2. Modelling of Underwater Robots

"We have Einstein's space, de Sitter's spaces, expanding universes, contracting universes, vibrating universes, mysterious universes. In fact the pure mathematician may create universes just by writing down an equation, and indeed, if he is an individualist he can have an universe of his own".
J.J. Thomson, around 1919.

2.1 Introduction

In this Chapter the mathematical model of UVMSs is derived. Modeling of rigid bodies moving in a fluid or underwater manipulators has been studied in literature by, among others, [137, 156, 157, 174, 182, 189, 203, 242, 255, 256, 285, 286], where a deeper discussion of specific aspects can be found. In [224], the model of two UVMSs holding the same rigid object is derived.

2.2 Rigid Body's Kinematics

A rigid body is completely described by its position and orientation with respect to a reference frame $\Sigma_i, O-\boldsymbol{xyz}$ that it is supposed to be earth-fixed and inertial. Let define $\boldsymbol{\eta}_1 \in \mathbb{R}^3$ as

$$\boldsymbol{\eta}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the vector of the body position coordinates in a earth-fixed reference frame. The vector $\dot{\boldsymbol{\eta}}_1$ is the corresponding time derivative (expressed in the earth-fixed frame). If one defines

$$\boldsymbol{\nu}_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

as the linear velocity of the origin of the body-fixed frame $\Sigma_b, O_b - \boldsymbol{x}_b\boldsymbol{y}_b\boldsymbol{z}_b$ with respect to the origin of the earth-fixed frame expressed in the body-fixed frame (from now on: body-fixed linear velocity) the following relation between the defined linear velocities holds:

G. Antonelli: Underwater Robots, 2nd Edition, STAR 2, pp. 15–44, 2006.

$$\boldsymbol{\nu}_1 = \boldsymbol{R}_I^B \dot{\boldsymbol{\eta}}_1 , \tag{2.1}$$

where $\boldsymbol{R}_I^B$ is the rotation matrix expressing the transformation from the inertial frame to the body-fixed frame.

In the following, two different attitude representations will be introduced: Euler angles and Euler parameters or quaternion. In marine terminology is common the use of Euler angles while several control strategies use the quaternion in order to avoid the representation singularities that might arise by the use of Euler angles.

Table 2.1. Common notation for marine vehicle's motion

		forces and moments	$\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$	$\boldsymbol{\eta}_1, \boldsymbol{\eta}_2$
motion in the x-direction	surge	X	u	x
motion in the y-direction	sway	Y	v	y
motion in the z-direction	heave	Z	w	z
rotation about the x-axis	roll	K	p	ϕ
rotation about the y-axis	pitch	M	q	θ
rotation about the z-axis	yaw	N	r	ψ

2.2.1 Attitude Representation by Euler Angles

Let define $\boldsymbol{\eta}_2 \in \mathbb{R}^3$ as

$$\boldsymbol{\eta}_2 = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

the vector of body Euler-angle coordinates in a earth-fixed reference frame. In the nautical field those are commonly named roll, pitch and yaw angles and corresponds to the elementary rotation around x, y and z in fixed frame [254]. The vector $\dot{\boldsymbol{\eta}}_2$ is the corresponding time derivative (expressed in the inertial frame). Let define

$$\boldsymbol{\nu}_2 = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

as the angular velocity of the body-fixed frame with respect to the earth-fixed frame expressed in the body-fixed frame (from now on: body-fixed angular velocity). The vector $\dot{\boldsymbol{\eta}}_2$ does not have a physical interpretation and it is related to the body-fixed angular velocity by a proper Jacobian matrix:

$$\boldsymbol{\nu}_2 = \boldsymbol{J}_{k,o}(\boldsymbol{\eta}_2)\dot{\boldsymbol{\eta}}_2 . \tag{2.2}$$

The matrix $\boldsymbol{J}_{k,o} \in \mathbb{R}^{3\times 3}$ can be expressed in terms of Euler angles as:

$$\boldsymbol{J}_{k,o}(\boldsymbol{\eta}_2) = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & c_\theta s_\phi \\ 0 & -s_\phi & c_\theta c_\phi \end{bmatrix} , \tag{2.3}$$

where c_α and s_α are short notations for $\cos(\alpha)$ and $\sin(\alpha)$, respectively. Matrix $\boldsymbol{J}_{k,o}(\boldsymbol{\eta}_2)$ is not invertible for every value of $\boldsymbol{\eta}_2$. In detail, it is

$$\boldsymbol{J}_{k,o}^{-1}(\boldsymbol{\eta}_2) = \frac{1}{c_\theta} \begin{bmatrix} 1 & s_\phi s_\theta & c_\phi s_\theta \\ 0 & c_\phi c_\theta & -c_\theta s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} , \tag{2.4}$$

that it is singular for $\theta = (2l+1)\frac{\pi}{2}$ rad, with $l \in \mathbb{N}$, i.e., for a pitch angle of $\pm\frac{\pi}{2}$ rad.

