end{bmatrix} \tilde{v} = 0$; as a result, $\mathcal{A}_\infty = \ker B^{+}(\sigma) \bullet$. ■

Remark 7.3. Let $\mathbf{K}$ be *any differential field.* The equality in the statement of Theorem 7.4 still makes sense, thus this equality becomes the *definition* of $\mathcal{A}_\infty$ (for the construction of $\mathcal{B}_\infty$ when $\mathbf{K}$ is a *differential ring*, see [8]).

Proposition 7.6. *[8] Let* $\mathbf{K}$ *be any differential field; for any natural integer* $\mu, \tilde{C}_\mu^{*} = \tilde{C}_\mu$.

Proof. For $\mu = 0$, $\tilde{C}_\mu = \tilde{C}_\mu^{*} = 0$. For $\mu \geq 1$, $\tilde{C}_\mu^{*}$ is the set of all elements $x \in \tilde{\Delta}$ such that $\sigma^\mu x = 0$. Obviously, $\tilde{\delta}^{(i-1)}$ belongs to $\tilde{C}_\mu^{*}$ if, and only if $1 \leq i \leq \mu$. By (7.4), $\tilde{C}_\mu^* \subset \tilde{C}_\mu$. Let us prove by induction the reverse inclusion. By (7.3), for any $a \in \mathbf{K}$, $\sigma a \tilde{\delta} = \left(a^\beta \sigma - \sigma a^{\beta\gamma} \sigma\right) \tilde{\delta} = 0$, which implies that $\tilde{C}_1 = \mathbf{K}\tilde{\delta} \subset \tilde{C}_1^*$. Assuming that $\tilde{C}_\mu \subset \tilde{C}_\mu^*, \mu \geq 1$, let $a \in \mathbf{K}$; then, $\sigma^{\mu+1} a \tilde{\delta}^{(\mu)} = \sigma^\mu \left(a^\beta - \sigma a^{\beta\gamma}\right) \tilde{\delta}^{(\mu-1)}$; by hypothesis, $a^\beta \tilde{\delta}^{(\mu-1)}$ and $\sigma a^{\beta\gamma} \tilde{\delta}^{(\mu-1)}$ belong to $\tilde{C}_\mu^*$, thus $\sigma^{\mu+1} a \tilde{\delta}^{(\mu)} = 0$, which implies that $\tilde{C}_{\mu+1} \subset \tilde{C}_{\mu+1}^*$. ■

As $\tilde{\Delta}$ is a cogenerator of ${}_{\mathbf{S}}\mathbf{Mod}$ (see Section 7.2.4, Exercise 7.2), the following result is a consequence of ([7], Section 6.5.2, Proposition 6.28) and of Proposition 7.6, assuming that the impulsive system M^+ has the structure in Definition 7.4:

Proposition 7.7. *(i) There exist sub-behaviors $\mathcal{A}_{\infty,c} \simeq (\Phi^+)^*$ and $\mathcal{A}_{\infty,u} \simeq (\mathcal{T}(M^+))^*$ of $\mathcal{A}_\infty$ such that $\mathcal{A}_\infty = \mathcal{A}_{\infty,c} \oplus \mathcal{A}_{\infty,u}$. (ii) The sub-behavior $\mathcal{A}_{\infty,c}$ satisfying this property is unique and such that $\mathcal{A}_{\infty,c} \cong_{\mathbf{E}} \tilde{\Delta}^{k-r}$ ($\mathcal{A}_{\infty,c}$ is called the "controllable pseudo-impulsive behavior"). (iii) $\mathcal{A}_{\infty,u} \cong_{\mathbf{E}} \prod_{i=s+1}^{r} \tilde{C}_{\mu_i}$ (this sub-behavior, unique up to* **E***-isomorphism, is said to be "uncontrollable").*

Remark 7.4. (i) According to Theorem 7.4, $\mathcal{A}_\infty$ is a "behavior" in the sense specified in ([7], Section 6.5.2), *i.e.* a *kernel*, whereas $\mathcal{B}_\infty$ (deduced from $\mathcal{A}_\infty$ using Definition 7.15) cannot be expressed in a so simple way (in this meaning, the expression "impulsive behavior" is an abuse of language). The notion of "sub-behavior" of a pseudo-impulsive behavior $\mathcal{A}_\infty$ is defined in accordance with the general definition in ([7], Section 6.5.2). (ii) A construction of $\mathcal{B}_\infty$ using the "causal Laplace transform" (in the continuous-time case) and the "anti-causal Z-transform" (in the discrete-time case) is developed in [5] and [6]. This construction has connections with the pioneer work of Verghese [29], and with the recent contributions [21] and [20] (where the approach is quite different and limited to the case of systems with constant coefficients).

