Convergence of Iterations

Paul Cull

Computer Science Dept., Oregon State University, Corvallis, OR 97331 USA
pc@cs.orst.edu

Abstract. Convergence is a central problem in both computer science and in population biology.

Will a program terminate? Will a population go to an equilibrium?

In general these questions are quite difficult – even unsolvable.

In this paper we will concentrate on very simple iterations of the form

$$x_{t+1} = f(x_t)$$

where each x_t is simply a real number and $f(x)$ is a reasonable real function with a single fixed point. For such a system, we say that an iteration is "globally stable" if it approaches the fixed point for all starting points. We will show that there is a simple method which assures global stability. Our method uses bounding of $f(x)$ by a self-inverse function. We call this bounding "enveloping" and we show that **enveloping implies global stability.** For a number of standard population models, we show that local stability implies enveloping by a self-inverse linear fractional function and hence global stability. We close with some remarks on extensions and limitations of our method.

1 Introduction

Simple population growth models have a pleasant property, they display global convergence if they have local convergence. This fact was established for a number of models by Fisher *et al* [1,2] who constructed an explicit Lyapunov function for each model they studied. Since then a number of workers have created a variety of sufficient conditions to demonstrate global stability. [3,4,5,6,7,8] Each of these methods suffer from the difficulty that either the method does not apply to one of the commonly used models or the method is computationally difficult to apply.

In this paper, we describe a simple condition which is satisfied by all the commonly used simple population models, and we show that for these models the computation for the method is not difficult. Our simple condition is that the population models are *enveloped* by *linear fractional functions.* No single linear fractional serves for all models. Instead the linear fractionals depend on a single parameter which must be adjusted for the particular model. In some cases, this parameter will also change depending on the parameters of the model. This parameter dependence may be why this simple condition has not been discovered before.

R. Moreno Díaz et al. (Eds.): EUROCAST 2005, LNCS 3643, pp. 457–466, 2005.

Our pleasure with this result is not solely mathematical. There is also a psychological component. We suspect that the original creators of these population models were good biologists and not sophisticated mathematicians. If the similarity among these models required deep and complicated mathematics, we would feel that we had not captured the simple vision of the original modelers. We will argue that the usual way of writing these models suggests an implicit constraint that will force enveloping by a linear fractional.

2 Background and Definitions

In the most general sense, we want to study difference equations of the form

$$x_{t+1} = f(x_t)$$

but with this degree of generality, little can be said. If we require that f is a function which is defined for all values of x, then given an initial condition x_0, we can show that there is a unique solution to the difference equation, that is, x_t traces out a well-defined trajectory. To obtain stronger results, we will assume that f is continuous and has as many continuous derivatives as necessary. As we will see in the examples, we will assume even more structure for a population model. Intuitively, if there is no population now, there will be no population later. If the population is small, we expect it to be growing. If the population is large, we expect it to be decreasing. These ideas suggest that there should be an *equilibrium point* where the population size will remain constant. We expect the function f to be *single-humped*, that is, f should rise to a maximum and then decrease. For some models, f will go to 0 for some finite x, but for other models f will continually decrease toward 0.

We want to know what will happen to x_t for large values of t. Clearly we expect that if x_0 is near $\overline{x}$ then x_t will overshoot and undershoot $\overline{x}$. Possibly this oscillation will be sustained, or possibly x_t will settle down at $\overline{x}$. The next definitions codify these ideas. A population model is ***globally stable*** if and only if for all x_0 such that $f(x_0) > 0$ we have

$$\lim_{t\to\infty} x_t = \overline{x}$$

where $\overline{x}$ is the unique equilibrium point of $x_{t+1} = f(x_t)$. A population model is ***locally stable*** if and only if for every small enough neighborhood of $\overline{x}$ if x_0 is in this neighborhood, then x_t is in this neighborhood for all t, and

$$\lim_{t\to\infty} x_t = \overline{x}.$$

How can we decide if a model has one of these properties? The following well-known theorem gives one answer.

Theorem 1. *If $f(x)$ is differentiable then, a population model is locally stable if $|f'(\overline{x})| < 1$, and if the model is locally stable then $|f'(\overline{x})| \leq 1$.*

For global stability, a slight modification of a very general theorem of Sarkovskii [9] gives:

Theorem 2. *A continuous population model is globally stable iff it has no cycle of period 2. (That is, there is no point except $\overline{x}$ such that $f(f(x)) = x$.)*

This theorem has been noted by Cull[7] and Rosenkranz[4].