The rotation matrix $\boldsymbol{R}_I^B$, needed in (2.1) to transform the linear velocities, is expressed in terms of Euler angles by the following:

$$\boldsymbol{R}_I^B(\boldsymbol{\eta}_2) = \begin{bmatrix} c_\psi c_\theta & s_\psi c_\theta & -s_\theta \\ -s_\psi c_\phi + c_\psi s_\theta s_\phi & c_\psi c_\phi + s_\psi s_\theta s_\phi & s_\phi c_\theta \\ s_\psi s_\phi + c_\psi s_\theta c_\phi & -c_\psi s_\phi + s_\psi s_\theta c_\phi & c_\phi c_\theta \end{bmatrix} . \tag{2.5}$$

Table 2.1 shows the common notation used for marine vehicles according to the SNAME notation ([272]), Figure 2.1 shows the defined frames and the elementary motions.

2.2.2 Attitude Representation by Quaternion

To overcome the possible occurrence of representation singularities it might be convenient to resort to non-minimal attitude representations. One possible choice is given by the quaternion. The term *quaternion* was introduced by Hamilton in 1840, 70 years after the introduction of a four-parameter rigid-body attitude representation by Euler. In the following, a short introduction to quaternion is given.

By defining the mutual orientation between two frames of common origin in terms of the rotation matrix

$$\boldsymbol{R}_{\boldsymbol{k}}(\delta) = \cos\delta \boldsymbol{I}_3 + (1 - \cos\delta)\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}} - \sin\delta \boldsymbol{S}(\boldsymbol{k}) ,$$

where δ is the angle and $\boldsymbol{k} \in \mathbb{R}^3$ is the unit vector of the axis expressing the rotation needed to align the two frames, $\boldsymbol{I}_3$ is the (3×3) identity matrix, $\boldsymbol{S}(\boldsymbol{x})$ is the matrix operator performing the cross product between two (3×1) vectors

$$\boldsymbol{S}(\boldsymbol{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} , \tag{2.6}$$

the unit quaternion is defined as

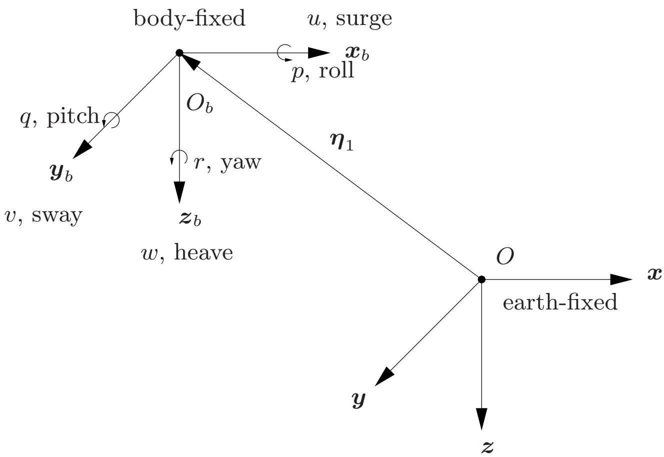

Fig. 2.1. Frames and elementary vehicle's motion

$$\mathcal{Q} = \{\boldsymbol{\varepsilon}, \eta\}$$

with

$$\boldsymbol{\varepsilon} = \boldsymbol{k}\sin\frac{\delta}{2},$$
$$\eta = \cos\frac{\delta}{2},$$

where $\eta \geq 0$ for $\delta \in [-\pi, \pi]$ rad. This restriction is necessary for uniqueness of the quaternion associated to a given matrix, in that the two quaternion $\{\boldsymbol{\varepsilon}, \eta\}$ and $\{-\boldsymbol{\varepsilon}, -\eta\}$ represent the same orientation, i.e., the same rotation matrix.