Assuming that $\mathbf{K} = \mathrm{Re}$, set for any integer $\mu \geq 1$

$$C_\mu = \tau^{-1}\left(\tilde{C}_\mu\right) = \oplus_{i=1}^{\mu} \mathrm{Re}\delta^{(i-1)}. \tag{7.24}$$

The following theorem is an obvious consequence of Proposition 7.7:

Theorem 7.5. *The impulsive behavior $\mathcal{B}_\infty$ can be expressed as: $\mathcal{B}_\infty = \mathcal{B}_{\infty,c} \oplus \mathcal{B}_{\infty,u}$, where $\mathcal{B}_{\infty,c} := \tau_{(k)}^{-1} \mathcal{A}_{\infty,c} \cong_{\mathrm{Re}} \Delta^\kappa$ and $\mathcal{B}_{\infty,u} = \tau_{(k)}^{-1} \mathcal{A}_{\infty,u} \cong_{\mathrm{Re}} \prod_{i=1}^{\rho} C_{\mu_i}$ (the space $\mathcal{B}_{\infty,c}$, which is uniquely determined, is called the "controllable impulsive behavior", and the impulsive behavior $\mathcal{B}_{\infty,u}$, unique up to* Re-*isomorphism, is said to be "uncontrollable").*

7.4.5 Generalization of the Notion of Temporal System

Up to now, the notion of temporal system has been defined when the coefficient field is $\mathbf{K} = \mathrm{Re}$. Our aim in this section is to generalize this notion to a coefficient field $\mathbf{K}$ which is any differential field.

Strict Equivalence

Let $\mathbf{D}$ be a principal ideal domain[2] and M_i be a f.g. $\mathbf{D}$-module with matrix of definition $B_i \in \mathbf{D}^{q_i \times k_i}$ $(i = 1, 2)$, assumed to be full row rank. The following result is classical ([12], Section 0.6, Theorem 6.2 and Proposition 6.5):

Proposition 7.8. *The two following properties are equivalent: (i)* $M_1 \cong M_2$*; (ii)* B_1 *and* B_2 *satisfy a "comaximal relation", i.e. there exist two matrices* $A_1 \in \mathbf{D}^{q_1 \times q_2}$ *and* $A_2 \in \mathbf{D}^{k_1 \times k_2}$ *such that*

$$\begin{bmatrix} B_1 & A_1 \end{bmatrix} \begin{bmatrix} -A_2 \\ B_2 \end{bmatrix} = 0, \tag{7.25}$$

$\begin{bmatrix} B_1 & A_1 \end{bmatrix}$ *is right-invertible (i.e.* $\{B_1, A_1\}$ *are left-coprime) and* $\begin{bmatrix} -A_2 \\ B_2 \end{bmatrix}$ *is left-invertible (i.e.* $\{B_2, A_2\}$ *are right-coprime).*

Definition 7.18. *[9] Consider two PMDs* $\{D_i, N_i, Q_i, W_i\}$ *with matrices over* $\mathbf{R} = \mathbf{K}[\partial; \alpha, \gamma]$*, partial state* ξ_i*, input* u *and output* y $(i = 1, 2)$*. These PMD are said to be* strictly equivalent *if the diagram below is commutative:*

$$\begin{array}{ccc} [\xi_1, u]_{\mathbf{R}} & \xrightarrow{\chi} & [\xi_2, u]_{\mathbf{R}} \\ i_1 \nwarrow & & i_2 \nearrow \\ & [u, y]_{\mathbf{R}} & \end{array} \tag{7.26}$$

where $[u, y]_{\mathbf{R}} = [u]_{\mathbf{R}} + [y]_{\mathbf{R}}$*,* i_1 *and* i_2 *are the canonical injections and* χ *is an* $\mathbf{R}$*-isomorphism.*

Proposition 7.9. *The two above PMDs are strictly equivalent if, and only if there exist matrices* M_1, X_1, M_2, X_2 *over* $\mathbf{R}$*, of appropriate dimension, such that*

$$\begin{bmatrix} M_1 & 0 \\ -X_1 & I_p \end{bmatrix} \begin{bmatrix} D_2 & -N_2 \\ Q_2 & W_2 \end{bmatrix} = \begin{bmatrix} D_1 & -N_1 \\ Q_1 & W_1 \end{bmatrix} \begin{bmatrix} M_2 & X_2 \\ 0 & I_m \end{bmatrix}, \tag{7.27}$$

$$\left\{ \begin{bmatrix} D_1 & -N_1 \end{bmatrix}, M_1 \right\} \text{ are left coprime over } \mathbf{R}, \tag{7.28}$$

$$\{D_2, M_2\} \text{ are right coprime over } \mathbf{R}. \tag{7.29}$$

[2] More general rings can be considered: see [12].

