# 4

# Stability Analysis of Time-delay Systems: A Lyapunov Approach

Kequin Gu[1] and Silviu-Iulian Niculescu[2]

[1] Department of Mechanical and Industrial Engineering, Southern Illinois University Edwardsville, Edwardsville, Illinois 62026-1805, USA
E-mail: kgu@siue.edu

[2] HEUDIASYC (UMR CNRS 6599), Université de Technologie de Compiègne Centre de Recherche de Royallieu, BP 20529 60205 Compiègne, France
E-mail: Silviu.Niculescu@hds.utc.fr

**Summary.** This chapter is devoted to the stability problem of time-delay systems using time-domain approach. Some basic concepts of time-delay systems are introduced. Then, some simple Lyapunov-Krasovskii funtionals, complete Quadratic Lyapunov-Krasovskii functional and discretization scheme are introduced, with connections and extent of conservatism compared. The issue of time-varying delays are also discussed. The concept of Razumikhin Theorem is introduced. An alternative model of coupled difference-differential equations and its stability problem are also introduced.

## 4.1 Introduction

It is a common pratice to use ordinary differential equations to describe the evolution of physical, engineering or biological system. However, it is also known that such a mathematical description is inadequate for many systems. Indeed, delay-differential equations (or more generally, functional differential equations) are often needed to reflect the fact that the future evolution of system variables not only depends on their current values, but also depends on their past history. Such systems are often known as time-delay systems (also known as hereditary systems, systems with time lag, or systems with aftereffects). This chapter is intended to serve as a tutorial to cover some of the basic ideas of time-delay systems, especially, the stability analysis using Lyapunov approach.

Time-delay systems are distributed parameter systems, or infinite-dimensional systems. To bring out the idea, compare the ordinary differential equation

$$\dot{x}(t) = ax(t), \tag{4.1}$$

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with a simple time-delay system

$$\dot{x}(t) = ax(t - r). \tag{4.2}$$

In these two systems, $a$ and $r$ are constant scalars, and $x$, a scalar function of time $t$, is the state variable. It is well known that for the system represented by (4.1), given any time $t_0$, then the future value of the state $x(t)$, $t > t_0$ is completely determined by $x(t_0)$, a scalar, which indicates that the system (4.1) is a 1-dimensional system. On the other hand, for the system (4.2), to completely determine $x(t)$, $t > t_0$, it is necessary to know $x(t)$ for all $t_0 - r \leq t \leq t_0$. Therefore, the *state* at time $t_0$ is an element of the infinite-dimensional functional space $\{x(t) \mid t_0 - r \leq t \leq t_0\}$, and the system (4.2) is an infinite-dimensional system.

Examples of time-delay systems abound in various disciplines of science, engineering and mathematics. Kolmanovskii and Myshkis gave many examples [18]. Other books also contain many practical examples, see, for example, [11] [13] [23]. Here, we will mention only two examples.

*Example 4.1.* **Network control**. The popularity of internet has brought to the network control problem to prominence. One of the model studied in the literature is the simplified fluid approximation proposed by Kelly [17]

$$\dot{x}(t) = k[w - x(t - \tau)p(x(t - \tau))],$$

where $p$ is a continuously differentiable and strictly increasing function bounded by 1, and $k$ and $w$ are positive constant. The delay $\tau$ represents the round-trip time. The function $p$ can be interpreted as the fraction of packets the presence of (potential) congestion. For more details of network model, see [5] [17] [28].

*Example 4.2.* **Transport delay in chemical reactions.** This example was discussed in [20] and [21]. Consider a first order, exothermic and irreversible chemical reaction from $A$ to $B$. In practice, the conversion from $A$ to $B$ is not complete. To increase the conversion rate and reduce the costs, a recycle stream is used. The time it takes to transport from the output to the input introduces time delay. The resulting process can be described by the following equations

$$\begin{aligned}
\frac{dA(t)}{dt} &= \frac{q}{V}[\lambda A_0 + (1-\lambda)A(t-\tau) + A(t)] - K_0 e^{-Q/T} A(t) \\
\frac{dT(t)}{dt} &= \frac{1}{V}[\lambda T_0 + (1-\lambda)T(t-\tau) - T(t)]\frac{\Delta H}{C\rho} - K_0 e^{-Q/T} A(t) \\
&\quad - \frac{1}{VC\rho}U(T(t) - T_w,)
\end{aligned}$$

where $A(t)$ is the concentration of the component $A$, $T(t)$ is the temperature, and $\lambda \in [0, 1]$ is the recycle coefficient ($\lambda = 1$ represents no recycle), and $\tau$ is the transport delay. The case without time delay $\tau$ has been discussed in [2] and [26].

The rest of the chapter is organized as follows. Section 4.2 introduces some basic concepts of time-delay systems. Section 4.3 introduces the concept of stability and Lyapunov-Krasovskii stability Theorem. Section 4.4 introduces some simple Lyapunov-Krasovskii functionals Section 4.5 covers the complete quadratic Lyapunov-Krasovskii functional and its discratization. Section 4.6 compares different Lyapunov functionals, with numerical examples. Section 4.7 discusses time-varying delays. Section 4.8 discusses Razumikhin Theorem. Section 4.9 discusses coupled difference-differential equations and stability. Section 4.10 contains conclusions and discussions.

## 4.2 Basic Concepts of Time-delay Systems

### 4.2.1 Systems of Retarded Type

We will concentrate on time-delay systems of *retarded type* in this article. A retarded time-delay system can be represented as

$$\dot{x}(t) = f(t, x_t) \tag{4.3}$$

where $x(t) \in \mathbb{R}^n$, $x_t$ is a function defined in the interval $[-r, 0]$ as

$$x_t(\theta) = x(t+\theta), \qquad -r \le \theta \le 0,$$

$r$ is the maximum delay, and $f$ is a *functional*, or a function of functions. In other words, the value of $f$ can be determined if the value of $t$ and the function $x_t$ are given. It is common practice to restrict $x_t$ to be a continuous function. Let $\mathcal{C}$ be the set of all continuous functions defined in the interval $[-r, 0]$, then the initial condition of (4.3) can be expressed as

$$x_{t_0} = \phi, \text{ for some } \phi \in \mathcal{C} \tag{4.4}$$

which means that

$$x(t_0 + \theta) = \phi(\theta), \text{ for } \theta \in [-r, 0].$$

With this notation, the domain of definition of $f$ is $\mathbb{R}\times\mathcal{C}$. The solution of (4.3) with initial condition (4.4) is often denoted as $x(t, t_0, \phi)$, or $x(t, \phi)$ if $t_0$ is understood.

In some context, it is beneficial to consider the initial condition as consisting of two parts, $x(t_0)$ and $x(t)$ for $t_0 - r \le t < t_0$. This may be convenient to accommodate the case of a discontinuous $\phi$ in the initial condition.

Other types of time-delay systems are discussed in, for example, [13] and [18]. For example, if $\dot{x}(t)$ also depends on derivative of $x$ at a time $\tau < t$, then the system is of *neutral type*.

### 4.2.2 Pointwise Delays

An important special case in pratice can be expressed as

$$\dot{x}(t) = f(t, x(t), x(t-r)). \tag{4.5}$$

In other words, $\dot{x}(t)$ only depends on $x$ at current time and at the time of maximum delay, and is independent of $x(t+\theta)$, $-r < \theta < 0$. Let's consider the case of $t_0 = 0$, so that the initial condition becomes

$$x_0 = \phi, \text{ or } x(t) = \phi(t), \ -r \leq t \leq 0.$$

Such a system admits a simple *method of steps* to generate the future trajectories: Since $x(t-r)$ is already known as the initial condition for $t \in [0, r]$, the equation (4.5) can be considered as an ordinary differential equation in this interval, and $x(t)$, $t \in [0, r]$ can be generated by solving this ordinary differential equation. Once $x(t)$, $t \in [0, r]$ is available, $x(t-r)$, $t \in [r, 2r]$ is also available, and therefore, one can further generate $x(t)$, $t \in [r, 2r]$ by solving ordinary differential equation. Continue this process will allow us to generate $x(t)$ for $t \in [0, \infty)$.

Similarly, we say the system

$$\dot{x}(t) = f(t, x(t), x(t-r_1), x(t-r_2), ..., x(t-r_k)) \tag{4.6}$$

is of *multiple delays*. Furthermore, if there is a common factor $\tau$ which divides all delays $r_j$, $j = 1, 2, ..., k$, then we say the system is of *commensurate delays*. Without loss of generality, we may assume $r_j = j\tau$ in this case. If there does not exist such a factor, in other words, we can find two delays $r_i$ and $r_j$ such that $r_i/r_j$ is irrational, then we say that the delays are *incommensurate*. Obviously, the method of steps can also be used in systems of multiple delays.