The unit quaternion satisfies the condition

$$\eta^2 + \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon} = 1\,. \tag{2.7}$$

The relationship between $\boldsymbol{\nu}_2$ and the time derivative of the quaternion is given by the *quaternion propagation equations*

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}\eta\boldsymbol{\nu}_2 + \frac{1}{2}\boldsymbol{S}(\boldsymbol{\varepsilon})\boldsymbol{\nu}_2\,, \tag{2.8}$$

$$\dot{\eta} = -\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\nu}_2\,, \tag{2.9}$$

that can be rearranged in the form:

$$\begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \dot{\eta} \end{bmatrix} = \boldsymbol{J}_{k,oq}(\mathcal{Q})\boldsymbol{\nu}_2 = \frac{1}{2}\begin{bmatrix} \eta & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & \eta & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & \eta \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 \end{bmatrix}\boldsymbol{\nu}_2\,. \tag{2.10}$$

The matrix $\boldsymbol{J}_{k,oq}(\mathcal{Q})$ satisfies:

$$\boldsymbol{J}_{k,oq}^{\mathrm{T}} \boldsymbol{J}_{k,oq} = \frac{1}{4} \boldsymbol{I}_3 \,,$$

that allows to invert the mapping (2.10) yielding:

$$\boldsymbol{\nu}_2 = 4 \boldsymbol{J}_{k,oq}^{\mathrm{T}} \begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \dot{\eta} \end{bmatrix} .$$

For completeness the rotation matrix $\boldsymbol{R}_I^B$, needed to compute (2.1), in terms of quaternion is given:

$$\boldsymbol{R}_I^B(\mathcal{Q}) = \begin{bmatrix} 1-2(\varepsilon_2^2+\varepsilon_3^2) & 2(\varepsilon_1\varepsilon_2+\varepsilon_3\eta) & 2(\varepsilon_1\varepsilon_3-\varepsilon_2\eta) \\ 2(\varepsilon_1\varepsilon_2-\varepsilon_3\eta) & 1-2(\varepsilon_1^2+\varepsilon_3^2) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\eta) \\ 2(\varepsilon_1\varepsilon_3+\varepsilon_2\eta) & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\eta) & 1-2(\varepsilon_1^2+\varepsilon_2^2) \end{bmatrix} . \tag{2.11}$$

2.2.3 Attitude Error Representation

Let now define $\boldsymbol{R}_B^I \in \mathrm{I\!R}^{3\times3}$ as the rotation matrix from the body-fixed frame to the earth-fixed frame, which is also described by the quaternion $\mathcal{Q}$, and $\boldsymbol{R}_d^I \in \mathrm{I\!R}^{3\times3}$ the rotation matrix from the frame expressing the desired vehicle orientation to the earth-fixed frame, which is also described by the quaternion $\mathcal{Q}_d = \{\boldsymbol{\varepsilon}_d, \eta_d\}$. One possible choice for the rotation matrix necessary to align the two frames is

$$\widetilde{\boldsymbol{R}} = \boldsymbol{R}_B^{I^{\mathrm{T}}} \boldsymbol{R}_d^I = \boldsymbol{R}_I^B \boldsymbol{R}_d^I \,,$$

where $\boldsymbol{R}_I^B = \boldsymbol{R}_B^{I\,\mathrm{T}}$. The quaternion $\widetilde{\mathcal{Q}} = \{\tilde{\boldsymbol{\varepsilon}}, \tilde{\eta}\}$ associated with $\widetilde{\boldsymbol{R}}$ can be obtained directly from $\widetilde{\boldsymbol{R}}$ or computed by composition (quaternion product): $\widetilde{\mathcal{Q}} = \mathcal{Q}^{-1} * \mathcal{Q}_d$, where $\mathcal{Q}^{-1} = \{-\boldsymbol{\varepsilon}, \eta\}$:

$$\tilde{\boldsymbol{\varepsilon}} = \eta \boldsymbol{\varepsilon}_d - \eta_d \boldsymbol{\varepsilon} + \boldsymbol{S}(\boldsymbol{\varepsilon}_d)\boldsymbol{\varepsilon}, \tag{2.12}$$

$$\tilde{\eta} = \eta \eta_d + \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\varepsilon}_d \,. \tag{2.13}$$

Since the quaternion associated with $\widetilde{\boldsymbol{R}} = \boldsymbol{I}_3$ (i.e. representing two aligned frames) is $\widetilde{\mathcal{Q}} = \{\boldsymbol{0}, 1\}$, it is sufficient to represent the attitude error as $\tilde{\boldsymbol{\varepsilon}}$.

The quaternion propagation equations can be rewritten also in terms of the error variables:

$$\dot{\tilde{\boldsymbol{\varepsilon}}} = \frac{1}{2} \tilde{\eta} \tilde{\boldsymbol{\nu}}_2 + \frac{1}{2} \boldsymbol{S}(\tilde{\boldsymbol{\varepsilon}}) \tilde{\boldsymbol{\nu}}_2 \,, \tag{2.14}$$

$$\dot{\tilde{\eta}} = -\frac{1}{2} \tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}} \tilde{\boldsymbol{\nu}}_2 \,, \tag{2.15}$$

where $\tilde{\boldsymbol{\nu}}_2 = \boldsymbol{\nu}_{2,d} - \boldsymbol{\nu}_2$ is the angular velocity error expressed in body-fixed frame. Defining

$$\boldsymbol{z} = \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}} \\ \tilde{\eta} \end{bmatrix} ,$$

the relations in (2.14)–(2.15) can be rewritten in the form:

$$\dot{\boldsymbol{z}} = \frac{1}{2}\begin{bmatrix} \tilde{\eta}\boldsymbol{I}_3 + \boldsymbol{S}(\tilde{\boldsymbol{\varepsilon}}) \\ -\tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}} \end{bmatrix} \tilde{\boldsymbol{\nu}}_2 = \boldsymbol{J}_{k,oq}(\boldsymbol{z})\tilde{\boldsymbol{\nu}}_2 \tag{2.16}$$