Proof . 1) Set $\tilde{B}_i = \begin{bmatrix} D_i & -N_i \end{bmatrix}$ $(i = 1, 2)$. By Proposition 7.8, the isomorphism χ in (7.26) exists if, and only if the two matrices $\tilde{B}_1$ and $\tilde{B}_2$ satisfy a comaximal relation, *i.e.* there exist two matrices $A_1 = M_1$ and $A_2 = \begin{bmatrix} M_2 & X_2 \\ Y_2' & Z_2 \end{bmatrix}$ such that $\tilde{B}_1 A_2 = M_1 \tilde{B}_2$ with the suitable coprimeness properties. 2) The existence of the canonical injections i_1 and i_2 in (7.26) means that (a) we have $\begin{bmatrix} \xi_1 \\ u \end{bmatrix} = \begin{bmatrix} M_2 & X_2 \\ Y_2' & Z_2 \end{bmatrix} \begin{bmatrix} \xi_2 \\ u \end{bmatrix}$, thus $Y_2' = 0$ and $Z_2 = I_m$; (b) we have $y = Q_2 \xi_2 + W_2 u = Q_1 \xi_1 + W_1 u$. The latter quantity can be expressed as: $Q_1 \xi_1 + W_1 u = Q_1 (M_2 \xi_2 + X_2 u) + W_1 u + X_1 (D_2 \xi_2 - N_2 u)$, thus $Q_2 = Q_1 M_2 + X_1 D_2$, $W_2 = Q_1 X_2 + W_1 - X_1 N_2$, *i.e.* the equality (7.27) holds. 3) (7.28) and (7.29) mean that $\left\{ \tilde{B}_1, A_1 \right\}$ and $\left\{ \tilde{B}_2, A_2 \right\}$ are, respectively, left- and right-coprime. ■

When $\mathbf{K} = \mathrm{Re}$, Proposition 7.9 is essentially due to Fuhrmann [17]; other equivalent formulations have been developed (see, *e.g.*, [19], Section 8.2).

Full Equivalence

Let us consider the matrix of definition $B(\partial)$ in (7.14), associated with a PMD, let $(A(\sigma), B^+(\sigma))$ be a left-coprime factorization of $B(\partial)$ over $\mathbf{S}$, and write

$$B^+(\sigma) = \begin{bmatrix} D^+(\sigma) & -N^+(\sigma) & Z^+(\sigma) \\ Q^+(\sigma) & W^+(\sigma) & Y^+(\sigma) \end{bmatrix} \tag{7.30}$$

according to the sizes in (7.14). The impulsive system associated with $B(\partial)$ is $M^+ = \operatorname{coker} \bullet B^+(\sigma)$ (see Section 7.4.4, Definition 7.16), and $M^+ = [\xi^+, u^+, y^+]_{\mathbf{S}}$ where

$$B^+(\sigma) \begin{bmatrix} \xi^+ \\ u^+ \\ y^+ \end{bmatrix} = 0.$$

Definition 7.19. *Consider two PMDs* $\{D_i, N_i, Q_i, W_i\}$ *with matrices over* $\mathbf{R}$ *and denote by* M_i^+ *the associated impulsive system* $(i = 1, 2)$. *These PMDs are* fully *equivalent if (i) they are strictly equivalent and (ii) there exists an* $\mathbf{S}$*-isomorphism* $M_1^+ \cong M_2^+$.

Exercise 7.11. (i) Write the comaximal relation (7.27) (over $\mathbf{R}$) in the form (7.25) (where each B_i $(i = 1, 2)$ is of the form (7.14) with matrices $\{D_i, N_i, Q_i, W_i\}$) and determine the form of A_i $(i = 1, 2)$. (ii) Let $\left(J_1, \begin{bmatrix} B_1^+ & A_1^+ \end{bmatrix}\right)$ be a left-coprime factorization over $\mathbf{S}$ of $\begin{bmatrix} B_1 & A_1 \end{bmatrix}$ and $\left(\begin{bmatrix} -A_2^+ \\ B_2^+ \end{bmatrix}, J_2\right)$ be a right-coprime factorization over $\mathbf{S}$ of $\begin{bmatrix} -A_2 \\ B_2 \end{bmatrix}$. Prove that

$$\begin{bmatrix} B_1^+ & A_1^+ \end{bmatrix} \begin{bmatrix} -A_2^+ \\ B_2^+ \end{bmatrix} = 0. \tag{7.31}$$