Systems with either single delay or multiple delays are known as of *pointwise delays*, *concentrated delays*, or *discrete delay*.

### 4.2.3 Linear Systems

If the functional $f$ is linear with respect to $x_t$ in (4.3), then we say that the system is *linear*. If it is independent of $t$, then we say it is *time-invariant*. For a linear time-invariant system, we may define *fundamental solution* $X(t)$ as the solution with initial condition

$$\begin{aligned} x(0) &= I; \\ x(t) &= 0, \ -r \leq t < 0. \end{aligned}$$

where $I$ is the identity matrix of appropriate dimension. If the system is $n$-dimensional, then $X(t)$ is an $n \times n$-dimensional matrix function of time.

Fundamental solution plays an important role in the study of linear time-delay systems.

Consider, for example, the linear system with single delay

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - r) \tag{4.7}$$

It can be shown, using linearity, that the solution of (4.7) under initial condition $x_0 = \phi$ can be expressed as

$$x(t, \phi) = X(t)\phi(0) + \int_{-r}^{0} X(t - r - \theta) A_1 \phi(\theta) d\theta \tag{4.8}$$

### 4.2.4 Characteristic Quasipolynomials

A linear time-invariant time-delay system is associated with a corresponding *characteristic quasipolynomial* through Laplace Transform. For the system (4.7), the characteristic quasipolynomial is

$$p(s) = \det(sI - A_0 - e^{-rs} A_1).$$

It can be shown that the characteristic quasipolynomial is directly related to the Laplace Transform of the fundamental solution,

$$p(s) = \det(\mathcal{L}[X(t)]).$$

Similar to systems of finite dimension, a time-delay system of retarded type is stable if and only if all the *poles*, or the roots of the characteristic quasipolynomial, are on the left half of the complex plane. However, unlike finite-dimensional systems, a time-delay system has an infinite number of poles, and charaterizing and finding these poles are much more challenging due to the fact that a quasipolynomial involves transcendental functions.

For a linear time-invariant system with multiple delays, the characteristic quasipolynomial can be considered as a polynomial of $s$, $e^{-r_1 s}$, $e^{-r_2 s}$, ..., $e^{-r_k s}$. For commensurate delays, since $e^{-r_j s} = (e^{-\tau s})^{l_j}$, we can further consider the characteristic quasipolynomial as a polynomial of two variables $s$ and $e^{-\tau s}$. This fact made the stability problem of systems with commensurate delays a much easier problem.

Since the focus in this chapter is on Lyapunov approach, we will not pursue further the stability analysis based on poles.

## 4.3 Stability

We will start with a formal definition of stability.

**Definition 4.1.** *For a time-delay system described by (4.3), the trivial solution* $x(t) = 0$ *is said to be* stable *if for any given* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that* $||x_\tau||_c < \delta$ *implies* $||x(t)|| < \varepsilon$ *for all* $t \geq \tau$*. It is said to be* asymptotically stable *if it is stable, and for any given* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$*, there exists, in addition, a* $\delta_a > 0$*, such that* $||x_\tau||_c < \delta_a$ *implies* $\lim_{t\to\infty} x(t) = 0$*. It is said to be* uniformly stable *if it is stable, and* $\delta$ *can be made independent of* $\tau$*. It is* uniformly asymptotically stable *if it is uniformly stable and there exists a* $\delta_a > 0$ *such that for any* $\eta > 0$*, there exists a* $T$ *such that* $||x_\tau||_c < \delta_a$ *implies* $||x(t)|| < \eta$ *for* $t > \tau + T$*. It is* globally (uniformly) asymptotically stable *if it is (uniformly) asymptotically stable and* $\delta_a$ *can be made arbitrarily large.*

In the above, $||\cdot||$ represents the vector 2-norm, and $||\cdot||$ is defined as

$$||\phi||_c = \max_{-r\leq\theta\leq 0} ||\phi(\theta)||.$$

The above definition is obviously analogous to finite-dimensional systems. The stability relative to any given solution other than the trivial solution can be transformed to one relative to the trivial solution through a change of variable.

Corresponding to Lyapunov function $V(t,x)$ for finite-dimensional systems, here we need a Lyapunov-Krasovskii functional $V(t,x_t)$ due to the fact that the state is $x_t$. We have the following Lyapunov-Krasovskii Stability Theorem.

**Theorem 4.1.** *Suppose* $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ *in (4.3) maps* $\mathbb{R}\times$*(bounded sets in* $\mathcal{C}$*) into bounded sets in* $\mathbb{R}^n$*, and that* $u, v, w : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ *are continuous nondecreasing functions. In addition,* $u(s)$ *and* $v(s)$ *are positive for positive* $s$*, and* $u(0) = v(0) = 0$*. If there exists a continuous differentiable functional* $V : \mathbb{R}\times\mathcal{C} \to \mathbb{R}$ *such that*

$$u(||\phi(0)||) \leq V(t,\phi) \leq v(||\phi||_c), \tag{4.9}$$

*and*

$$\dot{V}(t,\phi) \leq -w(||\phi(0)||), \tag{4.10}$$

*then the trivial solution of (4.3) is uniformly stable. If* $w(s) > 0$ *for* $s > 0$*, then it is uniformly asymptotically stable. If, in addition,* $\lim_{s\to\infty} u(s) = \infty$*, then it is globally uniformly asymptotically stable.*

In the above, $\bar{\mathbb{R}}_+$ is the set of nonnegative real scalars. The notation $\dot{V}(t,\phi)$ is defined as

$$\dot{V}(t,\phi) \triangleq \frac{d}{dt} V(t,x_t)|_{x_t=\phi}$$

In other words, we can think of $\dot{V}(\tau,\phi)$ as the derivative of $V(t,x_t)$ with respect to time $t$, evaluated at the time $t = \tau$, where $x_t$ is the solution of (4.3) with initial condition $x_\tau = \phi$. Indeed, (4.9) and (4.10) are often written as

$$u(||x(t)||) \leq V(t, x_t) \leq v(||x_t||_c),$$
$$\dot{V}(t, x_t) \leq -w(||x(t)||),$$

Notice, although the "state" in this case is $x_t$, the lower bound of $V(t, x_t)$ and the upper bound of $\dot{V}(t, x_t)$ only need to be functions of $||x(t)||$, and not necessarily be function of $||x_t||_c$. For a proof, the readers are referred to [11], [13] or [18].

## 4.4 Some Simple Lyapunov-Krasovskii Functionals

This section discusses some simple Lyapunov-Krasovskii functionals for the stability analysis of time-delay systems. The materials of this section may be found from [11] [23] [3].

### 4.4.1 Delay-independent Stability

Consider the time-delay system (4.7). We may consider the Lyapunov-Krasovskii functional

$$V(x_t) = x^T(t)Px(t) + \int_{-r}^{0} x^T(t+\theta)Sx(t+\theta)d\theta.$$

Where, $P$ and $R$ are symmetric matrices. Obviously,

$$P > 0, \tag{4.11}$$
$$S \geq 0, \tag{4.12}$$

are sufficient to ensure the satisfaction of (4.9). In the above (4.11) means $P$ is positive definite, and (4.12) means $S$ is positive semi-definite. Similarly, we also use "$< 0$" and "$\leq 0$" to indicate a matrix is negative definite or semi-definite. Calculating the derivative of $V$ along the system trajectory yields,

$$\dot{V}(x_t) = \begin{pmatrix} x^T(t) & x^T(t-r) \end{pmatrix} \begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-r) \end{pmatrix}.$$

To satisfy (4.10), it is sufficient that

$$\begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} < 0. \tag{4.13}$$

Therefore, we can conclude the following.

**Proposition 4.1.** *The system (4.7) is asymptotically stable if there exists symmetric matrices $P$ and $S$ of appropriate dimension such that (4.11) and (4.13) are satisfied.*

Notice that (4.12) is already implied by (4.13). Inequalities (4.11) and (4.13) are examples of *linear metrix inequalities (LMI)*, where parameters (in this case symmetric matrices $P$ and $S$). An important development in recent years is that effective numerical methods have been developed to solve LMIs, see [4] for details, and Appendix B of [11] for some facts of LMI most useful for time-delay systems. A number of software packages are available to solve LMIs, see, for example, [8] for LMI Toolbox for MATLAB ®.

It should be observed that the stability conditions (4.11) and (4.13) is independent of the delay $r$. Such conditions are known as delay-independent stability conditions. Such conditions are obviously have very limited application, because it cannot account for a very common practical situation: a system often tolerate a small delay without losing stability, while a large delay destabilizes the system.