The equations above are given in terms of the body-fixed angular velocity. In fact, they will be used in the control laws of Chap. 7. The generic expression of the propagation equations is the following:

$$\dot{\widetilde{\boldsymbol{\varepsilon}}}^a_{ba} = \frac{1}{2}\boldsymbol{E}(\widetilde{\mathcal{Q}}_{ba})\widetilde{\boldsymbol{\omega}}^a_{ba}\,,$$
$$\dot{\widetilde{\eta}}_{ba} = -\frac{1}{2}\widetilde{\boldsymbol{\varepsilon}}^{a\mathrm{T}}_{ba}\widetilde{\boldsymbol{\omega}}^a_{ba}\,,$$

with

$$\boldsymbol{E}(\widetilde{\mathcal{Q}}_{ba}) = \widetilde{\eta}_{ba}\boldsymbol{I}_3 - \boldsymbol{S}(\widetilde{\boldsymbol{\varepsilon}}^a_{ba}).$$

where $\widetilde{\mathcal{Q}}_{ba} = \{\widetilde{\boldsymbol{\varepsilon}}^a_{ba}, \widetilde{\eta}_{ba}\}$ is the quaternion associated to $\boldsymbol{R}^a_b = \boldsymbol{R}^{\mathrm{T}}_a\boldsymbol{R}_b$ and the angular velocity $\widetilde{\boldsymbol{\omega}}^a_{ba} = \boldsymbol{R}^{\mathrm{T}}_a(\boldsymbol{\omega}_b - \boldsymbol{\omega}_a)$ of the frame Σ_b relative to the frame Σ_a, expressed in the frame Σ_a.

Quaternion from Rotation Matrix

It can be useful to recall the procedure needed to extract the quaternion from the rotation matrix [127, 261].

Given a generic rotation matrix $\boldsymbol{R}$:

1. compute the trace of $\boldsymbol{R}$ according to:
$$R_{4,4} = \mathrm{tr}(\boldsymbol{R}) = \sum_{j=1}^{3} R_{j,j}$$
2. compute the index i according to:
$$R_{i,i} = \max(R_{1,1}, R_{2,2}, R_{3,3}, R_{4,4})$$
3. define the scalar c_i as:
$$|c_i| = \sqrt{1 + 2R_{i,i} - R_{4,4}}$$
in which the sign can be plus or minus.
4. compute the other three values of c by knowing the following relationships:
$$\begin{aligned} c_4c_1 &= R_{3,2} - R_{2,3} \\ c_4c_2 &= R_{1,3} - R_{3,1} \\ c_4c_3 &= R_{2,1} - R_{1,2} \\ c_2c_3 &= R_{3,2} + R_{2,3} \\ c_3c_1 &= R_{1,3} + R_{3,1} \\ c_1c_2 &= R_{2,1} + R_{1,2} \end{aligned}$$
simply dividing the equations in which c_i is involved by c_i itself.

5. compute the quaternion $\mathcal{Q}$ by the following:

$$[\,\boldsymbol{\varepsilon} \quad \eta\,]^{\mathrm{T}} = \frac{1}{2}[\,c_1 \quad c_2 \quad c_3 \quad c_4\,]^{\mathrm{T}}.$$

Quaternion from Euler Angles

The transformation from Euler angles to quaternion is always possible, i.e., it is not affected by the occurrence of representation singularities [127]. This implies that the use of quaternion to control underwater vehicles is compatible with the common use of Euler angles to express the desired trajectory of the vehicle.

The algorithm consists in computing the rotation matrix expressed in Euler angles by (2.5) and using the procedure described in the previous subsection to extract the corresponding quaternion.