Unfortunately, this global stability condition may be difficult to test. Further, there is no obvious connection between the local and global stability conditions.

Various authors have demonstrated global stability for some population models. Fisher *et al* [1] and Goh [2] used Lyapunov functions [10] to show global stability. This technique suffers from the drawbacks that a different Lyapunov function is needed for each model and that there is no systematic method to find these functions. Singer [3] used the negativity of the Schwarzian to show global stability. This technique does not cover all the models we will consider, and it even requires modification to cover all the models it was claimed to cover. Rosenkranz [4] noted that no period 2 was implied by $|f'(x)f'(f(x))| < 1$ and showed that this condition held for a population genetics model. This condition seems to be difficult to test for the models we will consider. Cull [7,6,5,8] developed two conditions **A** and **B** and showed that each of the models we will consider satisfied at least one of these conditions. These conditions used the first through third derivatives and so were difficult to apply. Also, as Hwang [11] pointed out these conditions required continuous differentiability. All of these methods are relatively mathematically sophisticated, and so it is not clear how biological modelers could intuitively see that these conditions were satisfied.

If we return to the condition for local stability, we see that it says if for x slightly less than 1, $f(x)$ is below a straight line with slope -1, and if for x slightly greater than 1, $f(x)$ is above the same straight line, then the model is locally stable. If we consider the model

$$x_{t+1} = x_t e^{2(1-x_t)},$$

we can see that the local stability bounding line is $2 - x$. Somewhat suprisingly, this line is an upper bound on $f(x)$ for all x in $[0, 1)$ and a lower bound for all $x > 1$. (See Figure 1a). Since $2 - (2 - x) = x$, the bounding by this line can be used to argue that for this model there are no points of period 2, and hence the model is globally stable. From this example, we abstract the following definition. A function $\phi(x)$ ***envelops*** a function $f(x)$ if and only if

$$\begin{array}{llll} \phi(x) > f(x) & \text{for} & x \in (0,1) & \\ \phi(x) < f(x) & \text{for} & x > 1 & \text{such that} \quad \phi(x) > 0 \quad \text{and} \quad f(x) > 0 \end{array}$$

We will use the notation $\phi(x) \bowtie f(x)$ to symbolize this enveloping.

As we will see, our example population models have one or more parameters, and a model with one choice of parameters will envelop the same model with a different choice of parameters. For example, the function $xe^{2(1-x)}$ envelops all the functions of the form $xe^{r(1-x)}$ for $r \in (0, 2)$.

While a straight line was sufficient to envelop $xe^{2(1-x)}$, a straight line fails to envelop the closely related function $x[1 + 2(1 - x)]$. To get a more general enveloping function, we consider the ratio of two linear functions and assume

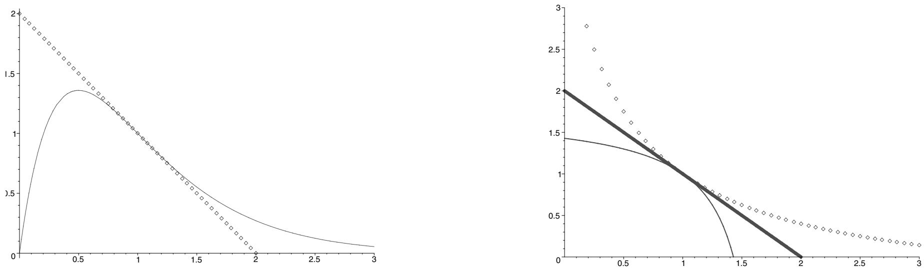

Fig. 1. (a) The function $xe^{2(1-x)}$ is enveloped by the straight line $2-x$ which is the linear fractional with $\alpha = 1/2$. (See Model I in Section 4.). (b) Three types of linear fractionals. Dotted line $\alpha = 1/4$. Heavy line $\alpha = 1/2$. Light line $\alpha = .7$.

that the ratio is 1 when $x = 1$ and the derivative of this function is -1 when $x = 1$, which gives the following definition.