Such simple stability conditions can be extended to more general systems. For example, for systems with multiple delays,

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{k} A_j x(t - r_j), \tag{4.14}$$

we can choose the Lyapunov-Krasovskii functional

$$V(x_t) = x^T(t) P x(t) + \sum_{j=1}^{k} \int_{-r_j}^{0} x^T(t+\theta) S_j x(t+\theta) d\theta. \tag{4.15}$$

Since its derivative along the system trajectory is

$$\dot{V}(x_t) = \psi^T(t) \Pi \psi(t),$$

where

$$\psi^T(t) = \left( x^T(t)\ x^T(t-r_1)\ \dots\ x^T(t-r_k) \right),$$

$$\Pi = \begin{pmatrix} PA_0 + A_0^T P + \sum\limits_{j=1}^{k} S_j & PA_1 & PA_2 & \dots & PA_k \\ A_1^T P & -S_1 & 0 & \dots & 0 \\ A_2^T P & 0 & -S_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_k^T P & 0 & 0 & \dots & -S_k \end{pmatrix}, \tag{4.16}$$

we arrive at the following stability conditions.

**Proposition 4.2.** *The system (4.14) is asymptotically stable is there exist symmetric matrices $P$, $S_j$, $j = 1, 2, ..., k$, such that*

$$P > 0,$$
$$\Pi < 0$$

*are satisfied, where $\Pi$ is defined in (4.16).*

A further extension is to systems with distributed delays

$$\dot{x}(t) = A_0 x(t) + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta. \tag{4.17}$$

Analogous to (4.15) for multiple delay case, we can choose

$$V(x_t) = x^T(t)Px(t) + \int_{-r}^{0} [\int_{\theta}^{0} x^T(t+\tau)S(\theta)x(t+\tau)d\tau]d\theta.$$

This gives

$$\begin{aligned}\dot{V}(x_t) = x^T(t)[PA_0 + A_0^T P &+ \int_{-r}^{0} S(\theta)d\theta]x(t)\\ &+ 2x^T(t)P\int_{-r}^{0} A(\theta)x(t+\theta)d\theta\\ &- \int_{-r}^{0} x^T(t+\theta)S(\theta)x(t+\theta)d\theta.\end{aligned}$$

Add and subtract $x^T(t)\int_{-r}^{0} R(\theta)d\theta x(t)$, where $R(\theta)$ is a symmetric matrix function, we obtain

$$\begin{aligned}\dot{V}(x_t) &= x^T(t)[PA_0 + A_0^T P + \int_{-r}^{0} R(\theta)d\theta]x(t)\\ &+ \int_{-r}^{0} \begin{pmatrix} x^T(t) & x^T(t+\theta) \end{pmatrix} \begin{pmatrix} S(\theta) - R(\theta) & PA(\theta) \\ A^T(\theta)P & -S(\theta) \end{pmatrix} \begin{pmatrix} x(t) \\ x(t+\theta) \end{pmatrix} d\theta.\end{aligned}$$

From this, we arrive at the following stability conditions.

**Proposition 4.3.** *The system (4.17) is asymptotically stable if there exist symmetric matrix $P$, and symmetric matrix functions $S$ and $R : [-r, 0] \to \mathbb{R}^{n\times n}$, such that*

$$P > 0,$$

$$PA_0 + A_0^T P + \int_{-r}^{0} R(\theta)d\theta < 0,$$

*and*

$$\begin{pmatrix} S(\theta) - R(\theta) & PA(\theta) \\ A^T(\theta)P & -S(\theta) \end{pmatrix} \le 0 \text{ for all } \theta \in [-r, 0]$$

*are satisfied.*

### 4.4.2 Delay-dependent Stability Using Model Transformation

A simple way of bringing delay $r$ into stability conditions of (4.7) is to transform it to a distributed time-delay system. This is done using the Newton-Raphson formula

$$x(t-r) = x(t) - \int_{-r}^{0} \dot{x}(t+\theta)d\theta$$

for the term $x(t-r)$ in (4.7), and using (4.7) for $\dot{x}(t+\theta)$ in the integral. This result in a new system

$$\dot{x}(t) = (A_0 + A_1)x(t) - A_1 A_0 \int_{-r}^{0} x(t+\theta)d\theta - A_1^2 \int_{-2r}^{-r} x(t+\theta)d\theta. \quad (4.18)$$

The process of obtaining (4.18) from (4.7) is sometimes known as *model transformation*. Before we go on to analyze (4.18), we should point out that the stability of the two systems expressed by (4.7) and (4.18) are not equivalent. Althought the stability of (4.18) implies that of (4.7), the reverse is not necessarily true. It can be seen that the maximum delay of (4.18) is $2r$ rather than $r$. Indeed, the characteristic equation of (4.7) is

$$\Delta_o(s) = \det(sI - A_0 - e^{-rs}A_1) = 0,$$

and that of (4.18) is

$$\Delta_t(s) = \Delta_a(s)\Delta_o(s) = 0,$$

where

$$\Delta_a(s) = \det\left(I - \frac{1-e^{-rs}}{s}A_1\right).$$

The factor $\Delta_a(s)$ represents *additional dynamics*. It is possible that all the zeros of $\Delta_o(s)$ are on the left half plane while some zeros of $\Delta_a(s)$ are on the right half plane. See [12] for detailed analysis, and [11] and the references therein for additional dynamics in more general setting.

To study the stability of (4.18), we notice that it is in the form of (4.17), and therefore, can use Proposition 4.3, which in this case becomes

$$P > 0,$$

$$P(A_0 + A_1) + (A_0 + A_1)^T P + \int_{-2r}^{0} R(\theta)d\theta < 0,$$

and

$$\begin{pmatrix} S(\theta) - R(\theta) & -PA_1A_0 \\ -(A_1A_0)^T P & -S(\theta) \end{pmatrix} \leq 0 \text{ for all } \theta \in [-r, 0],$$

$$\begin{pmatrix} S(\theta) - R(\theta) & -PA_1^2 \\ (A_1^2)^T P & -S(\theta) \end{pmatrix} \leq 0 \text{ for all } \theta \in [-2r, -r).$$

We may choose

$$R(\theta) = \begin{cases} R_0, & -r \leq \theta \leq 0, \\ R_1, & -2r \leq \theta < -r, \end{cases}$$

$$S(\theta) = \begin{cases} S_0, & -r \leq \theta \leq 0, \\ S_1, & -2r \leq \theta < -r, \end{cases}$$

to obtain the following stability conditions.

**Proposition 4.4.** *The system (4.18) is asymptotically stable (which implies that the system (4.17) is asymptotically stable) if there exists symmetric matrices $P$, $S_0$, $S_1$, $R_0$, and $R_1$ such that*

$$P > 0,$$

$$P(A_0 + A_1) + (A_0 + A_1)^T P + r(R_0 + R_1) < 0,$$

$$\begin{pmatrix} S_0 - R_0 & -PA_1A_0 \\ -(A_1A_0)^T P & -S_0 \end{pmatrix} \leq 0,$$

$$\begin{pmatrix} S_1 - R_1 & -PA_1^2 \\ (A_1^2)^T P & -S_1 \end{pmatrix} \leq 0.$$

We may write the above in a different form by eliminating $R_0$ and $R_1$.

