Tree-Structured Legendre Multi-wavelets

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Abstract. We address the problem of constructing multi-wavelets, that is, wavelets with more than one scaling and wavelet function. We generalize the algorithm, proposed by Alpert [1] for generating discrete Legendre multi-wavelets to the case of arbitrary, non-dyadic time interval splitting.

1 Introduction

In the past two decades there has been a considerable interest to the wavelet analysis, a tool that emerged from mathematics and was quickly adopted by diverse fields of science and engineering [7]. Wavelets are being applied to a wide and growing range of applications such as signal processing, data and image compression, solution of partial differential equations, and statistics.

Recently, multi-wavelets, that is, wavelets with more than one scaling and wavelet function have gained a considerable interest in the area of signal processing. Promising results have been obtained with multi-wavelets in signal de-noising and compression [8].

The first multi-wavelets were introduced by Alpert [1], who constructed Legendre type of wavelets on the interval [0,1) with several scaling functions $\phi_0, \phi_1, \ldots, \phi_{N-1}$, instead of just one scaling function ϕ_0. This difference enabled high-order approximation with basis functions supported on non-overlapping intervals. In the particular case of $N = 1$, Alpert's multi-wavelets coincide with the Haar basis.

The discrete analogue of continuous Legendre multi-wavelets was also introduced by Alpert [1]. The structure of this analogue is essentially similar to that of continuous multi-wavelet bases, but the discrete construction is more convenient when the representation of a function (and its related operations) is based on its values at a finite set of points [1].

In [3], a new basis, called the tree-structured Haar (TSH) basis was introduced. It is a generalization of the classical Haar basis to arbitrary time and scale splitting. The idea behind such a construction is to adapt the basis to the signal on hand. Discrete orthogonal TSH transform has been successfully applied to the problem of de-noising signals [5].

In [4], the TSH structure has been extended to the multi-wavelet bases by an analog to Alpert's construction. The construction of the basis functions assured their orthogonality in the continuous space. However, the question regarding the construction of discrete orthogonal bases has been left unexplored. In this

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paper we address the problem of constructing the discrete counterpart of tree-structured multi-wavelets, so called, tree-structured Legendre (TSL) transform, by generalizing the construction procedure, that has been introduced in [1].

2 Wavelet Fundamentals

The key concept behind the wavelet theory is *multiresolution analysis* (MRA). A multiresolution analysis in $L^2(R)$ is given by a nested sequence of subspaces:

$$\begin{array}{c} \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots, \\ \longleftarrow Coarser \quad Finer \longrightarrow \end{array} \tag{1}$$

such that $clos_{L^2}(\bigcup_{j\in Z} V_j) = L^2(R)$ (completeness), $\bigcap_{j\in Z} V_j = \{0\}$ (uniqueness), and $f(t) \in V_j \leftrightarrow f(2t) \in V_{j+1}, j \in Z$ (scaling property).

It can be shown that the following relations hold for MRA. There exists a function $\phi(t)$:

$$\phi(t) = \sum_{k=-\infty}^{\infty} p_k \phi(2t-k), \tag{2}$$

called *scaling function*, such that $V_j =$ linear span $\{\phi_{j,k}\};\ j,k \in Z$, where

$$\phi_{j,k}(t) = \sqrt{2^j}\phi(2^j t - k);\ j,k \in Z \tag{3}$$

are dilated and translated versions of $\phi(t)$ (we refer to Mallat [6] for detailed explanation).

Given a set of nested subspaces V_j, there exists a set of subspaces W_j, such that

$$V_{j+1} = V_j \bigoplus W_j,\ W_{j+1} \perp W_{j'},\ \text{ if } j \neq j';\ j,j' \in Z. \tag{4}$$