A ***linear fractional function*** is a function of the form

$$\phi(x) = \frac{1-\alpha x}{\alpha - (2\alpha - 1)x} \qquad \text{where } \alpha \in [0, 1)\ .$$

These functions have the properties

$$\begin{aligned} \phi(1) &= 1 \\ \phi'(1) &= -1 \\ \phi(\phi(x)) &= x \\ \phi'(x) &< 0. \end{aligned}$$

The shape of our linear fractional functions changes markedly as α varies. For $\alpha = 0$, $\phi(x) = 1/x$, which has a pole at $x = 0$, and decreases with an always positive second derivative. For $\alpha \in (0, 1/2)$, $\phi(x)$ starts (for $x = 0$) at $1/\alpha$ and decreases with a positive second derivative. For $\alpha = 1/2$, $\phi(x) = 2 - x$, which starts at 2 and decreases to 0 with a zero second derivative. For $\alpha \in (1/2, 1)$, $\phi(x)$ starts at $1/\alpha$, decreases with a negative second derivative, and hits 0 at $1/\alpha$ which is greater than 1. We are only interested in these functions when $x > 0$ and $\phi(x) > 0$, so we do not care about the pole in these linear fractionals because the pole occurs outside the area of interest. Figure 1b shows the three different shapes of linear fractional functions.

3 Theorems

We are now in a position to state the necessary theorems. In what follows, we will assume that our model is $x_{t+1} = f(x_t)$, and that the model has been normalized so that the equilibrium point is 1, that is $f(1) = 1$. We will use the notation $f^{(k)}(x)$ to mean that the function f has been applied k times to x. This notation can be recursively defined by $f^{(0)}(x) = x$ and $f^{(i)}(x) = f(f^{(i-1)}(x))$ for $i \geq 1$.

Theorem 3. *Let $\phi(x)$ be a monotone decreasing function which is positive on $(0, x_-)$ and so that $\phi(\phi(x)) = x$. Assume that $f(x)$ is a continuous function such that:*

$$\begin{aligned} \phi(x) > f(x) \quad &on \quad (0,1) \\ \phi(x) < f(x) \quad &on \quad (1, x_-) \\ f(x) > x \quad &on \quad (0,1) \\ f(x) < x \quad &on \quad (1,\infty) \\ f(x) > 0 \quad &on \quad (1, x_\infty) \end{aligned}$$

then for all $x \in (0, x_\infty)$, $\lim_{k\to\infty} f^{(k)}(x) = 1$.

A slight recasting of the above gives:

Corollary 1. *If $f_1(x)$ is enveloped by $f_2(x)$, and $f_2(x)$ is globally stable, then $f_1(x)$ is globally stable.*

Corollary 2. *If $f(x)$ is enveloped by a linear fractional function then $f(x)$ is globally stable.*

A function $h(z)$ is ***doubly positive*** iff

1. $h(z)$ has a power series $\sum_{i=0}^{\infty} h_i z^i$
2. $h_0 = 1$, $h_1 = 2$
3. For all $n \geq 1 \quad h_n \geq h_{n+1}$
4. For all $n \geq 2 \quad h_n - 2h_{n+1} + h_{n+2} \geq 0$

Theorem 4. *Let $x_{t+1} = f(x_t)$ where $f(x) = xh(1-x)$ and $h(z)$ is doubly positive, then $f(x)$ is enveloped by the linear fractional function*

$$\phi(x) = \frac{1 - \alpha x}{\alpha + (1 - 2\alpha)x}$$

where $\alpha = \frac{3-h_2}{4-h_2} \geq \frac{1}{2}$ and the model $x_{t+1} = f(x_t)$ is globally stable.

While this doubly positive condition will be sufficient for a number of models, it is not sufficient for all the examples. The following observation will be useful in many cases.

Observation 1. *Let $\phi(x) = A(x)/B(x)$, $f(x) = C(x)/D(x)$ and $G(x) = A(x)D(x) - B(x)C(x)$. If $G(1) = 0$, $G'(1) = 0$, and $G''(x) > 0$ on $(0,1)$ and $G''(x) < 0$ for $x > 1$, then $\phi(x)$ envelops $f(x)$. (We are implicitly assuming that A, B, C, D are all positive, and all functions are twice continuously differentiable.)*

4 Simple Models of Population Growth

In this section we will apply the techniques of the previous section to 7 models from the literature. Models I, II, III, IV all turn out to be doubly positive and so we just give the model and the enveloping fractional.