2.2.4 6-DOFs Kinematics

It is useful to collect the kinematic equations in 6-dimensional matrix forms. Let us define the vector $\boldsymbol{\eta} \in \mathbb{R}^6$ as

$$\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} \tag{2.17}$$

and the vector $\boldsymbol{\nu} \in \mathbb{R}^6$ as

$$\boldsymbol{\nu} = \begin{bmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \end{bmatrix}, \tag{2.18}$$

and by defining the matrix $\boldsymbol{J}_e(\boldsymbol{R}_B^I) \in \mathbb{R}^{6\times 6}$

$$\boldsymbol{J}_e(\boldsymbol{R}_B^I) = \begin{bmatrix} \boldsymbol{R}_I^B & \boldsymbol{O}_{3\times 3} \\ \boldsymbol{O}_{3\times 3} & \boldsymbol{J}_{k,o} \end{bmatrix}, \tag{2.19}$$

where the rotation matrix $\boldsymbol{R}_I^B$ given in (2.5) and $\boldsymbol{J}_{k,o}$ is given in (2.3), it is

$$\boldsymbol{\nu} = \boldsymbol{J}_e(\boldsymbol{R}_B^I)\dot{\boldsymbol{\eta}}. \tag{2.20}$$

The inverse mapping, given the block-diagonal structure of $\boldsymbol{J}_e$, is given by:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}_e^{-1}(\boldsymbol{R}_B^I)\boldsymbol{\nu} = \begin{bmatrix} \boldsymbol{R}_B^I & \boldsymbol{O}_{3\times 3} \\ \boldsymbol{O}_{3\times 3} & \boldsymbol{J}_{k,o}^{-1} \end{bmatrix} \boldsymbol{\nu}, \tag{2.21}$$

where $\boldsymbol{J}_{k,o}^{-1}$ is given in (2.4).

On the other side, it is possible to represent the orientation by means of quaternions. Let us define the vector $\boldsymbol{\eta}_q \in \mathbb{R}^7$ as

$$\boldsymbol{\eta}_q = \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\varepsilon} \\ \eta \end{bmatrix} \tag{2.22}$$

and the matrix $\boldsymbol{J}_{e,q}(\boldsymbol{R}_B^I) \in \mathbb{R}^{6\times 7}$

$$\boldsymbol{J}_{e,q}(\boldsymbol{R}_B^I) = \begin{bmatrix} \boldsymbol{R}_I^B & \boldsymbol{O}_{3\times 4} \\ \boldsymbol{O}_{3\times 3} & 4\boldsymbol{J}_{k,oq}^{\mathrm{T}} \end{bmatrix}, \tag{2.23}$$

where $\boldsymbol{J}_{k,oq}$ is given in (2.10); it is

$$\boldsymbol{\nu} = \boldsymbol{J}_{e,q}(\boldsymbol{R}_B^I)\dot{\boldsymbol{\eta}}_e . \tag{2.24}$$

The inverse mapping is given by:

$$\dot{\boldsymbol{\eta}}_e = \begin{bmatrix} \boldsymbol{R}_B^I & \boldsymbol{O}_{3\times 3} \\ \boldsymbol{O}_{4\times 3} & \boldsymbol{J}_{k,oq} \end{bmatrix} \boldsymbol{\nu} . \tag{2.25}$$

2.3 Rigid Body's Dynamics

Several approaches can be considered when deriving the equations of motion of a rigid body. In the following, the Newton-Euler formulation will be briefly summarized.

The motion of a generic system of material particles subject to external forces can be described by resorting to the *fundamental principles of dynamics* (Newton's laws of motion). Those relate the resultant force and moment to the time derivative of the linear and angular momentum.

Let ρ be the density of a particle of volume dV of a rigid body $\mathcal{B}$, ρdV is the corresponding mass denoted by the position vector $\boldsymbol{p}$ in an inertial frame $O - \boldsymbol{xyz}$. Let also $V_{\mathcal{B}}$ be the the body volume and

$$m = \int_{V_{\mathcal{B}}} \rho dV$$

be the total mass. The *center of mass* of $\mathcal{B}$ is defined as

$$\boldsymbol{p}_C = \frac{1}{m}\int_{V_{\mathcal{B}}} \boldsymbol{p}\rho dV.$$

The *linear momentum* of the body $\mathcal{B}$ is defined as the vector

$$\boldsymbol{l} = \int_{V_{\mathcal{B}}} \dot{\boldsymbol{p}}\rho dV = m\dot{\boldsymbol{p}}_C.$$

For a system with constant mass, the Newton's law of motion for the linear part

$$\boldsymbol{f} = \dot{\boldsymbol{l}} = m\frac{d}{dt}\dot{\boldsymbol{p}}_C \tag{2.26}$$

can be rewritten simply by the *Newton's equations of motion*:

$$\boldsymbol{f} = m\ddot{\boldsymbol{p}}_C \tag{2.27}$$

where $\boldsymbol{f}$ is the resultant of the external forces.