These subspaces give an orthogonal decomposition of $L^2(R)$, namely

$$L^2(R) = \bigoplus_{j\in Z} W_j. \tag{5}$$

Moreover, W_j's inherit the scaling property from Vj: $f(t) \in W_j \leftrightarrow f(2t) \in W_{j+1};\ j \in Z$. For a scaling function ϕ in V_0, there exists its counterpart ψ in W_0, called *wavelet*, such that $\{\psi_{j,k}\}$ generate W_j, where

$$\psi_{j,k}(t) = \sqrt{2^j}\psi(2^j t - k); j,k \in Z. \tag{6}$$

Since $\psi \in W_0 \subset V_1$, it can be written in terms of $\phi(2t-k)$. A pair

$$\begin{cases} \phi(t) = \sum_{k=-\infty}^{\infty} p_k \phi(2t-k), \\ \psi(t) = \sum_{k=-\infty}^{\infty} q_k \phi(2t-k) \end{cases} \tag{7}$$

is called *two-scale relations*.

3 Continuous Legendre Multi-wavelets

We now describe briefly the Alpert's design of Legendre type of multi-wavelets on the interval [0,1). The two-scale relations for N *Legendre scaling functions* of order $(N-1)$, $\phi_0, \phi_1, \ldots, \phi_{N-1}$, are defined as

$$\phi_i(t) = \sum_{j=0}^{N-1} p_{i,j}\phi_j(2t) + \sum_{j=0}^{m} p_{i,N+j}\phi_j(2t-1);\ i \in \{0, 1, \ldots, N-1\}. \quad (8)$$

Here as well as below we assume $t \in [0, 1)$. The $i-th$ scaling function, ϕ_i, is an $i-th$ order polynomial, and all ϕ_i's form an orthonormal basis, that is,

$$\phi_i(t) = \sum_{k=0}^{i} a_{i,k}x^k;\ i \in \{0, 1, \ldots, N-1\}, \quad (9)$$

where

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_k(t)dt = \delta_{i,k};\ i, k \in \{0, 1, \ldots, N-1\}. \quad (10)$$

The coefficients $p_{i,j}$ in (8) are determined uniquely from the conditions (9),(10).

Remarks:

1. Since $\phi_i(t)$ is the $i-th$ order polynomial, if follows that $p_{i,j} = p_{i,N+j} = 0$, for $i < j$.
2. The two-scale relations for the Legendre scaling function of order $n < N-1$ is a subset of the first n two-scale relations for ϕ_i for $i \in \{0, 1, \ldots, n\}$ from the $(N-1)-th$ order two-scale relations.

The two-scale relations for the $(N-1)-th$ order Legendre wavelets are in the form:

$$\psi_i(t) = \sum_{j=0}^{N-1} q_{i,j}\phi_j(2t) + \sum_{j=0}^{N-1} q_{i,N+j}\phi_j(2t-1);\ i \in \{0, 1, \ldots, N-1\}. \quad (11)$$

Since there are $2N^2$ unknown coefficients $q_{i,j}$ in (11), we need $2N^2$ independent conditions to determine the two-scale relations. Among many possible choices for these conditions that would determine different wavelets, the orthonormality and vanishing moment conditions were selected by Alpert [2]:

$$\int_{-\infty}^{\infty} \psi_i(t)\psi_k(t)dt = \delta_{i,k};\ i, k \in \{0, 1, \ldots, N-1\}, \quad (12)$$

$$\int_{-\infty}^{\infty} \psi_i(t)t^j dt = 0,\ i \in \{0, 1, \ldots, N-1\}; j \in \{0, 1, \ldots, i+N-1\}. \quad (13)$$

4 Continuous Legendre Multi-wavelets: Non-dyadic Interval Splitting

In [4], the concept of the dyadic two-scale relations between Legenrde scaling functions and wavelets has been generalized to two-scale relations having an arbitrary interval-splitting point, namely, α. The two-scale relations for the $(N-1)$-th order non-dyadic Legendre scaling functions and wavelets are:

$$\phi_i(t) = \sum_{j=0}^{N-1} p_{i,j}\phi_j(\alpha t) + \sum_{j=0}^{N-1} p_{i,N+j}\phi_j(\tfrac{\alpha t-1}{\alpha-1}), \quad (14)$$

$$\psi_i(t) = \sum_{j=0}^{N-1} q_{i,j}\psi_j(\alpha t) + \sum_{j=0}^{N-1} q_{i,N+j}\phi_j(\tfrac{\alpha t-1}{\alpha-1}), \tag{15}$$

for $i \in \{0, 1, \ldots, N-1\}$, $t \in [0, 1)$. The unknown coefficients $p_{i,j}$ and $q_{i,j}$ are uniquely defined based on the same considerations, as for the dyadic Legendre multi-wavelets.