Model I: The model $x_{t+1} = x_t e^{r(1-x_t)}$ is widely used (see, for example [12–14]). This model with $r = 2$ is enveloped by $\phi(x) = 2 - x$ and hence local and global stability coincide.
Model II: The model $x_{t+1} = x_t[1+r(1-x_t)]$ is widely used [12] and is sometimes considered to be a truncation of Model I. The enveloping function has $\alpha = \frac{3}{4}$ and is

$$\phi(x) = \frac{4-3x}{3-2x}.$$

Model III: The model $x_{t+1} = x_t[1 - r \ln x_t]$ is attributed to Gompertz and studied by Nobile *et al*[13]. As with the preceding two models $0 < r \leq 2$ is the necessary condition for local stability. The enveloping function has $\alpha = 2/3$ and is $\phi(x) = \frac{3-2x}{2-x}$.
Model IV: This model from [14] is

$$x_{t+1} = x_t(\frac{1}{b+cx_t} - d).$$

It differs from the previous three in that there are two parameters, b and d, remaining after the carrying capacity has been normalized to 1. The enveloping function is

$$\phi(x) = \frac{4d-(3d-1)x}{3d-1+2(1-d)x}.$$

We note that $\phi(x)$ has a pole, but $\phi(x)$ goes to zero before the pole, so we can simply ignore the pole. Of course, we only need $\phi(x)$ to bound $f(x)$ on the interval $(0, \frac{4d}{3d-1})$ where $\phi(x)$ is positive.
Model V: This model from Pennycuick *et al* [15] has

$$f(x) = \frac{(1+ae^b)x}{1+ae^{bx}}.$$

This and the following two model are more complicated than the previous models because we have to consider different enveloping functions for different parameter ranges. For $b \leq 2$, $xe^{b(1-x)}$ envelops $f(x)$. But $xe^{b(1-x)}$ is just Model I, and as we showed it is enveloped by $2 - x$.

For larger values of b, we use $a(b-2)e^b = 2$ from local stablity, and show that the enveloping linear fractional is

$$\phi(x) = \frac{b-(b-1)x}{(b-1)-(b-2)x}$$

by using the Observation.
Model VI: Model VI is from Hassel [16] and has

$$f(x) = \frac{(1+a)^b x}{(1+ax)^b} \qquad \text{with } a > 0,\ b > 0.$$

There are two cases to consider $0 < b \leq 2$ and $b > 2$. The enveloping function for $b \leq 2$ is $\phi(x) = 1/x$. Cross multiplication shows that we want $(1+ax)^b \bowtie$

$(1+a)^b x^2$. Taking b^{th} roots and rearranging shows that we want $1-x+ax(1-x^{\frac{2-b}{b}}) \bowtie 0$. Clearly, each of the two terms is positive (nonnegative) below 1 and negative (nonpositive) above 1, and so enveloping is established. For $b > 2$, we need to use the Obsevation to establish enveloping.

Model VII: Model VII is due to Maynard Smith [17] and has

$$f(x) = \frac{rx}{1+(r-1)x^c}.$$

This seems to be the hardest to analyze model in our set of examples. For example, this model does not satisfy the Schwarzian derivative condition or Cull's condition **A**. Even for our enveloping analysis, we will need to consider this model as three subcases.

Similar to previous models, local stability implies $r(c-2) \leq c$, and it is easy to show that this model with smaller values of r is enveloped by this model with larger values of r. For $c > 2$, we use $r = \frac{c}{c-2}$, and

$$\phi(x) = \frac{c-1-(c-2)x}{c-2-(c-3)x}.$$

For $c > 3$, the Observation shows enveloping, but for $c \in (2,3)$ consideration of the third derivative is needed to show enveloping.