(iii) Using the result to be proved in Exercise 7.10 (Section 7.4.4), show that (7.31) is a comaximal relation if, and only if $\{B_1, A_1\}$ and $\{A_2, B_2\}$ are, respectively, left- and right-coprime at infinity. (iv) Show that the following properties are equivalent: (a) $\left(J_1, B_1^+\right)$ is a left-coprime factorization over $\mathbf{S}$ of B_1; (b) $\delta_M\left(B_1\right) = \delta_M\left(\begin{bmatrix} B_1 & A_1 \end{bmatrix}\right)$. (v) Similarly, show that the following properties are equivalent: (a') $\left(B_2^+, J_2\right)$ is a right-coprime factorization over $\mathbf{S}$ of B_2; (b') $\delta_M\left(B_2\right) = \delta_M\left(\begin{bmatrix} -A_2 \\ B_2 \end{bmatrix}\right)$. (vi) Finally, using (7.21) and (7.23), conclude that two PMDs $\{D_i, N_i, Q_i, W_i\}$ $(i = 1, 2)$ are *fully equivalent* (written $\{D_1, N_1, Q_1, W_1\} \overset{f}{\sim} \{D_2, N_2, Q_2, W_2\}$) if, and only if (1) they are strictly equivalent, (2) $\{B_1, A_1\}$ and $\{A_2, B_2\}$, as defined in (i), are, respectively, left- and right-coprime at infinity, (3) the MacMillan degree conditions in (iv) and (v) are satisfied. (Hint: for (vi), $\left(B_2^+, J_2\right)$ is a right-coprime factorization of B_2 over $\mathbf{S}$ if, and only if $c_\infty\left(\begin{bmatrix} B_2^+ \\ J_2 \end{bmatrix}\right) = 0$, *i.e.* $c_\infty\left(\begin{bmatrix} B_2 \\ I \end{bmatrix}\right) = -c_\infty\left(J_2\right)$: see Exercise 7.6 (Section 7.3.2). In addition, $c_\infty\left(\begin{bmatrix} B_2 \\ I \end{bmatrix}\right) = \delta_M\left(\begin{bmatrix} B_2 \\ I \end{bmatrix}\right) = \delta_M\left(B_2\right)$ from Remark 7.2 and Exercise 7.7(iii) (Section 7.3.2). Conclude.)

Full equivalence is defined in [26] (see also [18]) in the case $\mathbf{K} = \mathrm{Re}$ in accordance with the necessary and sufficient condition to be proved in Exercise 7.11(vi) –in a slightly different but equivalent form.

An Algebraic Definition of a Temporal System

Definition 7.20. *A* temporal input-output system $\mathbf{\Sigma}$ *is an* equivalence class of fully equivalent PMDs. *The* impulsive system M^+ *of* $\mathbf{\Sigma}$ *is defined (up to* $\mathbf{S}$*-isomorphism) according to Definition 7.16 in Section 7.4.4, its* pseudo-impulsive behavior $\mathcal{A}_\infty$ *is defined according to Remark 7.3 in Section 7.4.4, and its* system M *is defined (up to* $\mathbf{R}$*-isomorphism) as* $[\xi, u]_{\mathbf{R}}$, *where* ξ *is the partial state of any PMD belonging to* $\mathbf{\Sigma}$.

7.5 Poles and Zeros at Infinity

In this section, $\mathbf{K}$ is any differential field and we consider an input-output temporal system $\mathbf{\Sigma}$ (in the meaning of Definition 7.20). The transfer matrix of $\mathbf{\Sigma}$ is $G\left(\partial\right) = Q\left(\partial\right) D^{-1}\left(\partial\right) N\left(\partial\right) + W\left(\partial\right)$, where $\{D, N, Q, W\}$ is any PMD belonging to $\mathbf{\Sigma}$.

The *transmission poles and zeros at infinity* of $\mathbf{\Sigma}$ are those of its input-output system M (see Definition 7.9, Section 7.3.2).

The definition of the other poles and zeros at infinity of $\mathbf{\Sigma}$ is made from its *impulsive system* M^+ [11], and it is similar to the definition of the various *finite poles and zeros* from its *system* M ([7], Section 6.4.2). Poles and zeros at infinity, as defined below, are torsion $\mathbf{S}$-modules (transmission poles and zeros at infinity can also be defined in this manner: see Exercise 7.17 below). Let T^+ be any of these modules; the associated pseudo-impulsive behavior is $(T^+)^* = \operatorname{Hom}_{\mathbf{S}}\left(T^+, \tilde{\Delta}\right)$, from which the associated impulsive behavior can be calculated in the case $\mathbf{K} = \mathrm{Re}$, using the commutative diagram (7.18). For explicit calculations, the matrix of definition $B^+(\sigma)$ of M^+ is assumed to be given by (7.30).