It has been mentioned in [4], that by the construction (14), (15), there is no more scale invariance. No integer shifts at the same scale (like in the dyadic case) form the subspaces V_j. In other words, the nested subspaces cannot anymore be indexed by their scale. However, the functions $\phi_j(\alpha t)$ and $\phi_j(\frac{\alpha t-1}{\alpha-1})$ are orthogonal and hence form a basis for V_0 and for its orthogonal complement, W_0 [4].

In the particular case of $\alpha = 2$, the new construction coincides with dyadic Legendre multi-wavelets.

Example 1. Linear non-dyadic Legendre multi-wavelets. If $N = 2$, the two-scale relations for scaling functions and wavelets take the form:

$$\begin{pmatrix} \phi_0(t) \\ \phi_1(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -\frac{\sqrt{3}(\alpha-1)}{\alpha} & \frac{1}{\alpha} & \frac{\sqrt{3}}{\alpha} & \frac{\alpha-1}{\alpha} \end{pmatrix} \begin{pmatrix} \phi_0(\alpha t) \\ \phi_1(\alpha t) \\ \phi_0(\frac{\alpha t-1}{\alpha-1}) \\ \phi_1(\frac{\alpha t-1}{\alpha-1}) \end{pmatrix}, \tag{16}$$

$$\begin{pmatrix} \psi_0(t) \\ \psi_1(t) \end{pmatrix} = \begin{pmatrix} \frac{(\alpha-2)\sqrt{\alpha-1}}{\sqrt{\alpha^2-\alpha+1}} & \frac{-\sqrt{3}\sqrt{\alpha-1}}{\sqrt{\alpha^2-\alpha+1}} & \frac{(\alpha-2)\sqrt{\alpha-1}}{(\alpha-1)\sqrt{\alpha^2-\alpha+1}} & \frac{\sqrt{3}\sqrt{\alpha-1}}{\sqrt{\alpha^2-\alpha+1}} \\ \frac{\sqrt{3}(\alpha-1)\sqrt{\alpha-1}}{\alpha\sqrt{\alpha^2-\alpha+1}} & \frac{(\alpha+1)(\alpha-1)\sqrt{\alpha-1}}{\alpha\sqrt{\alpha^2-\alpha+1}} & \frac{-\sqrt{3}\sqrt{\alpha-1}}{\alpha\sqrt{\alpha^2-\alpha+1}} & \frac{(2\alpha-2)\sqrt{\alpha-1}}{\alpha(\alpha-1)\sqrt{\alpha^2-\alpha+1}} \end{pmatrix} \times$$
$$\times \begin{pmatrix} \phi_0(\alpha t) \\ \phi_1(\alpha t) \\ \phi_0(\frac{\alpha t-1}{\alpha-1}) \\ \phi_1(\frac{\alpha t-1}{\alpha-1}) \end{pmatrix}. \tag{17}$$

If $\alpha = 2$, (16), (17) will be identical to the linear two-scale relations for dyadic Alpert's multi-wavelets.

The wavelets ψ_0 and ψ_1 for the case of $\alpha = 3, N = 2$ are presented in Fig. 1.

5 Splitting Point Selection

For general non-dyadic splitting, the scaling parameter, α, is selected according to so called *binary interval splitting tree* (BIST), that has been introduced in [3]. BIST is a binary tree, where each nonterminal node has either two children (splitting node), or a single child (non-splitting node). The root of the tree is associated to the entire interval $[0, 1)$. The outedges are labelled as below:

1. if the node is splitting, its left outedge is labelled by $\lambda = 0$, and its right outedge is labelled by $\lambda = 1$;
2. if the node is non-splitting, its only outedge is labelled by $\lambda = 2$.