**Corollary 4.1.** *The system (4.18) (and (4.17)) is asymptotically stable if there exist symmetric matrices $P$, $S_0$ and $S_1$ such that*

$$P > 0,$$

$$\begin{pmatrix} M & -PA_1A_0 & -PA_1^2 \\ & -S_0 & 0 \\ \textit{Symmetric} & & -S_1 \end{pmatrix} < 0,$$

*where*

$$M = \frac{1}{r}[P(A_0 + A_1) + (A_0 + A_1)^T P] + S_0 + S_1.$$

*Proof*. We make the conditions in Corollary 4.4 slightly more stringent by replacing "$\leq$" by "$<$", and eliminating $R_0$ and $R_1$ using the technique discussed in [9] or Appendix B of [11] to obtain the resulting LMIs. ∎

### 4.4.3 Implicit Model Transformation

It is also possible to obtain relatively simple delay-dependent stability conditions without explicit model transformation and with less conservatism, although it still uses maximum delay of $2r$. We call such a process as *implicit model transformation*. Here, we will discuss a method very similar to the one proposed by Park [25]. Consider again the system described by (4.7). Choose Lyapunov-Krasovskii functional

$$V(x_t) = x^T(t)Px(t) + \int_{-r}^{0}\int_{\theta}^{0} f^T(x_{t+\xi})Zf(x_{t+\xi})d\xi d\theta + \int_{-r}^{0} x^T(t+\theta)Sx(t+\theta)d\theta$$

where $f(x_t)$ represents the right hand side of (4.7), and by a change of time variable,

$$f(x_{t+\xi}) = A_0x(t+\xi) + A_1x(t+\xi-r).$$

Using the fact that for any defferentiable function $\psi$ and $\theta < 0$,

$$\frac{d}{dt}\int_{\theta}^{0}\psi(f(x_{t+\xi}))d\xi = \psi(f(x_t)) - \psi(f(x_{t+\theta})),$$

we obtain

$$\begin{aligned}\dot{V}(x_t) = {} & \phi_{0r}^T \begin{pmatrix} M & PA_1 + rA_0^TZA_1 \\ [PA_1 + rA_0^TZA_1]^T & rA \end{pmatrix} \phi_{0r} \\ & - \int_{-r}^{0} f^T(x_{t+\theta})Zf(x_{t+\theta})d\theta, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} M &= PA_0 + A_0^TP + rA_0^TZA_0 + S, \\ \phi_{0r}^T &= \begin{pmatrix} x^T(t) & x^T(t-r) \end{pmatrix}. \end{aligned}$$

For

$$\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} > 0, \tag{4.20}$$

we have

$$\begin{aligned} 0 &< \int_{-r}^{0} \begin{pmatrix} x^T(t) & \dot{x}^T(t+\theta) \end{pmatrix} \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t+\theta) \end{pmatrix} d\theta \\ &= rx^T(t)Xx(t) + 2x^T(t)Y(x(t) - x(t-r)) + \int_{-r}^{0} \dot{x}^T(t+\theta)Z\dot{x}(t+\theta)d\theta. \end{aligned} \tag{4.21}$$

Adding (4.21) to (4.19) and using

$$\dot{x}(t+\theta) = f(x_{t+\theta}), \tag{4.22}$$

we obtain

$$\dot{V}(x_t) \leq \phi_{0r}^T \begin{pmatrix} N & PA_1 + rA_0^T Z A_1 - Y \\ \text{Symmetric} & -S + rA_1^T Z A_1 \end{pmatrix} \phi_{0r},$$

where

$$N = PA_0 + A_0^T P + rA_0^T Z A_0 + S + rX + Y + Y^T. \tag{4.23}$$

Therefore, we conclude the following.

**Proposition 4.5.** *The system (4.7) is asymptotically stable if there exist matrix $Y$ and symmetric matrices $P$, $X$ and $Z$ such that*

$$P > 0,$$

$$\begin{pmatrix} N & PA_1 + rA_0^T Z A_1 - Y \\ \mathit{Symmetric} & -S + rA_1^T Z A_1 \end{pmatrix} < 0,$$

*and (4.20) are satisfied. In the above, $N$ is expressed in (4.23).*

Notice, due to the usage of (4.22) for $\theta \in [-r, 0)$, this process involves $x(t+\xi)$ for $-2r \leq \xi \leq 0$, and implicitly involves model transformation in some sense. It can shown that this stability condition is indeed less conservative than both Propositions 4.1 and 4.4. It can also be written in a number of different forms, see [11] for details.

## 4.5 Complete Quadratic Lyapunov-Krasovskii Functional

It will be shown by numerical examples later on in this section that all the methods discussed in the previous section involves substantial conservatism. Further more, all of them requires the system to be stable if the delay is set to zero. However, there are many practical cases where delay may be used to stabilize the system. See [1] for a simple example. Indeed, a finite difference approximation of derative in control implementation will introduce time delays, which are often used to stabilize the system.

To obtain necessary and sufficient condiiton for stability, it is necessary to use *complete quadratic Lyapunov-Krasovskii functional* as pointed out by Repin [27], Infante and Castelan [15].

### 4.5.1 Analytical Expression

Recall that the finite dimensional system

$$\dot{x}(t) = Ax(t) \tag{4.24}$$

is asymptotically stable if and only if for any given positive definite $W$, the Lyapunov equation

$$PA + A^T P = -W$$

has a positive definite solution. Indeed, a quadratic Lyapunov function can be constructed from the solution $P$,

$$V(x) = x^T Px,$$

which achieves

$$\dot{V}(x) = -x^T Wx.$$

Furthermore, the solution $P$ can be explicitly expressed as

$$P = \int_0^\infty X^T(t)WX(t)dt,$$

where $X(t)$ is the fundamental solution of (4.24), which satisfy

$$\dot{X}(t) = AX(t),$$
$$X(0) = I.$$

Let $x(t, \phi)$ be the solution of (4.24) with initial condition $x(0) = \phi$, then $x(t, \phi) = X(t)\phi$, and therefore, we may further write

$$V(\phi) = \int_0^\infty x^T(\tau, \phi)Wx(\tau, \phi)d\tau.$$

For a stable time-delay system (4.7), it is also possible to construct a Lyapunov-Krasovskii functional $V(x_t)$ such that

$$\dot{V}(x_t) = -x^T(t)Wx(t).$$

Indeed, let $x(t, \phi)$ be the solution of (4.7) with initial condition $x_0(\theta) = \phi(\theta)$, $\theta \in [-r, 0]$, then we can still write

$$V(\phi) = \int_0^\infty x^T(\tau, \phi)Wx(\tau, \phi)d\tau.$$

Through some algebra, we can expression $V(\phi)$ explicitly as a quadratic functional of $\phi$,

$$\begin{aligned} V(\phi) &= \phi^T(0)U(0)\phi(0) \\ &\quad + 2\phi^T(0)\int_{-r}^0 U(-r-\theta)A_1\phi(\theta)d\theta \\ &\quad + \int_{-r}^0\int_{-r}^0 \phi^T(\theta_1)A_1^T U(\theta_1-\theta_2)A_1\phi(\theta_2)d\theta_1 d\theta_2 \end{aligned} \tag{4.25}$$

where $U : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined as

$$U(\tau) = \int_0^\infty X^T(t) W X(t+\tau) dt.$$

In order to have $U(\tau)$ well defined, we agree that $X(t) = 0$ for $t < 0$. It can be shown that

$$U^T(\tau) = U(-\tau).$$

The readers are referred to [11] for more details.

### 4.5.2 Discretization

The analytical expression (4.25) indicates that for any asymptotically stable system, we can always find a complete quadratic Lyapunov-Krasovskii functional. In other words, the existence of such a functional is necessary and sufficient for stability. In order for numerical calculation, we enlarge the class of quadratic Lyapunov-Krasovskii functionals to the form

$$\begin{aligned} V(x_t) &= x^T(t) P x(t) \\ &+ 2x^T(t) \int_{-r}^0 Q(\theta) x(t+\theta) d\theta \\ &+ \int_{-r}^0 \int_{-r}^0 x^T(t+\xi) R(\xi,\eta) x(t+\eta) d\xi d\eta \\ &+ \int_{-r}^0 x^T(t+\xi) S(\xi) x(t+\xi) d\xi, \end{aligned} \tag{4.26}$$

where

$$P = P^T,$$

and for all $\xi \in [-r, 0]$, $\eta \in [-r, 0]$,

$$\begin{aligned} Q(\xi) &\in \mathbb{R}^{n\times n}, \\ R(\xi,\eta) = R^T(\eta,\xi) &\in \mathbb{R}^{n\times n}, \\ S(\xi) = S^T(\xi) &\in \mathbb{R}^{n\times n}. \end{aligned}$$

Since $V(x_t)$ is clearly upper-bounded, sufficient conditions for asymptotic stability (we can show they are also necessary) are

$$V(x_t) \geq \varepsilon ||x(t)||^2, \tag{4.27}$$

$$\dot{V}(x_t) \leq -\varepsilon ||x(t)||^2, \tag{4.28}$$

for some $\varepsilon > 0$.