Let us define the *Inertia tensor* of the body $\mathcal{B}$ relative to the pole O:

$$\boldsymbol{I}_O = \int_{V_{\mathcal{B}}} \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{p})\boldsymbol{S}(\boldsymbol{p})\rho dV,$$

where $\boldsymbol{S}$ is the skew-symmetric operator defined in (2.6). The matrix $\boldsymbol{I}_O$ is symmetric and positive definite. The positive diagonal elements I_{Oxx}, I_{Oyy}, I_{Ozz} are the *inertia moments* with respect to the three coordinate axes of the reference frame. The off diagonal elements are the *products of inertia*.

The relationship between the inertia tensor in two different frames $\boldsymbol{I}_O$ and $\boldsymbol{I}'_O$, related by a rotation matrix $\boldsymbol{R}$, with the same pole O, is the following:

$$\boldsymbol{I}_O = \boldsymbol{R}\boldsymbol{I}'_O\boldsymbol{R}^{\mathrm{T}}.$$

The change of pole is related by the *Steiner's theorem*:

$$\boldsymbol{I}_O = \boldsymbol{I}_C + m\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{p}_C)\boldsymbol{S}(\boldsymbol{p}_C),$$

where $\boldsymbol{I}_C$ is the inertial tensor relative to the center of mass, when expressed in a frame parallel to the frame in which $\boldsymbol{I}_O$ is defined.

Notice that O can be either a fixed or moving pole. In case of a fixed pole the elements of the inertia tensor are function of time. A suitable choice of the pole might be a point fixed to the rigid body in a way to obtain a constant inertia tensor. Moreover, since the inertia tensor is symmetric positive definite is always possible to find a frame in which the matrix attains a diagonal form, this frame is called *principal frame*, also, if the pole coincides with the center of mass, it is called *central frame*. This is true also if the body does not have a significant geometric symmetry.

Let Ω be any point in space and $\boldsymbol{p}_\Omega$ the corresponding position vector. Ω can be either moving or fixed with respect to the reference frame. The *angular momentum* of the body $\mathcal{B}$ relative to the pole Ω is defined as the vector:

$$\boldsymbol{k}_\Omega = \int_{V_{\mathcal{B}}} \dot{\boldsymbol{p}} \times (\boldsymbol{p}_\Omega - \boldsymbol{p})\, \rho dV. \tag{2.28}$$

Taking into account the definition of center of mass, (2.28) can be rewritten in the form:

$$\boldsymbol{k}_\Omega = \boldsymbol{I}_C\boldsymbol{\omega} + m\dot{\boldsymbol{p}}_C \times (\boldsymbol{p}_\Omega - \boldsymbol{p}_C)\,, \tag{2.29}$$

where $\boldsymbol{\omega}$ is the angular velocity.

The resultant moment $\boldsymbol{\mu}_\Omega$ with respect to the pole Ω of a rigid body subject to n external forces $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ is:

$$\boldsymbol{\mu}_\Omega = \sum_{i=1}^{n} \boldsymbol{f}_i \times (\boldsymbol{p}_\Omega - \boldsymbol{p}_i)\,.$$

In case of a system with constant mass and rigid body, the angular part of the Newton's law of motion

$$\boldsymbol{\mu}_\Omega = \dot{\boldsymbol{k}}_\Omega$$

yields the *Euler equations of motion*:

$$\boldsymbol{\mu}_\Omega = \boldsymbol{I}_\Omega \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{I}_\Omega \boldsymbol{\omega}) \,. \tag{2.30}$$

The right-hand side of the Newton and Euler equations of motion, (2.27) and (2.30), are defined inertial forces and inertial moments, respectively.

2.3.1 Rigid Body's Dynamics in Matrix Form

To derive the equations of motion in matrix form it is useful to refer the quantities to a body-fixed frame $O_b - \boldsymbol{x}_b \boldsymbol{y}_b \boldsymbol{z}_b$ using the body-fixed linear and angular velocities that has been introduced in Section 2.2.

The following relationships hold:

$$\boldsymbol{p}_\Omega - \boldsymbol{p}_C = \boldsymbol{R}_B^I \boldsymbol{r}_C^b \tag{2.31}$$

$$\dot{\boldsymbol{R}}_B^I \cdot = \boldsymbol{\omega} \times (\boldsymbol{R}_B^I \cdot) \tag{2.32}$$

$$\boldsymbol{R}_I^B \left(\boldsymbol{\omega} \times \boldsymbol{R}_B^I \cdot \right) = \boldsymbol{\nu}_2 \times \cdot \tag{2.33}$$

$$\boldsymbol{\omega} = \boldsymbol{R}_B^I \boldsymbol{\nu}_2 \tag{2.34}$$

$$\dot{\boldsymbol{\omega}} = \boldsymbol{R}_B^I \dot{\boldsymbol{\nu}}_2 \tag{2.35}$$

$$\dot{\boldsymbol{p}}_C = \boldsymbol{R}_B^I (\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 \times \boldsymbol{r}_C^b) \tag{2.36}$$

$$\boldsymbol{I}_C = \boldsymbol{R}_B^I \boldsymbol{I}_C^b \boldsymbol{R}_I^B \tag{2.37}$$

where, according to the (2.31), $\boldsymbol{r}_C^b$ is the vector position from the origin of the body-fixed frame to the center of mass expressed in the body-fixed frame ($\dot{\boldsymbol{r}}_C^b = \boldsymbol{0}$ for a rigid body).