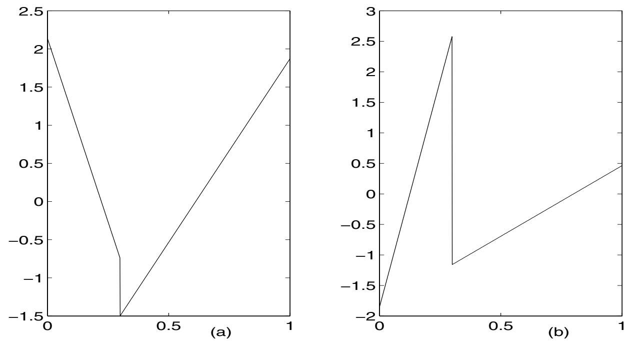

Fig. 1. Non-symmetric Legendre multi-wavelets for $\alpha = 3$: a) wavelet $\psi_0(t)$, b) wavelet $\psi_1(t)$

Each node, a, is indexed by a ternary vector $(\lambda_1(a), \lambda_2(a), \ldots, \lambda_k(a))$, where λ_j are the outedge labels from the root to this node, and $depth(a) = k$. Additionally, the node a is labelled by the number l of leaves of the subtree rooted in that node (all leaves are labelled with 1). The described tree determines splitting of $[0, 1)$ interval into subintervals, each defined by a node of the tree:

1. assign $I_{root} = [0, 1)$;
2. if $I_{\lambda_1,\lambda_2,\ldots,\lambda_k} = [0, 1)$ for a node $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, then

$$\begin{cases} I_{\lambda_1,\lambda_2,\ldots,\lambda_k,0} = [c, c + \frac{l_{\lambda_1,\lambda_2,\ldots,\lambda_k}}{l_{\lambda_1,\lambda_2,\ldots,\lambda_k,0}}(d - c)), \\ I_{\lambda_1,\lambda_2,\ldots,\lambda_k,1} = [c + \frac{l_{\lambda_1,\lambda_2,\ldots,\lambda_k}}{l_{\lambda_1,\lambda_2,\ldots,\lambda_k,0}}(d - c), d), \end{cases} \tag{18}$$

   if $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a splitting node; and

$$I_{\lambda_1,\lambda_2,\ldots,\lambda_k,2} = I_{\lambda_1,\lambda_2,\ldots,\lambda_k}, \tag{19}$$

   if $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a non-splitting node.

A BIST is illustrated in Fig. 2. The parameter α determines the interval splitting ratio:

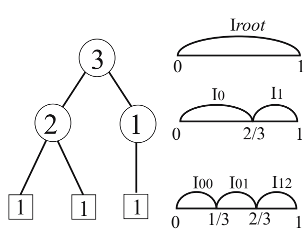

Fig. 2. Binary interval splitting tree

$$\begin{cases} \frac{|I_{\lambda_1,\lambda_2,\ldots,\lambda_k}|}{|I_{\lambda_1,\lambda_2,\ldots,\lambda_k,0}|} = \frac{l_{\lambda_1,\lambda_2,\ldots,\lambda_k}}{l_{\lambda_1,\lambda_2,\ldots,\lambda_k,0}} = \alpha, \\ \frac{|I_{\lambda_1,\lambda_2,\ldots,\lambda_k}|}{|I_{\lambda_1,\lambda_2,\ldots,\lambda_k,1}|} = \frac{l_{\lambda_1,\lambda_2,\ldots,\lambda_k}}{l_{\lambda_1,\lambda_2,\ldots,\lambda_k,1}} = \frac{\alpha}{\alpha-1}. \end{cases} \quad (20)$$

Given such a rule for interval splitting, a multi-wavelet basis with non-dyadic splitting points can be generated. For each splitting node, two sets of functions, $\phi_0, \phi_1, \ldots, \phi_{N-1}$ and $\psi_0, \psi_1, \ldots, \psi_{N-1}$ can be assigned according to α, defined from (20).

Construction of the basis functions, described by (16), (17) assures their orthogonality.