5 Enveloping by a Linear Fractional Is Only Sufficient

Here we want to give a simple model which has global stability, but cannot be enveloped by any linear fractional. Define $f(x)$ by

$$f(x) = \begin{cases} 6x & 0 \leq x < 1/2 \\ 7-8x & 1/2 \leq x < 3/4 \\ 1 & 3/4 \leq x. \end{cases}$$

then $x_{t+1} = f(x_t)$ has $x = 1$ as its globally stable equilibrium point because if $x_t \geq 1$ then $x_{t+1} = 1$, for $x_t \in [1/2, 1)$, $x_{t+1} > 1$ and $x_{t+2} = 1$, and for $x_t \in (0, 1/2)$, the subsequent iterates grow by multiples of 6 and eventually surpass 1/2. This $f(x)$ cannot be enveloped by a linear fractional because $f(1/2) = 3$ which implies that the linear fractional would have $\alpha \leq -1$ and hence have a pole in $(0,1)$ and thus it could not envelop a positive function. On the other hand, the self-inverse function

$$\phi(x) \;=\; \begin{cases} 5-4x & x \leq 1 \\ (5-x)/4 & x > 1 \end{cases}$$

does envelop $f(x)$ and so demonstrates global stability.

6 Extensions

The previous sections have worked with the usual applied math assumption that real phenomena are as smooth and as differentiable as necessary to get a good theorem or estimate. Of course, everyone who has ever applied mathematics knows that this assumption is false, but they also know that it serves as a useful "rule of thumb." That is, in some cases the smoothness assumption may lead to bad estimates, but in many, many cases the smooth estimate is very close to observed (experimental) values. In a few cases, a result which was initially proved assuming smoothness has been shown to hold when some of the smoothness assumptions are dropped. Here we want to mention that **enveloping implies global stability** does *not* require continuity, even though we originally assumed continuity. Further, the assumption that x_{t+1} is a *function* of x_t is also superfluous. The enveloping result will also hold for *multi-functions*, which are mappings in which $f(x)$ may return any one of several values or any value within some range. Discontinuous functions can have points y so that for some x_0's, $\lim_{k\to\infty} f^{(k)}(x_0) = y$, but $f(y) \neq y$. We call such y's *limiting points.* To apply our theorem, one must show that no such limiting points exist within the range of interest.

6.1 General Theorem

Although our enveloping method was devised for the population models discussed above, the method can also be applied to other iterations. Not all iterations are normalized so that the fixed point is at $x = 1$. In many cases, the iteration is designed to compute the fixed point.

Theorem 5. *If the iteration $x_{t+1} = f(x_t)$ obeys*

$$\begin{array}{lcl} f(x) > x & \text{on} & (a\,,\,p) \\ f(x) < x & \text{on} & (p\,,\,b) \end{array}$$

where $f(x)$ may be a discontinuous multifunction but has p as its only fixed point or limiting point in $(a\,,\,b)$, and if there is a self-inverse function $\phi(x)$ so that

$$\begin{array}{lcl} \phi(x) > f(x) & \text{on} & (a\,,\,p) \\ f(x) > \phi(x) & \text{on} & (p\,,\,b) \end{array}$$

then $\lim_{k\to\infty} f^{(k)}(x_0) = p$ for every $x_0 \in (a\,,\,b)$.

6.2 Some Newton Iterations

For our examples, we'll consider the Newton iterations for square root and for reciprocal. As is well known [18], $\sqrt{A}$ can be computed by the iteration

$$x_{t+1} = \frac{x_t^2 + A}{2x_t}.$$

Clearly this iteration has $\sqrt{A}$ as its sole fixed point on $(0\,,\,\infty)$ and the continuous function obeys

$$\begin{array}{lll} f(x) \;>\; x & \text{on} & (0\,,\,\sqrt{A}) \\ f(x) \;<\; x & \text{on} & (\sqrt{A}\,,\,\infty). \end{array}$$

We take $\phi(x) \;=\; A/x$ and its easy to check that $\phi(x)$ does envelop $f(x)$ on $(0\,,\,\infty)$. So we conclude that for any $x_0 \;\in\; (0\,,\,\infty)$, this Newton iteration will converge to $\sqrt{A}$.

For a slightly more complicated example, we use the well known [18] iteration

$$x_{t+1} \;=\; x_t(2 - Ax_t).$$

to compute $1/A$. Here $f(x)$ has $1/A$ as its sole fixed point in $(0\,,\,2/A)$. Notice that $f(0) = f(2/A) = 0$ so this iteration will not converge to $1/A$ when it is started at either of these fixed points. We can take the straight line $\phi(x) \;=\; 2/A \;-\; x$ and show that this $\phi(x)$ does envelop $f(x)$ on $(0\,,\,2/A)$ and hence that this iteration converges to $1/A$ when started at any point within $(0\,,\,2/A)$.