The search for the existence of functions $Q$, $R$ and $S$ (in addition to matrix $P$) is clearly an infinite-dimensional problem. It can be viewed as an infinite-dimensional LMI. To make numerical computation feasible, we will constrain these matrix functions to be *piecewise linear*. Specifically, divide the interval $[-r, 0]$ into $N$ intervals of equal length (nonuniform mesh is also possible, but we will not discuss it here)

$$h = \frac{r}{N},$$

and let the dividing points be denoted as

$$\theta_p = -ph = -\frac{pr}{N},\ p = 0, 1, 2, ..., N.$$

Let

$$\begin{aligned} Q_p &= Q(\theta_p), \\ S_p &= S(\theta_p), \\ R_{pq} &= R(\theta_p, \theta_q). \end{aligned}$$

Then, for $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, we restrict, for $p = 1, 2, ..., N$,

$$\begin{aligned} Q(\theta_p + \alpha h) &= (1-\alpha) Q_p + \alpha Q_{p-1}, \\ S(\theta_p + \alpha h) &= (1-\alpha) S_p + \alpha S_{p-1}, \end{aligned}$$

and

$$\begin{aligned} &R(\theta_p + \alpha h, \theta_q + \beta h) \\ &= \begin{cases} (1-\alpha) R_{pq} + \beta R_{p-1,q-1} + (\alpha - \beta) R_{p-1,q},\ \alpha \geq \beta, \\ \\ (1-\beta) R_{pq} + \alpha R_{p-1,q-1} + (\beta - \alpha) R_{p,q-1},\ \alpha < \beta. \end{cases} \end{aligned}$$

Through a rather tedious process, we can reduce (4.27) and (4.28) to LMIs. This approach is known as the *discretized Lyapunov functional method*. Here, we will only give the resulting LMI for the case of $N = 1$ in the following. The readers are referred to [11] for the general case.

**Proposition 4.6.** *The system is asymptotically stable if there exist $n \times n$ real matrices $P = P^T$, $Q_p$, $S_p = S_p^T$, $R_{pq} = R_{qp}^T$, $p = 0, 1$; $q = 0, 1$, such that*

$$\begin{pmatrix} P & Q_0 & Q_1 \\ & R_{00} + S_0 & R_{01} \\ \textit{Symmetric} & & R_{11} + S_1 \end{pmatrix} > 0,$$

*and*

$$\begin{pmatrix} \Delta_{00} & Q_1 - PA_1 & D_0^s & D_0^a \\ & S_1 & D_1^s & D_1^a \\ & & h(R_{00} - R_{11}) + S_0 - S_1 & 0 \\ \textit{Symmetric} & & & 3(S_0 - S_1) \end{pmatrix} > 0,$$

*where*

$$\begin{aligned}
\Delta_{00} &= -PA_0 - A_0^T P - Q_0 - Q_0^T - S_0,\\
D_0^s &= \frac{r}{2}A_0^T(Q_0+Q_1) + \frac{r}{2}(R_{00}+R_{01}) - (Q_0 - Q_1),\\
D_1^s &= \frac{r}{2}A_1^T(Q_0+Q_1) - \frac{r}{2}(R_{10}+R_{11}),\\
D_0^a &= -\frac{r}{2}A_0^T(Q_0-Q_1) - \frac{r}{2}(R_{00}-R_{01}),\\
D_1^a &= -\frac{r}{2}A_1^T(Q_0-Q_1) + \frac{r}{2}(R_{10}-R_{11}).
\end{aligned}$$

## 4.6 A Comparison of Lyapunov-Krasovskii Functionals

Obviously, the delay-independent stability condition in Proposition 4.1 is very conservative if the delay is known. Although the simple delay-dependent condition in Proposition 4.4 is intended to improve the situation, it is not necessarily less conservative in all the situations. There are indeed systems which satisfy the conditions in Proposition 4.1 but do not satisfy those in Proposition 4.4. See [12] for an example.

As mentioned earlier, it can be shown that the method with implicit model transformation discussed in Proposition 4.5 is indeed less conservative than both Proposition 4.1 and Proposition 4.4.

The discretized Lyapunov functional method can approach analytical results very quickly, and is the least conservative among these methods. The following example is often used in the literature.

*Example 4.3.* Consider the system

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r)$$

Various methods are used to estimate the maximum delay $r_{\max}$ without losing stability, and the results are listed in the following table. In the first line, "Analytical" indicates the true maximum delay obtained by the first time a pair of roots of the characteristic quasipolynomial crosses the imaginary axis as the delay increases; "Explicit" means the delay-dependent stability conditions in Propostion 4.4 which uses explicit model transformation; "Implicit" denotes the delay-dependent stability conditions in Proposition 4.5 which uses implicit model transformation, the remaining three columns are the results using discretized Lyapunov functional method with different $N$, with $N = 1$ covered in Proposition 4.6.

| Methods | Analytical | Explicit | Implicit | $N = 1$ | $N = 2$ | $N = 3$ |
|---|---|---|---|---|---|---|
| $r_{\max}$ | 6.17258 | 1.00 | 4.359 | 6.059 | 6.165 | 6.171 |

The next example shows that there are indeed systems that are unstable without delay, but may becomes stable for some nonzero delays.

*Example 4.4.* Consider the system

$$\ddot{x}(t) - 0.1\dot{x}(t) + x(t) = -r\frac{x(t) - x(t-r)}{r}$$

The left hand side may be considered as a second order system with negative damping, and the right hand side can be considered as a control to stabilize the system by providing sufficient positive damping and using finite difference to approximate the derivative. If the derivative is used instead of finite difference, then obviously the system would be stable for $r > 0.1$. For such systems, the stability conditions covered in Propositions 4.4 and 4.5 are not applicable since they requires the system to be stable for zero delay. We now write the system in a state space form

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x(t-r) \\ \dot{x}(t-r) \end{pmatrix}.$$

The system is stable for $r \in (r_{\min}, r_{\max})$. The following table lists the estimated values using discretized Lyapunov functional method with different $N$, as well as the analytical values. It can be seen, again, that discretized Lyapunov functional method can approach the analytical results with a rather modest $N$.

| $N$ | 1 | 2 | 3 | Analytical |
|---|---|---|---|---|
| $r_{\min}$ | 0.1006 | 0.1003 | 0.1003 | 0.1002 |
| $r_{\max}$ | 1.4272 | 1.6921 | 1.7161 | 1.7178 |

## 4.7 Dealing with Time-varying Delays

Consider a system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - r(t)), \tag{4.29}$$

where the time-varying delay $r(t)$ satisfies

$$r_m \le r(t) \le r_M, \tag{4.30}$$

$$\dot{r}(t) \le \rho, \tag{4.31}$$

where $\rho$ is a known constant, $0 \le \rho < 1$.

We choose a complete quadratic Lyapunov-Krasovskii functional

$$V(t, x_t) = V_1(x_t) + V_2(t, x_t),$$

where,

$$\begin{aligned} V_1(x_t) &= x^T(t)Px(t) \\ &+ 2x^T(t)\int_{-r_m}^{0} Q(\theta)x(t+\theta)d\theta \\ &+ \int_{-r_m}^{0}\int_{-r_m}^{0} x^T(t+\xi)R(\xi,\eta)x(t+\eta)d\xi d\eta \\ &+ \int_{-r_m}^{0} x^T(t+\xi)S(\xi)x(t+\xi)d\xi. \end{aligned} \tag{4.32}$$

Let $V_1^*(x_t)$ indicates the derivative of (4.32) along the comparison system

$$\dot{x}(t) = A_0x(t) + A_1x(t - r_m), \tag{4.33}$$

which is in an identical form discussed in the last section. Then,

$$\begin{aligned} \dot{V}_1(x_t) &= V_1^*(x_t) + 2x^T(t)PA_1[x(t - r(t)) - x(t - r_m)] \\ &\quad + 2[x(t - r(t)) - x(t - r_m)]^T A_1^T \int_{-r^m}^{0} Q(\theta)x(t+\theta)d\theta. \end{aligned}$$

Let

$$\begin{aligned} V_2(t, x_t) &= \int_{-r_M}^{-r_m} [\int_{\theta}^{0} x^T(t+\zeta)K_1x(t+\zeta)d\zeta]d\theta \\ &+ \int_{-r_M}^{-r_m} [\int_{\theta-r(t+\theta)}^{0} x^T(t+\zeta)K_2x(t+\zeta)d\zeta]d\theta. \end{aligned}$$

Then

$$\begin{aligned} \dot{V}_2(t, x_t) &= (r_M - r_m)x^T(t)(K_1 + K_2)x(t) \\ &- \int_{-r_M}^{-r_m} x^T(t+\theta)K_1x(t+\theta)d\theta \\ &- \int_{-r_M}^{-r_m} (1 - \dot{r}(t+\theta))x^T(t+\theta - r(t+\theta))K_2x(t+\theta - r(t+\theta))d\theta. \end{aligned}$$