6 Discrete Construction

In [1], Alpert proposed a discrete bases construction for Legendre multi-wavelets. The structure of this analogue is essentially similar to that of the continuous bases, but the discrete construction is more convinient when representation of the function (and its related operators) is based on its values at a finite set of points [1]. The price is the loss of complete scale invariance: V_n's are no longer the dilates of a single space V_0, rather only nearly so.

In the more general case of non-dyadic spitting, the continuous multi-wavelet basis can be also transformed into a discrete basis. Below we present an algorithm for constructing such bases for non-dyadic Legendre multi-wavelets that includes Alpert's dyadic contruction as a special case. We will refer to these beses as to *tree-structured Legendre* (TSL) multi-wavelets.

Given a set of n discrete points, $\{x_1, x_2, \ldots, x_n\} \subset R$, our goal is to define an orthogonal basis for the n-dimensional space of functions, defined on $\{x_1, x_2, \ldots, x_n\}$. We assume, that $x_1 < x_2 < \ldots < x_n$, and $n = 2^m N$, where N is the required order of approximation, and m is a positive integer. The basis will have two fundamental properties:

1. all but N basis vectors will have N vanishing moments;
2. the basis vectors will be nonzero on different scales.

We start with constructing a binary tree, similar to BIST, though slightly different. Now a label at each node of the tree must be a multiple of N, where $N \geq 1$. Each label is equal the sum of labels of its children, however, all terminal nodes will now be labelled with N, instead of 1 as in the case of BIST. The root of the tree has label n. An example of such a tree with $N = 2$ is presented in Figure 3. By d we denote the depth of the tree, and let μ_j, $j \in \{0, 1, \ldots, d\}$ be the number of nodes on level j ($j = d$ corresponds to the leaves). The above tree is of depth $d = 4$, with $\mu_0 = 1, \mu_1 = 2, \mu_2 = 3, \mu_3 = 4$, and $\mu_4 = 6$. Let also $c(j, i)$, $i \in \{1, 2, \ldots, \mu_j\}$ be the nodes on level j, counted from the left to the right, and $\nu(c(j, i))$ be the label at node $c(j, i)$.

Let us fix the following notations. For a $(2k \times 2k)$ matrix S we let S^U and S^L denote the two $(k \times 2k)$ matrices: S^U, consisting of the upper k rows and S^L, consisting of the lower k rows of S. Suppose that the columns $s_1, s_2, \ldots, s_{2k}$ of S are linearly independent. We define $T = Orth(S)$ to be the matrix that

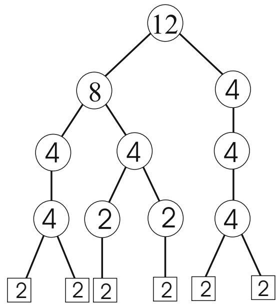

Fig. 3. TSL tree

results from the column-by-column Gram-Shmidt orthogonalization of S. Then, denoting the columns of T by $t_1, t_2, \ldots, t_{2k}$, we have:

$$\left.\begin{array}{r} \text{linear span}\{t_1, t_2, \ldots, t_i\} = \text{linear span}\{s_1, s_2, \ldots, s_i\} \\ t_i^T t_i = \delta_{ij} \end{array}\right\} i, j \in \{1, 2, \ldots, 2k\}. \tag{21}$$

Now we proceed to the definition of basis matrices. We construct so called *moment matrices*, $M_{1,i}$ for $i \in \{1, 2, \ldots, \mu_{d-1}\}$ as follows:

$$M_{1,i} = \begin{pmatrix} 1 & x_{u_i+1} & x_{u_i+1}^2 & \ldots & x_{u_i+1}^{2N-1} \\ 1 & x_{u_i+2} & x_{u_i+2}^2 & \ldots & x_{u_i+2}^{2N-1} \\ \vdots & & & & \vdots \\ 1 & x_{u_i+2N} & x_{u_i+2N}^2 & \ldots & x_{u_i+2N}^{2N-1} \end{pmatrix}, \tag{22}$$