7.5.1 Uncontrollable Poles at Infinity

The module of *uncontrollable poles at infinity* is $\mathcal{T}(M^+)$, according to Definition 7.16 (Section 7.4.4). Its elementary divisors are those of the matrix $B^+(\sigma)$.

Exercise 7.12. Assume that $\mathbf{K} = \mathrm{Re}$. Let $\mathbf{\Sigma}_1$ be the system $(\partial + 1)\, y = u$, $\mathbf{\Sigma}_2$ be the system $\bar{y} = \partial^2 \bar{u}$, and consider the interconnection $\bar{y} = u$ (*i.e.*, $\mathbf{\Sigma}_2 \to \mathbf{\Sigma}_1$), active only for $t \in \mathbb{T}_0$. (i) What kind of "pole-zero cancellation at infinity" does arise when the temporal system is formed? (ii) Check that $\mathcal{T}(M^+) \cong \tilde{C}_1 = \mathrm{Re}\,\tilde{\delta}$. (iii) "Where" in the temporal system does the uncontrollable impulsive behavior $\mathrm{Re}\,\delta$ arise?

7.5.2 System Poles at Infinity

Let $\breve{M}^+ = \mathbf{L} \bigotimes_{\mathbf{S}} M^+$ and $\breve{\varphi} : M^+ \to \breve{M}$ be the canonical map defined by $\breve{\varphi}(w^+) = \breve{w}^+ = 1_{\mathbf{L}} \otimes_{\mathbf{S}} w^+$, where $1_{\mathbf{L}}$ is the unit-element of $\mathbf{L}$.

Lemma 7.2. *(i) The $\mathbf{L}$-vector space $\breve{M}^+$ is of dimension m and $\breve{M}^+ = [\breve{u}^+]_{\mathbf{L}}$. (ii) The module $[u^+]_{\mathbf{S}}$ is free of rank m and $M^+/[u^+]_{\mathbf{S}}$ is torsion. (iii) Considering equality (a) in Theorem 7.1(i), $[u^+]_{\mathbf{S}} \subset \Phi^+$.*

Proof. The module M^+ is defined by $B^+(\sigma)\, w^+ = 0$, therefore $A^{-1}(\sigma)$ $B^+(\sigma)\, \breve{w}^+ = 0$, *i.e.* $B(\partial)\, \breve{w}^+ = 0$. Thus, from Section 7.3.1, $\breve{M}^+ = \mathbf{L} \bigotimes_{\mathbf{Q}} \hat{M} = \mathbf{L} \bigotimes_{\mathbf{Q}} [\hat{u}]_{\mathbf{Q}} = [\breve{u}^+]_{\mathbf{L}}$, and (i) is proved. By (i), $[u^+]_{\mathbf{S}}$ is of rank m, and $[u^+]_{\mathbf{S}}$ is free since $[u^+]_{\mathbf{S}}$ is generated by m elements[3]; in addition, $\mathbf{L} \bigotimes_{\mathbf{S}} (M^+/[u^+]_{\mathbf{S}}) = 0$, thus $M^+/[u^+]_{\mathbf{S}}$ is torsion. (iii) is a consequence of the freeness of $[u^+]_{\mathbf{S}}$ (see addendum 6 in Section 7.7). ■

[3] Over a left Ore domain, the rank of a module is the cardinal of a maximal linearly independent subset of that module ([12], Section 0.9).

Definition 7.21. *The module of* system poles at infinity*, denoted by* SP_∞*, is* $M^+/\left[u^+\right]_{\mathbf{S}}$.

The elementary divisors of $M^+/\left[u^+\right]_{\mathbf{S}}$ are those of the submatrix

$$\begin{bmatrix} D^+(\sigma) & Z^+(\sigma) \\ Q^+(\sigma) & Y^+(\sigma) \end{bmatrix}$$

of $B^+(\sigma)$.

Exercise 7.13. Show that the temporal system in Exercise 7.12 has one transmission pole at infinity of order 1 and that $SP_\infty \cong \tilde{C}_2$.

7.5.3 Hidden Modes at Infinity

The module of uncontrollable poles at infinity is also called the module of *input-decoupling zeros at infinity* and is denoted by IDZ_∞.

Definition 7.22. *The module of* output-decoupling zeros at infinity *(denoted by* ODZ_∞*) is* $M^+/\left[y^+, u^+\right]_{\mathbf{S}}$.

The elementary divisors of $M^+/\left[y^+, u^+\right]_{\mathbf{S}}$ are those of the submatrix $\begin{bmatrix} D^+(\sigma) \\ Q^+(\sigma) \end{bmatrix}$.