Equation (2.26) can be rewritten in terms of the linear body-fixed velocities as

$$\begin{aligned} \boldsymbol{f} &= m \frac{d}{dt} \left[\boldsymbol{R}_B^I \left(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 \times \boldsymbol{r}_C^b \right) \right] \\ &= m \boldsymbol{R}_B^I \left(\dot{\boldsymbol{\nu}}_1 + \dot{\boldsymbol{\nu}}_2 \times \boldsymbol{r}_C^b + \boldsymbol{\nu}_2 \times \dot{\boldsymbol{r}}_C^b \right) + m \boldsymbol{\omega} \times \boldsymbol{R}_B^I (\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 \times \boldsymbol{r}_C^b), \end{aligned}$$

Premultiplying by $\boldsymbol{R}_I^B$ and defining as

$$\boldsymbol{\tau}_1 = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix},$$

the resultant forces acting on the rigid body expressed in a body-fixed frame, and as

$$\boldsymbol{\tau}_2 = \begin{bmatrix} K \\ M \\ N \end{bmatrix},$$

the corresponding resultant moment to the pole O_b, one obtains:

$$\boldsymbol{\tau}_1 = m\dot{\boldsymbol{\nu}}_1 + m\dot{\boldsymbol{\nu}}_2 \times \boldsymbol{r}_C^b + m\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1 + m\boldsymbol{\nu}_2 \times (\boldsymbol{\nu}_2 \times \boldsymbol{r}_C^b).$$

Equation (2.29) is written in an inertial frame. It is possible to rewrite the angular momentum in terms of the body-fixed velocities:

$$\boldsymbol{k}_\Omega = \boldsymbol{R}_B^I \left(\boldsymbol{I}_C^b \boldsymbol{\nu}_2 + m\boldsymbol{\nu}_1 \times \boldsymbol{r}_C^b \right). \tag{2.38}$$

Derivating (2.38) one obtains:

$$\boldsymbol{\tau}_2^I = \boldsymbol{\omega} \times \boldsymbol{R}_B^I \left(\boldsymbol{I}_C^b \boldsymbol{\nu}_2 + m\boldsymbol{\nu}_1 \times \boldsymbol{r}_C^b \right) + \boldsymbol{R}_B^I \left(\boldsymbol{I}_C^b \dot{\boldsymbol{\nu}}_2 + m\dot{\boldsymbol{\nu}}_1 \times \boldsymbol{r}_C^b \right),$$

that, using the relations above, can be written in the form:

$$\boldsymbol{\tau}_2 = \boldsymbol{I}_C^b \dot{\boldsymbol{\nu}}_2 + \boldsymbol{\nu}_2 \times (\boldsymbol{I}_C^b \boldsymbol{\nu}_2) + m\boldsymbol{\nu}_2 \times (\boldsymbol{\nu}_1 \times \boldsymbol{r}_C^b) + m\dot{\boldsymbol{\nu}}_1 \times \boldsymbol{r}_C^b.$$

It is now possible to rewrite the Newton-Euler equations of motion of a rigid body moving in the space. It is:

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_v, \tag{2.39}$$

where

$$\boldsymbol{\tau}_v = \begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix}.$$

The matrix $\boldsymbol{M}_{RB}$ is constant, symmetric and positive definite, i.e., $\dot{\boldsymbol{M}}_{RB} = \boldsymbol{O}$, $\boldsymbol{M}_{RB} = \boldsymbol{M}_{RB}^{\mathrm{T}} > \boldsymbol{O}$. Its unique parametrization is in the form:

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\boldsymbol{I}_3 & -m\boldsymbol{S}(\boldsymbol{r}_C^b) \\ m\boldsymbol{S}(\boldsymbol{r}_C^b) & \boldsymbol{I}_{O_b} \end{bmatrix},$$

where $\boldsymbol{I}_3$ is the (3×3) identity matrix, and $\boldsymbol{I}_{O_b}$ is the inertia tensor expressed in the body-fixed frame.