In view of the fact that

$$\begin{aligned} &x(t - r(t)) - x(t - r_m) \\ &= \int_{-r(t)}^{-r_m} \dot{x}(t+\theta)d\theta \\ &= \int_{-r(t)}^{-r_m} [A_0x(t+\theta) + A_1x(t+\theta + r(t+\theta))]d\theta, \end{aligned}$$

we have

$$\dot{V}(t, x_t) = V_1^*(x_t) - 2x^T(t)PA_1 \int_{-r(t)}^{-r_m} [A_0 x(t+\theta) + A_1 x(t+\theta+r(t+\theta))]d\theta$$
$$- 2\int_{-r(t)}^{-r_m} [A_0 x(t+\theta) + A_1 x(t+\theta+r(t+\theta))]^T d\theta A_1^T \int_{-r_m}^{0} Q(\theta)x(t+\theta)d\theta$$
$$+ (r_M - r_m)x^T(t)(K_1 + K_2)x(t) - \int_{-r_M}^{-r_m} x^T(t+\theta)K_1 x(t+\theta)d\theta$$
$$- \int_{-r_M}^{-r_m} (1 - \dot{r}(t+\theta))x^T(t+\theta-r(t+\theta))K_2 x(t+\theta-r(t+\theta))d\theta.$$

Using (4.30) and (4.31), we can arrive at

$$\dot{V}(t, x_t) \le - \int_{-r(t)}^{-r_m} \begin{pmatrix} x^T(t) & x^T(t+\theta) \end{pmatrix} \begin{pmatrix} \hat{K}_{1a} & PA_1A_0 \\ A_0^T A_1^T P & K_{1a} \end{pmatrix} \begin{pmatrix} x(t) \\ x(t+\theta) \end{pmatrix} d\theta$$
$$- \int_{-r(t)}^{-r_m} \begin{pmatrix} \mu^T(t) & x^T(t+\theta) \end{pmatrix} \begin{pmatrix} \hat{K}_{1b} & A_1A_0 \\ A_0^T A_1^T & K_{1b} \end{pmatrix} \begin{pmatrix} \mu(t) \\ x(t+\theta) \end{pmatrix} d\theta$$
$$- \int_{-r(t)}^{-r_m} \begin{pmatrix} x^T(t) & \nu^T(t,\theta) \end{pmatrix} \begin{pmatrix} \hat{K}_{2a} & PA_1A_1 \\ A_1^T A_1^T P & (1-\rho)K_{2a} \end{pmatrix} \begin{pmatrix} x(t) \\ \nu(t,\theta) \end{pmatrix} d\theta$$
$$- \int_{-r(t)}^{-r_m} \begin{pmatrix} \mu^T(t) & \nu^T(t,\theta) \end{pmatrix} \begin{pmatrix} \hat{K}_{2b} & A_1A_1 \\ A_1^T A_1^T & (1-\rho)K_{2b} \end{pmatrix} \begin{pmatrix} \mu(t) \\ \nu(t,\theta) \end{pmatrix} d\theta$$
$$+ V_1^*(x_t) + (r_M - r_m)x^T(t)(K_1 + K_2)x(t)$$
$$+ (r(t) - r_m)x^T(t)(\hat{K}_{1a} + \hat{K}_{2a})x(t)$$
$$+ (r(t) - r_m)\mu^T(t)(\hat{K}_{1b} + \hat{K}_{2b})\mu(t),$$

where

$$\mu(t) = \int_{-r_m}^{0} Q(\theta)x(t+\theta)d\theta,$$
$$\nu(t,\theta) = x(t+\theta-r(t+\theta)),$$

and

$$K_{1a} + K_{1b} = K_1,$$
$$K_{2a} + K_{2b} = K_2.$$

If we choose $K_{1a}$, $K_{2a}$ (so that $K_{1b}$, $K_{2b}$ are also determined), and $\hat{K}_{1a}$, $\hat{K}_{1b}$, $\hat{K}_{2a}$, $\hat{K}_{2b}$ such that

$$\begin{pmatrix} \hat{K}_{1a} & PA_1A_0 \\ A_0^TA_1^TP & K_{1a} \end{pmatrix} \geq 0,$$

$$\begin{pmatrix} \hat{K}_{1b} & A_1A_0 \\ A_0^TA_1^T & K_{1b} \end{pmatrix} \geq 0,$$

$$\begin{pmatrix} \hat{K}_{2a} & PA_1A_1 \\ A_1^TA_1^TP & (1-\rho)K_{2a} \end{pmatrix} \geq 0,$$

$$\begin{pmatrix} \hat{K}_{2b} & A_1A_1 \\ A_1^TA_1^T & (1-\rho)K_{2b} \end{pmatrix} \geq 0,$$

then the four integrals are all less or equal to zero. Therefore, we conclude that the system is asymptotically stable if we can make

$$\begin{aligned} &V_1^*(x_t) + (r_M - r_m)x^T(t)(K_1 + K_2)x(t) \\ &+ (r(t) - r_m)x^T(t)(\hat{K}_{1a} + \hat{K}_{2a})x(t) \\ &+ (r(t) - r_m)\mu^T(t)(\hat{K}_{1b} + \hat{K}_{2b})\mu(t) \\ &\leq -\varepsilon||x(t)||^2. \end{aligned}$$

A discretized Lyapunov functional approach can be used to achieve this. The above development is similar to [14].

An alternative is to formulate the time-varying delay as a perturbation to a time-invariant delay, and formulate it as an uncertain feedback problem. See, for example, [11] and [23].

It is also possible to lift the restriction of derivative bound (4.31). One simple approach is to use the Razumikhin Theorem based methods. Other approaches include an alternative formulation of Lyapunov-Krasovskii functional method proposed in [7], and the input-output approach along the similar idea as [16].

## 4.8 Razumikhin Theorem

Razumikhin showed that it is still possible to use function rather than functionals in stability analysis of time-delay system. This is based on the following Razumikhin Theorem.

**Theorem 4.2.** *Suppose $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ in (4.3) takes $\mathbb{R} \times$ (bounded sets of $\mathcal{C}$) into bounded sets of $\mathbb{R}^n$, and $u$, $v$, $w : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$, $v$ strictly increasing. If there exists a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that*

$$u(||x||) \leq V(t, x) \leq v(||x||), \text{ for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n, \tag{4.34}$$

*and the derivative of $V$ along the solution $x(t)$ of (4.3) satisfies*

$$\dot{V}(t, x(t)) \leq -w(||x(t)||) \text{ whenever } V(t+\theta, x(t+\theta)) \leq V(t, x(t)), \quad (4.35)$$

*for $\theta \in [-r, 0]$, then the system (4.3) is uniformly stable. If, in addition, $w(s) > 0$ for $s > 0$, and there exists a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that condition (4.35) is strengthened to*

$$\dot{V}(t, x(t)) \leq -w(||x(t)||) \text{ whenever } V(t+\theta, x(t+\theta)) \leq p(V(t, x(t)), \quad (4.36)$$

*for $\theta \in [-r, 0]$, then the system (4.3) is uniformly asymptotically stable. If, in addition, $\lim_{s\to\infty} u(s) = \infty$, then the system (4.3) is globally uniformly asymptotically stable.*

The basic idea of the above Theorem is to consider the Lyapunov-Krasovskii functional

$$\bar{V}(x_t) = \max_{\theta\in[-r,0]} V(x+\theta),$$

and realize that if $V(x(t)) < \bar{V}(x_t)$, then $\bar{V}(x_t)$ does not grow at the instant $t$ even if $\dot{V}(x(t)) > 0$. Therefore, in order for $\bar{V}(x_t)$ to grow, one only needs to make sure that $\dot{V}(x(t))$ is not positive whenever $V(x(t)) = \bar{V}(x_t)$. For a proof, the readers are referred to [11], [13] or [18].

A direct application of Razumikhin Theorem to time-invariant time-delay systems typically results in a more conservative stability conditions than the counterpart obtained by using the Lyapunov-Krasovskii functional method. However, there are a number of situations where Razumikhin Theorem has advantage. For example, time-varying delay can be easily handled. Consider the following system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - r(t)). \quad (4.37)$$

This is the same as (4.7) except the delay is time-varying. Typically, the delay is known within certain range,

$$r_m \leq r(t) \leq r_M.$$

However, there is no restriction on the rate of change of $r(t)$. Let

$$V(x) = x^T P x, \; P > 0.$$

Then, we can calculate

$$\dot{V}(x(t)) = x^T(t)(PA_0 + A_0^T P)x(t) + 2x^T PA_1 x(t - r(t)).$$

The system is asymptotically stable if

$$\dot{V}(x(t)) \leq -\varepsilon ||x(t)||^2 \text{ for some small } \varepsilon > 0,$$

whenever

$$V(x(t-r(t)) \leq \beta V(x(t)) \text{ for some } \beta > 1.$$

In other words, it is sufficient that

$$\dot{V}(x(t)) - \alpha[\beta V(x(t-r(t)) - V(x(t))] \leq -\varepsilon ||x(t)||^2,$$

for some $\varepsilon > 0$, $\alpha \geq 0$ and $\beta > 1$. Using the expression for $V$ and $\dot{V}$, the above becomes

$$\begin{aligned} &\begin{pmatrix} x^T(t) \; x^T(t-r(t)) \end{pmatrix} \begin{pmatrix} PA_0 + A_0^T P + \alpha P & PA_1 \\ A_1^T P & -\alpha\beta P \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-r(t)) \end{pmatrix} \\ &\leq -\varepsilon x^T(t) x(t) \end{aligned}$$

from which we can conclude the following.