Exercise 7.14. Assume that $\mathbf{K} = \mathrm{Re}$. Consider the systems $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in Exercise 7.12 and the interconnection $y = \bar{u}$ (*i.e.*, $\mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$), active only for $t \in \mathbb{T}_0$. (i) What kind of "pole-zero cancellation at infinity" does arise when the temporal system is formed? (ii) Check that $M^+/\left[y^+, u^+\right]_{\mathbf{S}} \cong \tilde{C}_1 = \mathrm{Re}\,\tilde{\delta}$. (iii) "Where" in the temporal system does the "unobservable impulsive behavior" $\mathrm{Re}\,\delta$ arise?

Definition 7.23. *The module of input-output decoupling zeros at infinity (denoted by* $IODZ_\infty$*) is* $\mathcal{T}(M^+)/\left(\mathcal{T}(M^+) \cap \left[y^+, u^+\right]_{\mathbf{S}}\right)$.

Definition 7.24. *Considering equality (a) in Theorem 7.1(i) (Section 7.2.5), the module of hidden modes at infinity (denoted by* HM_∞*) is*

$$M^+/\left(\Phi^+ \cap \left[y^+, u^+\right]_{\mathbf{S}}\right).$$

Remark 7.5. The above module HM_∞ is uniquely determined up to isomorphism, as shown by Theorem 7.6 below.

Let us denote by $\varepsilon(T^+)$ the set of all elementary divisors of a f.g. torsion $\mathbf{S}$-module T^+.

Exercise 7.15. (i) Let T_1^+ and T_2^+ be submodules of a f.g. **S**-module, such that T_1^+ and T_2^+ are torsion and $T_1^+ \cap T_2^+ = 0$. Prove that $\varepsilon\left(T_1^+ \oplus T_2^+\right) = \varepsilon\left(T_1^+\right) \dot{\cup} \varepsilon\left(T_2^+\right)$, where $\dot{\cup}$ is the disjoint union[4]. (ii) Let M_1^+, M_2^+, M_3^+ be f.g. torsion **S**-modules such that $M_1^+ \subset M_2^+ \subset M_3^+$ and M_{i+1}^+/M_i^+ is torsion $(i = 1, 2)$. Prove that $\#\left(M_3^+/M_1^+\right) = \#\left(M_3^+/M_2^+\right) + \#\left(M_2^+/M_1^+\right)$ (Hint: for (i): denoting by B_i^+ a matrix of definition of $T_i^+, i = 1, 2$, the diagonal sum[5] $B_1^+ \oplus B_2^+$ is a matrix of definition of $T_1^+ \oplus T_2^+$; for (ii): use ([7], Proposition 6.7(iii)) and see the proof of ([9], Lemma(a)).)

The following result is classical ([1], Section II.1.5) and will be useful in the sequel.

Lemma 7.3. *Let* **A** *be a ring and* M_i *and* N_i *be submodules of an* **A***-module, such that* $N_i \subset M_i$ $(i = 1, 2)$ *and* $M_1 \cap M_2 = 0$. *Then*

$$\frac{M_1 \oplus M_2}{N_1 \oplus N_2} \cong \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}.$$

The theorem below is more precise than ([11], Theorem 2(1)) :

Theorem 7.6. *The following equality holds*[6]*:*

$$\varepsilon\left(HM_\infty\right) = \varepsilon\left(IDZ_\infty\right) \dot{\cup} \varepsilon\left(ODZ_\infty\right) \setminus \varepsilon\left(IODZ_\infty\right).$$

Proof. We have

$$\frac{M^+}{\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}} = \frac{\mathcal{T}\left(M^+\right) \oplus \Phi^+}{\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}} \cong \mathcal{T}\left(M^+\right) \oplus \frac{\Phi^+}{\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}}, \tag{7.32}$$

this isomorphism holding because $(\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}) \cap \mathcal{T}\left(M^+\right) = 0$. In addition,

$$\begin{aligned} \frac{M^+}{[y^+, u^+]_{\mathbf{S}}} &= \frac{\mathcal{T}\left(M^+\right) \oplus \Phi^+}{(\mathcal{T}\left(M^+\right) \cap [y^+, u^+]_{\mathbf{S}}) \oplus (\Phi^+ \cap [y^+, u^+]_{\mathbf{S}})} \\ &\cong \frac{\mathcal{T}\left(M^+\right)}{\mathcal{T}\left(M^+\right) \cap [y^+, u^+]_{\mathbf{S}}} \oplus \frac{\Phi^+}{\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}} \end{aligned} \tag{7.33}$$

by Lemma 7.3. The theorem is a consequence of (7.32), (7.33) and of the result to be proved in Exercise 7.15(i). ■

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[4] This notion was already used in [7]. For example, $\{x, y\} \dot{\cup} \{x, z\} = \{x, x, y, z\}$.

[5] See [7], footnote 10.