if $c(d-1, i)$ is a splitting node, and

$$M_{1,i} = \begin{pmatrix} 1 & x_{u_i+1} & x_{u_i+1}^2 & \ldots & x_{u_i+1}^{2N-1} \\ 1 & x_{u_i+2} & x_{u_i+2}^2 & \ldots & x_{u_i+2}^{2N-1} \\ \vdots & & & & \vdots \\ 1 & x_{u_i+N} & x_{u_i+N}^2 & \ldots & x_{u_i+N}^{2N-1} \end{pmatrix}, \tag{23}$$

if $c(d-1, i)$ is a non-splitting node. Here $u_i = \sum_{j=1}^{i-1} \nu(c(d-1, j))$ for $i > 1$, and $u_1 = 0$. Additionally, we define matrices $U_{1,i}$, $i \in \{1, 2, \ldots, \mu_{d-1}\}$:

$$U_{1,i} = \begin{cases} Orth(M_{1,i})^T, \text{ if } c(d-1, i) \text{ is a splitting node,} \\ \begin{pmatrix} I_N \\ 0_N \end{pmatrix}, \quad \text{otherwise.} \end{cases} \tag{24}$$

Here I_N and 0_N denote $(N \times N)$ identity and zero matrices, respectively. In this way, the lower N rows of $U_{1,i}$ will have at least N vanishing moments (since they are orthogonal to the first N columns of $M_{1,i}$). These N last rows of $U_{1,i}$ will be included into the final basis, while the first N rows will remain for further processing.

The first $(n \times n)$ basis matrix, U_1, is constructed by deleting all zero rows from the following auxiliary matrix $\tilde{U}_1$:

$$\tilde{U}_1 = \begin{pmatrix} U^U_{1,1} & & & \\ & U^U_{1,2} & & \\ & & \ddots & \\ & & & U^U_{1,\mu_{d-1}} \\ U^L_{1,1} & & & \\ & U^L_{1,2} & & \\ & & \ddots & \\ & & & U^L_{1,\mu_{d-1}} \end{pmatrix}. \tag{25}$$

The second basis matrix is $U_2 \times U_1$, with an $(n \times n)$ matrix U_2, defined by the formula

$$U_2 = \begin{pmatrix} U'_2 & \\ & I_{n-l_2} \end{pmatrix}, \tag{26}$$

where l_2 is the size of matrix U'_2. The latter is obtained by deleting zero rows from $\tilde{U}'_2$,

$$\tilde{U}'_2 = \begin{pmatrix} \tilde{U}^U_{2,1} & & & \\ & \tilde{U}^U_{2,2} & & \\ & & \ddots & \\ & & & \tilde{U}^U_{2,\mu_{d-2}} \\ \tilde{U}^L_{2,1} & & & \\ & \tilde{U}^L_{2,2} & & \\ & & \ddots & \\ & & & \tilde{U}^L_{2,\mu_{d-2}} \end{pmatrix}, \tag{27}$$

with

$$\tilde{U}_{2,i} = \begin{cases} U_{2,i}, & \text{if } c(d-2,i) \text{ is a splitting node,} \\ \begin{pmatrix} I_N \\ 0_N \end{pmatrix}, & \text{otherwise.} \end{cases} \tag{28}$$

If $c(d-2,i)$ is a splitting node with two children $c(d-1,k)$ and $c(d-1,k+1)$, then $U_{2,i} = Orth(M_{2,i})^T$, where $M_{2,i}$ is given by

$$M_{2,i} = \begin{pmatrix} U^U_{1,k} \times M_{1,k} \\ U^U_{1,k+1} \times M_{1,k+1} \end{pmatrix}. \tag{29}$$

Otherwise, if the node $c(d-2,i)$ is non-splitting, and $c(d-1,k)$ is its (only) child, then $U_{2,i} = U_{1,k}$ and $M_{2,i} = M_{1,k}$.