**Proposition 4.7.** *The system (4.37) is asymptotically stable if there exist a real scalar $\alpha > 0$ and symmetric matrix $P > 0$ such that*

$$\begin{pmatrix} PA_0 + A_0^T P + \alpha P & PA_1 \\ A_1^T P & -\alpha P \end{pmatrix} < 0. \tag{4.38}$$

Compared to Proposition 4.1, the above can be obtained from (4.13) by constraining $S = \alpha P$. Therefore, this is obviously more conservative if used for systems with a time-invariant delay. Computationally although (4.38) involves fewer parameters than (4.13), it is actually computationally more difficult because it is no longer an LMI due to the multiplicative term $\alpha P$. See [11] for handling such computational issue.

Parallel to the Lypunov-Krasovskii functional methods, we can also derive delay-dependent results using explicit and implicit model transformation. See [11] for details.

## 4.9 Coupled Difference-Differential Equations

### 4.9.1 Introducation

In this section, we will discuss the system described by coupled difference-differential equations,

$$\dot{x}(t) = Ax(t) + By(t-r), \tag{4.39}$$

$$y(t) = Cx(t) + Dy(t-r), \tag{4.40}$$

where $x(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^n$. This model is also known as the *lossless propagation model* due to the fact that it comes out naturally from simplifying some lossless propagation systems [24]. Most of the materials in this section are based on [10].

The equations (4.39) and (4.40) represent both neutral and retarded time-delay systems with commensurate multiple delays as special cases. For example, for the system described by

$$\sum_{k=0}^{p} F_k \dot{x}(t-kr) = \sum_{k=0}^{p} A_k x(t-kr), \qquad F_0 = I,$$

we may define

$$\begin{aligned} y_k(t) &= x(t-kr+r), \\ z(t) &= \sum_{k=0}^{p} F_k x(t-kr). \end{aligned}$$

This allows us to write the system as

$$\begin{aligned} \dot{z}(t) &= A_0 z(t) + \sum_{k=1}^{p} (A_k - A_0 F_k) y_k(t-r), \\ y_1(t) &= z(t) - \sum_{k=1}^{p} F_k y_k(t-r), \\ y_k(t) &= y_{k-1}(t-r), \qquad k = 2, 3, ..., p, \end{aligned}$$

which is in the standard form of (4.39) and (4.40).

Obviously, the future evolution of the system described by (4.39) and (4.40) is completely decided by $x(t)$ and $y(t+\theta)$, $-r \leq \theta < 0$. Naturally, the initial condition to be specified should be described by

$$x(0) = \psi, \tag{4.41}$$

$$y_0 = \phi. \tag{4.42}$$

In (4.42), we have used the notation that $y_t$ represents a time-shift and restriction of $y$ in the interval $[t-r, t)$ defined as

$$y_t(\theta) = y(t+\theta), \ -r \leq \theta < 0,$$

and $\phi : [-r, 0) \to \mathbb{R}^n$.

For the pair $(\psi, \phi)$, we also define the norm as

$$||(\psi, \phi)|| = \max\{||\psi||, \sup_{-r \leq \theta < 0} ||\phi(\theta)||\}.$$

We can describe the general Lyapunov-Krasovskii stability condition for the system described by (4.39) and (4.40) as follows, which is also similar to a neutral time-delay system.

**Theorem 4.3.** *Consider the system described by (4.39) and (4.40) with* $\rho(D) < 1$. *Let* $u, v, w : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ *be continuous and nondecreasing functions. In addition,* $u(s)$ *and* $v(s)$ *are positive for positive* $s$, *and* $u(0) = v(0) = 0$. *If there exists a continuous differentiable functional* $V : (\psi, \phi)$ *such that*

$$u(||\psi||) \leq V(\psi, \phi) \leq v(||(\psi, \phi)||),$$
$$\dot{V}(\psi, \phi) \leq -w(\psi),$$

*then the trivial solution of the system is stable. If, in addition,* $w(s) > 0$ *for* $s > 0$, *then it is asymptotically stable.*

We can prove the above in a very similar way to the standard neutral time-delay system (for example, Theorem 1.1 in Chapter 8 of [18]) using the fact that $\rho(D) < 1$.

### 4.9.2 Fundamental Solutions

As in the case of the sytem (4.7), a complete quadratic Lyapunov-Krasovakii functional is essential to give nonconservative stability conditions, and the analytical construction of such a Lyapunov-Krasovskii functional is based on the fundamental solutions.

We will write the solution of the equation

$$\dot{x}(t) = Ax(t) + By(t-r) + \delta(t)I, \tag{4.43}$$
$$y(t) = Cx(t) + Dy(t-r), \tag{4.44}$$

with zero initial conditions

$$x(0) = 0,\ y_0 = 0, \tag{4.45}$$

as

$$x(t) = X_x(t),$$
$$y(t) = Y_x(t).$$

Similarly, the solution of

$$\dot{x}(t) = Ax(t) + By(t-r), \tag{4.46}$$
$$y(t) = Cx(t) + Dy(t-r) + \delta(t)I, \tag{4.47}$$

with zero initial conditions (4.45) are denoted as

$$x(t) = X_y(t),$$
$$y(t) = Y_y(t).$$

We also agree that $X_x(t) = 0, Y_x(t) = 0, X_y(t) = 0, Y_y(t) = 0$ for $t < 0$. The solutions ($X_x(t)$, $Y_x(t)$, $X_y(t)$, $Y_y(t)$) are known as the fundamental solutions of the system described by (4.39) and (4.40). $(X_x(t), Y_x(t))$ can also be regarded as the solution of (4.39) and (4.40) with initial condition

$$x(0) = I, \; y_0 = 0.$$

Similary, $(X_y(t), Y_y(t))$ may be regarded as the solution of (4.39) and (4.40) with initial condition

$$x(0) = 0,$$
$$y(\theta) = \delta(\theta)I, \; -r < \theta \leq 0.$$

With this interpretation in mind, we may write $X_y(t)$ and $Y_y(t)$ in terms of $X_x(t)$ and $Y_x(t)$. Indeed, it is easy to see that the solution of (4.46) and (4.47) in the interval $[0, r)$ is $x(t) = 0$, $y(t) = \delta(t)$. Now consider the interval $[r, 2r)$, $y(t-r)$ is zero except the impulse at $t = r$, producing a step of $B$ at time $t = r$. Therefore, solution is $x(t) = X_x(t-r)B$ and $y(t) = Y_x(t-r)B$. Continuing this process yields

$$X_y(t) = \sum_{k=0}^{\infty} D^k X_x(t - kr - r)B \tag{4.48}$$

$$= \sum_{k=0}^{[t/r]-1} D^k X_x(t - kr - r)B, \tag{4.49}$$

$$Y_y(t) = \sum_{k=0}^{\infty} \delta(t - kr)D^k + \sum_{k=0}^{\infty} D^k Y_x(t - kr - r)B \tag{4.50}$$

$$= \sum_{k=0}^{[t/r]} \delta(t - kr)D^k + \sum_{k=0}^{[t/r]-1} D^k Y_x(t - kr - r)B, \tag{4.51}$$

where $[t/r]$ represents the largest integer not to exceed $t/r$.

