Could Any Graph Be Turned into a Small-World?

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1 Introduction

In the last decade, effective measurements of real interaction networks have revealed specific unexpected properties. Among these, most of these networks present a very small diameter and a high clustering. Furthermore, very short paths can be efficiently found between any pair of nodes without global knowledge of the network (i.e., in a decentralized manner) which is known as the small-world phenomenon [1]. Several models have been proposed to explain this phenomenon [2,3]. However, Kleinberg showed in [4] that these models lack the essential *navigability* property: in spite of a polylogarithmic diameter, decentralized routing requires the visit of a polynomial number of nodes in these models.

He proposed a navigable model defined as a grid augmented by additional random links according to a specific distribution. This raised an essential question to capture the small-world phenomenon: are there only specific graph metrics that can be turned into small-worlds by the addition of shortcuts? We show that the navigability property is not specific to the grid topology, and that a wide family of graphs can be turned into navigable small-worlds by the addition of one random link per node. In a second step, we try to catch the dimensional phenomenon showing that if two independent graphs can be augmented into two navigable small-worlds then their cartesian product can also be (e.g. any product of arbitrary length tori). In all the following, we consider infinite graphs, but our definitions and results apply as well to families of finite graphs (see [5]).

Turning graphs into small-worlds. In the following, an *underlying metric* δ_H of a graph G is the metric given by a spanning connected subgraph H, and $b_{H,\mathbf{u}}(r)$ is the number of nodes at distance $\leq r$ from $\mathbf{u}$ in H. Also, a routing algorithm for a graph H, augmented by random links, is said *decentralized* if it only uses δ_H and the knowledge of the nodes it has previously visited as well as their neighbors; crucially, it can only visit nodes that are neighbors of previously visited nodes. Definition 1 is inspired by the work of Kleinberg [4].

Definition 1. *A graph H augmented by random links is a* navigable small-world *if there exists a decentralized algorithm that, for any two nodes $\mathbf{u}$ and $\mathbf{v}$, computes a path, using the underlying metric δ_H, from $\mathbf{u}$ to $\mathbf{v}$ in the augmented graph by*

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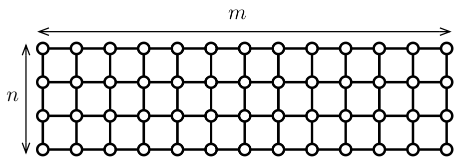
visiting an expected number of nodes bounded by a polylogarithmic function of $b_{H,\mathbf{v}}(\delta_H(\mathbf{u},\mathbf{v}))$.

Kleinberg showed in [4] that adding one random link according to a harmonic distribution whose exponent equals the dimension of the regular $n \times \cdots \times n$ grid, turns the grid into a navigable small-world. But this approach fails for instance on unbalanced $n \times m$ grids (with $m \gg n$), since the "dimension" varies with the distance: balls of small radius grow like r^2 but larger balls grow like $r^{1+\epsilon(r)}$, where $\epsilon(r) \to 0$ as r grows.



It appears that defining the random link distribution in terms of ball growth in the original base graph, rather than in terms of distance between nodes, allows to generalize Kleinberg's process to a wide class of graphs. Precisely, we say that a bounded degree infinite graph H (resp., family of finite graphs $\mathcal{H}$) is an *α-moderate growth graph* if the size of the ball centered on each node $\mathbf{u}$ of H (resp., of $H \in \mathcal{H}$) can be written as $b_{H,\mathbf{u}}(r) = r^{d_{\mathbf{u}}(r)}$, where the "apparent dimension from $\mathbf{u}$", $d_{\mathbf{u}}(r)$, verifies: $\frac{\partial d_{\mathbf{u}}(r)}{\partial r} \leq \frac{\alpha}{r \ln r}$.

This class includes all graphs with non-increasing "apparent dimension" $d_{\mathbf{u}}(r)$ (e.g., unbalanced grids). Thanks to the regularity of ball growth in this class, adding one random link of length r according to $1/(b_{H,\mathbf{u}}(r) \log^{1+\epsilon} r)$ yields navigable small-worlds. Our theorem below covers graphs whose balls grow like $r^{O(\log\log r)}$ or slower, and, in particular, all known Cayley graphs (since the existence of groups of intermediate growth, between polynomial and $e^{\sqrt{r}}$, is still open). We also show that the cartesian product of two moderate growth graphs can be turned into a small-world, even though it may not belong to the class itself. This reveals that the small-world phenomenon of a network may rely on multiple underlying structures.

Theorem 1. *Any α-moderate growth infinite graph H (resp., family of graphs $\mathcal{H}$), augmented by one long range link per node* $\mathbf{u}$*, pointing to a random node* $\mathbf{v}$ *at distance r with probability proportional to $\frac{1}{b_{H,\mathbf{u}}(r)\log^q r}$, for any $q > 1$, is a navigable small-world. any pair of nodes at distance ℓ from each other is* $O(\ln^{1+q+\alpha\ln 5} \ell)$.

Given any two moderate growth graphs, their cartesian product G augmented by one long range link per node $\mathbf{u}$*, pointing to a random node* $\mathbf{v}$ *at distance r with probability proportional to $\frac{1}{b_{G,\mathbf{u}}(r)\log^{q'} r}$, for all $q' > q_0$, for some constant $q_0 > 0$, is a navigable small-world.*

References

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