In general, the $m-th$ basis matrix, for $m \in \{2,\ldots,d\}$, is $U_m \times U_{m-1} \times \cdots \times U_1$ with U_j, $j \in \{2,\ldots,m\}$, defined by the formula

$$U_j = \begin{pmatrix} U'_j & \\ & I_{n-l_j} \end{pmatrix}, \tag{30}$$

where l_j is the size of U'_j, and U'_j are obtained by deleting zero rows from the following matrices $\tilde{U}'_j$:

$$\tilde{U}'_j = \begin{pmatrix} \tilde{U}^U_{j,1} & & \\ & \ddots & \\ & & \tilde{U}^U_{j,\mu_{d-j}} \\ \tilde{U}^L_{j,1} & & \\ & \ddots & \\ & & \tilde{U}^L_{j,\mu_{d-j}} \end{pmatrix}. \tag{31}$$

Here again

$$\tilde{U}_{j,i} = \begin{cases} U_{j,i}, & \text{if } c(d-j,i) \text{ is a splitting node,} \\ \begin{pmatrix} I_N \\ 0_N \end{pmatrix}, & \text{otherwise.} \end{cases} \tag{32}$$

$U_{j,i} = Orth(M_{j,i})^T$, and $M_{j,i}$ is

$$M_{j,i} = \begin{pmatrix} U^U_{j-1,k} \times M_{j-1,k} \\ U^U_{j-1,k+1} \times M_{j-1,k+1} \end{pmatrix}, \tag{33}$$

if $c(d-j,i)$ is a splitting node, and $c(d-j+1,k)$, $c(d-j+1,k+1)$ are its left and right children, respectively. $U_{j,i} = U_{j-1,k}$, $M_{j,i} = M_{j-1,k}$ if the node $c(d-j,i)$ is non-splitting, and $c(d-j+1,k)$ is its only child.

The final matrix, $U = U_d \times U_{d-1} \times \cdots \times U_1$, represents the discrete wavelet-like basis of parameter N on $x_1, x_2 \ldots, x_n$. As it has been mentioned in [1], some adjustments must be made to this formula to ensure numerical stability. These issues are discussed in [2] and we will not cover them here.

Example 2. For the TSL tree, illustrated in Fig. 3, the schematic final basis matrix showing the irregular structure of the construction is given below. Nonzero elements of the matrix are denoted by "$\star$".

$$U = \begin{pmatrix}
\star\star\star\star\star\star\star\star\star\star\star\star \\
\star\star\star\star\star\star\star\star\star\star\star\star \\
\star\star\star\star\star\star\star\star\star\star\star\star \\
\star\star\star\star\star\star\star\star\star\star\star\star \\
\star\star\star\star\star\star\star\star 0000 \\
\star\star\star\star\star\star\star\star 0000 \\
0000\star\star\star\star 0000 \\
0000\star\star\star\star 0000 \\
\star\star\star\star 00000000 \\
\star\star\star\star 00000000 \\
00000000\star\star\star\star \\
00000000\star\star\star\star
\end{pmatrix}.$$

7 Fast Implementation

Since the basis matrix U is represented as a product of sparse matrices, $U = U_d \times U_{d-1} \times \cdots \times U_1$, it is apparent, that for an arbitrary vector of length n, the application of matrix U can be accomplished in order $O(n)$ arithmetic operations. Thus, the described transform is equally efficient with Haar or TSH transofms. The inverse transform matrix, U^{-1} is factorized as $U^{-1} = U^T = U_1^T \times U_2^T \times U_d^T$ (since U is orthogonal), and thus the inverse transform is of the same complexity.

8 Conclusion

We have considered multi-wavelet bases, that is, bases with a finite set of scaling functions and wavelets. In particular, we have focused on Legendre type of multi-wavelets, introduced by Alpert in [1] and generalized later in [4] to the case of arbitrary (non-dyadic) time splitting. The latter can be appropriate for signals with some irregular structure. Knowing this structure, an adaptive wavelet-like basis can be constructed that can lead to a more efficient transform-domain expansion of the signal. Here we have proposed an algorithm for constructing discrete counterparts of such non-dyadic Legendre multi-wavelets. The discrete construction is more convenient, when representation of a function (and its related operators) is based on its values at a finite set of points.

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