With the fundamental solutions, it is easy to write the general solutions of (4.39) and (4.40). Let the solution of (4.39) and (4.40) with initial conditions (4.41) and (4.42) be denoted as

$$x(t) = x(t, \psi, \phi),$$
$$y(t) = y(t, \psi, \phi).$$

Then, using linearity, it is not difficult to see that

$$x(t,\psi,\phi) = X_x(t)\psi + \int_{-r}^{0} X_y(t+\theta)\phi(\theta)d\theta, \tag{4.52}$$

$$y(t,\psi,\phi) = Y_x(t)\psi + \int_{-r}^{0} Y_y(t+\theta)\phi(\theta)d\theta. \tag{4.53}$$

Using Expressions (4.48) and (4.50), they can also be expressed as

$$x(t,\psi,\phi) = X_x(t)\psi + \int_{-r}^{0} \sum_{k=0}^{[(t+\theta)/r]-1} D^k X_x(t+\theta-kr-r)B\phi(\theta)d\theta, \tag{4.54}$$

$$\begin{aligned} y(t,\psi,\phi) &= Y_x(t)\psi + D^{[t/r]+1}\phi(t-[t/r]r-r) \\ &+ \int_{-r}^{0} \sum_{k=0}^{[(t+\theta)/r]-1} D^k Y_x(t+\theta-kr-r)B\phi(\theta)d\theta. \end{aligned} \tag{4.55}$$

It can be observed from the above discussions that, for continuous $\phi(\theta)$, $x(t)$ is continuous. However $y(t)$ is in general discontinuous. This is typical of neutral time-delay systems. Also, for the system to be stable, a necessary condition is that the spectrum radius of matrix $D$ is less than 1, $\rho(D) < 1$, another well known fact for neutral time-delay systems.

On the other hand, if $\rho(D) < 1$, then the system would be exponentially stable if and only if $X_x(t)$ and $Y_x(t)$ are exponentially bounded. Indeed, in this case, for any given $\rho(D) < \gamma < 1$, there exists a $K > 0$ such that

$$||D^k|| \leq K\gamma^k.$$

Also,

$$X_x(t) \leq Me^{-\alpha t},\ M > 0,\ \alpha > 0,$$
$$Y_x(t) \leq Ne^{-\beta t},\ N > 0,\ \beta > 0.$$

Then for any bounded initial condition

$$||\psi|| \leq L,$$
$$||\phi(\theta)|| \leq L,\ -r \leq \theta < 0,$$

we have

$$\begin{aligned}
||x(t,\psi,\phi)|| &\le MLe^{-\alpha t} + \sum_{k=0}^{[t/r]-1} K\gamma^k \int_{-r}^{0} Me^{-\alpha(t-(k-2)r)}||B||Ld\theta \\
&= MLe^{-\alpha t} + \sum_{k=0}^{[t/r]-1} KMLr||B||\gamma^k e^{-\alpha(t-(k-2)r)} \\
&= MLe^{-\alpha t} + KMLr||B||e^{-\alpha(t+2r)} \sum_{k=0}^{[t/r]-1} (\gamma e^{\alpha r})^k \\
&= MLe^{-\alpha t} + KMLr||B||e^{-\alpha(t+2r)} \frac{(\gamma e^{\alpha r})^{[t/r]} - 1}{\gamma e^{\alpha r} - 1} \\
&= MLe^{-\alpha t} + KMLr||B|| \frac{\gamma^{[t/r]} e^{\alpha(r[t/r]-t-2r)} - e^{-\alpha(t+2r)}}{\gamma e^{\alpha r} - 1} \\
&\le MLe^{-\alpha t} + KMLr||B|| \frac{\gamma^{[t/r]} e^{\alpha(r[t/r]-t-2r)} + e^{-\alpha(t+2r)}}{|\gamma e^{\alpha r} - 1|}.
\end{aligned}$$

In the above, we have assumed $\gamma e^{\alpha r} \neq 1$, which can be satisfied by properly choosing $\gamma$. Since $e^{\alpha(r[t/r]-t-2r)} < 1$, and $e^{-\alpha t}$, $\gamma^{[t/r]}$ and $e^{-\alpha(t+2r)}$ all approach zero exponentially, $||x(t,\psi,\phi)|| \to 0$ exponentially. Similarly, we can show that $||y(t,\psi,\phi)|| \to 0$ exponentially.

### 4.9.3 Lyapunov-Krasovskii functional

We will assume the system described by (4.39) and (4.40) is exponentially stable. We will construct a Lyapunov-Krasovskii functional $V(x(t), y_t)$ such that

$$\dot{V}(x(t), y_t) = -x^T(t)Wx(t), \tag{4.56}$$

for any given positive definite matrix $W$. For this purpose, one may choose

$$V(\psi,\phi) = \int_0^\infty x^T(t,\psi,\phi)Wx(t,\psi,\phi)dt. \tag{4.57}$$

In other words,

$$V(x(t), y_t) = \int_0^\infty x^T(\xi, x(t), y_t)Wx(\xi, x(t), y_t)d\xi.$$

Then, it is easily shown that

$$\begin{aligned}
\dot{V}(x(t), y_t) &= \int_0^\infty \frac{\partial}{\partial t}[x^T(\xi, x(t), y_t) W x(\xi, x(t), y_t)] d\xi \\
&= \int_0^\infty \frac{\partial}{\partial t}[x^T(\xi + t, \psi, \phi) W x(\xi + t, \psi, \phi)] d\xi \\
&= \int_0^\infty \frac{\partial}{\partial \xi}[x^T(\xi + t, \psi, \phi) W x(\xi + t, \psi, \phi)] d\xi \\
&= \int_0^\infty \frac{\partial}{\partial \xi}[x^T(\xi, x(t), y_t) W x(\xi, x(t), y_t)] d\xi \\
&= x^T(\xi, x(t), y_t) W x(\xi, x(t), y_t)|_{\xi=0}^{\xi=\infty},
\end{aligned}$$

or

$$\dot{V}(x(t), y_t) = -x^T(t) W x(t). \tag{4.58}$$

Using the general solution (4.52) and (4.53), $V(\psi, \phi)$ can be expressed in an explicit quadratic form of $(\psi, \phi)$. Indeed, using (4.52) and (4.53) in (4.57), it is easily obtained that

$$\begin{aligned}
V(\psi, \phi) = {} & \psi^T U_{xx} \psi + 2\psi^T \int_{-r}^0 U_{xy}(\eta) \phi(\eta) d\eta \\
& + \int_{-r}^0 \int_{-r}^0 \phi^T(\xi) U_{yy}(\xi, \eta) \phi(\eta) d\xi d\eta,
\end{aligned} \tag{4.59}$$

where

$$U_{xx} = \int_0^\infty X_x^T(\theta) W X_x(\theta) d\theta, \tag{4.60}$$

$$U_{xy}(\eta) = \int_0^\infty X_x^T(\theta) W X_y(\theta - \eta) d\theta, \tag{4.61}$$

$$U_{yy}(\xi, \eta) = \int_0^\infty X_y^T(\theta - \xi) W X_y(\theta - \eta) d\theta. \tag{4.62}$$

These are clearly well defined and finite since both $X_x$ and $X_y$ are exponentially decaying matrix functions. Also, it is easy to see that $U_{xx}$ is positive definite.

### 4.9.4 Further Comments

The discussions so far established the following fact.

**Proposition 4.8.** *If the system described by (4.39) and (4.40) is exponentially stable, and $\rho(D) < 1$. Then, there exists a quadratic Lyapunov-Krasovskii functional in the form of*

$$\begin{aligned} V(x(t), y_t) &= x^T(t)Px(t) + x^T \int_{-r}^{0} Q(\eta)y(t+\eta)d\eta \\ &+ \int_{-r}^{0}\int_{-r}^{0} y^T(t+\xi)R(\xi,\eta)y(t+\eta)d\xi d\eta \\ &+ \int_{-r}^{0} y^T(t+\eta)S(\eta)y(t+\eta)d\eta, \end{aligned}$$

*such that*

$$\varepsilon||x(t)||^2 \leq V(x(t), y_t) \leq M||(x(t), y_t)||^2,$$

*and*

$$\dot{V}(x(t), y_t) \leq -\varepsilon||x(t)||^2,$$

*for some* $\varepsilon > 0$ *and* $M > 0$.

The quadratic form of $V$ and its derivative makes it possible for discretization in a similar scheme as described in [9]. It should be pointed out that even for retarded time-delay systems, the above description has its advantages. First, for systems with multiple commensurate delays, while it is possible to use the scheme described in Chapter 7 of [11] to handle this case, the formulation here is much simpler and the computation would be substantially reduced. Second, in many practical cases, the delay occurs only in a limited part of the system. For a system with single delay (4.7), this means that $A_1$ has significantly lower rank than the number of states. In this case, we may write $A_1 = FG$, where $F$ has full column rank and $G$ has full row rank. Then, we can write the system as

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + Fy(t-r), \\ y(t) &= Gx(t). \end{aligned}$$

In this way, since the dimension of $y$ is significantly lower than $x$, the dimension of LMI resulted from discretization is significantly reduced.

## 4.10 Conclusions

A number of basic ideas regarding Lyapunov approach of time-delay systems are discussed. The main emphasis is on the presentation of main ideas and motivations. The readers who wish to explore further are referred to references for technical details.

Another interesting topic is dealing with uncertainties. The readers are referred to [11] and [23].

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