7 Conclusion

Enveloping is a simple technique to demonstrate global stability for some one-dimensional difference equations. Enveloping was introduced by Cull and Chaffee [19,20,21]. We demonstrated that the usual population models can be enveloped by linear fractional functions. Such enveloping seems to capture the idea of *simple* function in that a "free-hand" drawing of a population model can usually be enveloped by a linear fractional. (Cull [22] gives a discussion of dynamical systems defined by linear fractionals.) As we showed by example, enveloping by a linear fractional is only a *sufficient* condition for global stability. The simplest population models which have local stability without global stability are discussed by Singer [3] and by Cull [5]. While most of the examples in this paper are all one-humped population models, **enveloping implies global stability** also holds for functions with multiple peaks, for discontinuous functions, and even for multi-functions.

References

1. Fisher, M., Goh, B., Vincent, T.: Some Stability Conditions for Discrete-time Single Species Models. Bulletin of Mathematical Biology **41** (1979) 861–875
2. Goh, B.S.: Management and Analysis of Biological Populations. Elsevier, New York (1979)
3. Singer, D.: Stable Orbits and Bifurcation of Maps of the Interval. SIAM Journal on Applied Mathematics **35** (1978) 260–267
4. Rosenkranz, G.: On Global Stability of Discrete Population Models. Mathematical Biosciences **64** (1983) 227–231

5. Cull, P.: Stability of Discrete One-dimensional Population Models. Bulletin of Mathematical Biology **50** (1988) 67–75
6. Cull, P.: Local andGlobal Stability for Population Models. Biological Cybernetics **54** (1986) 141–149
7. Cull, P.: Global Stability of Population Models. Bulletin of Mathematical Biology **43** (1981) 47–58
8. Cull, P.: Local and Global Stability of Discrete One-dimensional Population Models. In Ricciardi, L.M., ed.: Biomathematics and Related Computational Problems. Kluwer, Dordrecht (1988) 271–278
9. Sarkovskii, A.: Coexistence of Cycles of a Continuous Map of a Line to Itself. Ukr. Mat. Z. **16** (1964) 61–71
10. LaSalle, J.: The Stability of Dynamical Systems. SIAM, Philadelphia (1976)
11. Huang, Y.: A Counterexample for P. Cull's Theorem. Kexue Tongbao **31** (1986) 1002–1003
12. Smith, J.: Mathematical Ideas in Biology. Cambridge University Press, Cambridge (1968)
13. Nobile, A., Ricciardi, L., Sacerdote, L.: On Gompertz Growth Model and Related Difference Equations. Biological Cybernetics **42** (1982) 221–229
14. Utida, S.: Population Fluctuation, an Experimental and Theoretical Approach. Cold Spring Harbor Symposium on Quantitative Biology **22** (1957) 139–151
15. Pennycuick, C., Compton, R., Beckingham, L.: A Computer Model for Simulating theGrowth of a Population, or of Two Interacting Populations. Journal of Theoretical Biology **18** (1968) 316–329
16. Hassel, M.: Density Dependence in Single Species Populations. Journal of Animal Ecology **44** (1974) 283–296
17. Smith, J.: Models in Ecology. Cambridge University Press, Cambridge (1974)
18. Cull, P., Flahive, M., Robson, R.: Difference Equations: from Rabbits to Chaos. Springer, New York (2005)
19. Cull, P., Chaffee, J.: Stability in simple population models. In Trappl, R., ed.: Cybernetics and Systems 2000. Austrian Society for Cybernetics Studies (2000) 289–294
20. Cull, P., Chaffee, J.: Stability in discrete population models. In Dubois, D.M., ed.: Computing Anticipatory Systems: CASYS'99. Conference Proceedings 517, American Institute of Physics, Woodbury, NY (2000) 263–275
21. Cull, P.: Stability in one-dimensional models. Scientiae Mathematicae Japonicae **58** (2003) 349–357
22. Cull, P.: Linear Fractionals - Simple Models with Chaotic-like Behavior. In Dubois, D.M., ed.: Computing Anticipatory Systems:CASYS 2001 - Fifth International Conference, Conference Proceedings 627. American Institue of Physics, Woodbury, N.Y. (2002) 170–181