[6] The reader may notice that the same relation holds between the sets of all elementary divisors of the modules of *finite* hidden modes (defined in addendum 9, Section 7.7), i.d.z., o.d.z. and i.o.d.z. This relation is more precise than equality (6.37) in ([7], Section 6.4.2).

7.5.4 Invariant Zeros at Infinity

Definition 7.25. *The module of* invariant zeros at infinity *(denoted by IZ_∞) is $\mathcal{T}(M^+/[y^+]_{\mathbf{S}})$.*

The elementary divisors of $\mathcal{T}(M^+/[y^+]_{\mathbf{S}})$ are those of the submatrix $\begin{bmatrix} D^+(\sigma) & -N^+(\sigma) \\ Q^+(\sigma) & W^+(\sigma) \end{bmatrix}$.

Exercise 7.16. Assume that $\mathbf{K} = \mathrm{Re}$. Let $\mathbf{\Sigma}_1$ be the system $y = \partial^2 u$, $\mathbf{\Sigma}_2$ be the system $\partial^3 \bar{y} = \partial \bar{u}$, and consider the interconnection $y = \bar{u}$ (*i.e.*, $\Sigma_1 \to \Sigma_2$), active only for $t \in \mathbb{T}_0$. (i) What kind of "pole-zero cancellation at infinity" does arise when the temporal system is formed? (ii) Calculate IDZ_∞, ODZ_∞, $IODZ_\infty$, HM_∞, SP_∞ and IZ_∞. (Answers: $IDZ_\infty = IODZ_\infty = 0$, $ODZ_\infty \cong HM_\infty \cong SP_\infty \cong IZ_\infty \cong \tilde{C}_2$.)

7.5.5 System Zeros at Infinity

To the author's knowledge, a module system zeros at infinity was not defined However, the *degree of the system zeros at infinity* (denoted by $\#(SZ_\infty)$ is defined as follows:

Definition 7.26. $\#(SZ_\infty) = \#(TZ_\infty) + \#(HM_\infty)$.

7.5.6 Relations between the Various Poles and Zeros at Infinity

Exercise 7.17. Consider the two torsion modules TP_∞ and TZ_∞ below:

$$TP_\infty = \frac{\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}}{[u^+]_{\mathbf{S}}}, \quad TZ_\infty = \mathcal{T}\left(\frac{\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}}{\Phi^+ \cap [y^+]_{\mathbf{S}}}\right).$$

(i) Prove that these modules are uniquely determined up to isomorphism. (ii) Call TP_∞ and TZ_∞ the *module of transmission poles at infinity* and the *module of transmission zeros at infinity*, respectively. Prove that this definition is consistent with Definition 7.9 (Section 7.3.2). (iii) Notice that $HM_\infty \subset SP_\infty$ and that $HM_\infty \cong SP_\infty / TP_\infty$ (see ([7], Proposition 6.7(iii)). Is it possible to deduce from this isomorphism a connection between the *elementary divisors* of TP_∞, those of HM_∞ and those of SP_∞? (Hint: for (i), write

$$\frac{[y^+, u^+]_{\mathbf{S}}}{[y^+]_{\mathbf{S}}} \cong \frac{\mathcal{T}(M^+) \cap [y^+, u^+]_{\mathbf{S}}}{\mathcal{T}(M^+) \cap [y^+]_{\mathbf{S}}} \oplus \frac{\Phi^+ \cap [y^+, u^+]_{\mathbf{S}}}{\Phi^+ \cap [y^+]_{\mathbf{S}}},$$

and deduce from this isomorphism that TZ_∞ is uniquely determined up to isomorphism; do a similar reasoning for TP_∞. For (ii), see [9], Sections 3.3 and 3.6, where the modules of *finite* transmission poles and zeros are considered. For (iii), see Exercise 7.13: the answer is negative.)

Exercise 7.18. [11] Prove the following relations: (i) $\#(SP_\infty) = \#(TP_\infty) + \#(HM_\infty)$; (ii) $\#(TZ_\infty) + \#(IODZ_\infty) \leq \#(IZ_\infty) \leq \#(SZ_\infty)$; (iii) if $G(\partial)$ is full row rank, $\#(TZ_\infty) + \#(IDZ_\infty) \leq \#(IZ_\infty)$; (iv) if $G(\partial)$ is full column rank, $\#(TZ_\infty) + \#(ODZ_\infty) \leq \#(IZ_\infty)$; (v) if $G(\partial)$ is square and invertible, $\#(IZ_\infty) = \#(SZ_\infty)$. (Hint: use Lemma 7.3 and the results to be proved in exercises 7.15 and 7.17; see also the proof of ([9], Theorem 1 and 2).)

Exercise 7.19. [10] Assume that $\mathbf{K} = \mathrm{Re}(t)$. Consider the temporal system whose matrix of definition $B(\partial)$ is given by (7.14) with $D(\partial) = \begin{bmatrix} 1 & 0 \\ t\partial^3 & \partial^2 \end{bmatrix}$, $N(\partial) = \begin{bmatrix} 0 \\ (t-1)\partial \end{bmatrix}$, $Q(\partial) = \begin{bmatrix} t\partial & t^2\partial \end{bmatrix}$, $W(\partial) = t^2\partial$. Calculate SP_∞, IDZ_∞, ODZ_∞, $IODZ_\infty$, HM_∞, IZ_∞, TP_∞, ZP_∞ (see exercise 7.17) and $\#(SZ_\infty)$. Interpretation? (Answers: $SP_\infty \cong \tilde{C}_1 \oplus \tilde{C}_1$, $IDZ_\infty \cong ODZ_\infty \cong IODZ_\infty \cong HM_\infty \cong TP_\infty \cong \tilde{C}_1$, $ZP_\infty = 0$ and $\#(SZ_\infty) = 1$. For the detailed interpretations, see [10].)

7.6 Concluding Remarks

Using the algebraic tools explained in this chapter, several results, classical for linear time-invariant systems, have been extended to linear continuous- or discrete-time-varying systems, *e.g.*: (i) the necessary and sufficient condition for the proper model matching problem to have a solution (Exercise 7.8), (ii) the necessary and sufficient condition for two PMDs to be fully equivalent (Exercise 7.11), (iii) the relations between the various poles and zeros at infinity (Exercise 7.18). Hidden modes at infinity are related to "pole/zero cancellations at infinity" (exercises 7.12 and 7.16). The algebraic definition of a temporal system (Definition 7.20) is new.

Impulsive behaviors of linear continuous- or discrete-time-varying temporal systems are further studied in [8], assuming that $\mathbf{K}$ is a differential ring such as $\mathrm{Re}[t]$ and that a suitable regularity condition is satisfied.

7.7 *errata* and *addenda* for [7]

1) p. 240, 17th line from top, change "of a_i by f" to: "$f(\mathbf{A}\, a_i)$"

2) p. 242, change the two sentences beginning at 6th line from top to: "Let $\psi : \mathbf{A} \to M$ be the epimorphism $\lambda \to \lambda\, w$. As $\ker \varphi = \mathbf{a}$, there exists an isomorphism $M \cong \mathbf{A}/\mathbf{a}$ by Proposition 6.7(i). Conversely, a quotient $\mathbf{A}/\mathbf{a}$ is cyclic, generated by $\varphi(1)$, where $\varphi : \mathbf{A} \to \mathbf{A}/\mathbf{a}$ is the canonical epimorphism."

3) p. 248, 7th line from top, add after "r": "(this Jordan block is also denoted by J_π in the sequel)"

4) p. 249, 8th line from bottom, change ", since $|U|$ is a *rational function* of the entries of U" to: "in general"

5) p. 251, 4th line from top, change the sentence in parentheses to: "not necessarily square, but such that all its entries outside the main diagonal are zero"

6) p. 257, after 13th line from top, add: "Consider equality (a) in Theorem 6.5(i). We have $[u]_{\mathbf{R}} \subset \mathcal{T}(M) \oplus \varPhi$ and $[u]_{\mathbf{R}} \cap \mathcal{T}(M) = 0$, thus $[u]_{\mathbf{R}} \subset \varPhi$."

7) p. 259, 9th line from top, change "controllable quotient" to: "controllable subsystem"

8) p. 260, 11th line from top, change "is equivalent to" to: "implies"

9) p. 265, after 6th line from top, add: "Consider equality (a) in Theorem 6.5(i); the *module of hidden modes* is $M/\left([y,u]_{\mathbf{R}} \cap \varPhi\right)$."

10) p. 265, 10th line from top, change "is defined as:" to: "satisfies the equality"

11) p. 269, 2nd line from bottom and p. 270, 2nd line from top, change "$\operatorname{Hom}_{\mathbf{A}}\left(\mathbf{D}^k, W\right)$" to: "$\operatorname{Hom}_{\mathbf{A}}\left(\mathbf{A}^k, W\right)$"

12) p. 272, 9th line from bottom, change "$\mathbf{D}\,(p+k)$" to: "$\mathbf{D}\left(p^{n+k}\right)$"

13) p. 272, 8th line from bottom, change "$p+k$" to: "$n+k$"

14) p. 276, 1st line from top, add before "A *sub-behavior*": "A *subsystem* of M is a quotient M/N of M."

15) p. 276, 7th line from top, change "quotient" to "subsystem"

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