On the other hand, it does not exist a unique parametrization of the matrix $\boldsymbol{C}_{RB}$, representing the Coriolis and centripetal terms. It can be demonstrated that the matrix $\boldsymbol{C}_{RB}$ can always be parameterized such that it is skew-symmetrical, i.e.,

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = -\boldsymbol{C}_{RB}^{\mathrm{T}}(\boldsymbol{\nu}) \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6,$$

explicit expressions for $\boldsymbol{C}_{RB}$ can be found, e.g., in [127].

Notice that (2.39) can be greatly simplified if the origin of the body-fixed frame is chosen coincident with the central frame, i.e., $\boldsymbol{r}_C^b = \boldsymbol{0}$ and $\boldsymbol{I}_{O_b}$ is a diagonal matrix.

2.4 Hydrodynamic Effects

In this Section the major hydrodynamic effects on a rigid body moving in a fluid will be briefly discussed.

The theory of fluidodynamics is rather complex and it is difficult to develop a reliable model for most of the hydrodynamic effects. A rigorous analysis for incompressible fluids would need to resort to the Navier-Stokes equations (distributed fluid-flow). However, in this book modeling of the hydrodynamic effects in a context of automatic control is considered. In literature, it is well known that kinematic and dynamic coupling between vehicle and manipulator can not be neglected [182, 203, 204, 206], while most of the hydrodynamic effects have no significant influence in the range of the operative velocities.

2.4.1 Added Mass and Inertia

When a rigid body is moving in a fluid, the additional inertia of the fluid surrounding the body, that is accelerated by the movement of the body, has to be considered. This effect can be neglected in industrial robotics since the density of the air is much lighter than the density of a moving mechanical system. In underwater applications, however, the density of the water, $\rho \approx 1000\,\text{kg/m}^3$, is comparable with the density of the vehicles. In particular, at 0°, the density of the fresh water is $1002.68\,\text{kg/m}^3$; for sea water with 3.5% of salinity it is $\rho = 1028.48\,\text{kg/m}^3$.

The fluid surrounding the body is accelerated with the body itself, a force is then necessary to achieve this acceleration; the fluid exerts a reaction force which is equal in magnitude and opposite in direction. This reaction force is the added mass contribution. The added mass is not a quantity of fluid to add to the system such that it has an increased mass. Different properties hold with respect to the (6×6) inertia matrix of a rigid body due to the fact that the added mass is function of the body's surface geometry. As an example, the inertia matrix is not necessarily positive definite.

The hydrodynamic force along $\boldsymbol{x}_b$ due to the linear acceleration in the $\boldsymbol{x}_b$-direction is defined as:

$$X_A = -X_{\dot{u}}\dot{u} \qquad \text{where} \qquad X_{\dot{u}} = \frac{\partial X}{\partial \dot{u}},$$

where the symbol ∂ denotes the partial derivative. In the same way it is possible to define all the remaining 35 elements that relate the 6 force/moment components $[\,X \quad Y \quad Z \quad K \quad M \quad N\,]^{\text{T}}$ to the 6 linear/angular acceleration $[\,\dot{u} \quad \dot{v} \quad \dot{w} \quad \dot{p} \quad \dot{q} \quad \dot{r}\,]^{\text{T}}$. These elements can be grouped in the Added Mass matrix $\boldsymbol{M}_A \in \mathbb{R}^{6\times 6}$. Usually, all the elements of the matrix are different from zero.

There is no specific property of the matrix $\boldsymbol{M}_A$. For certain frequencies and specific bodies, such as catamarans, negative diagonal elements have been documented [127]. However, for completely submerged bodies it can be considered $\boldsymbol{M}_A > \boldsymbol{O}$. Moreover, if the fluid is ideal, the body's velocity is low, there are no currents or waves and frequency independence it holds [214]:

$$\boldsymbol{M}_A = \boldsymbol{M}_A^{\text{T}} > \boldsymbol{O}. \tag{2.40}$$

The added mass has also an *added* Coriolis and centripetal contribution. It can be demonstrated that the matrix expression can always be parameterized such that:

$$\boldsymbol{C}_A(\boldsymbol{\nu}) = -\boldsymbol{C}_A^{\mathrm{T}}(\boldsymbol{\nu}) \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6.$$

If the body is completely submerged in the water, the velocity is low and it has three planes of symmetry as common for underwater vehicles, the following structure of matrices $\boldsymbol{M}_A$ and $\boldsymbol{C}_A$ can therefore be considered:

$$\boldsymbol{M}_A = -\operatorname{diag}\{X_{\dot{u}}, Y_{\dot{v}}, Z_{\dot{w}}, K_{\dot{p}}, M_{\dot{q}}, N_{\dot{r}}\},$$

$$\boldsymbol{C}_A = \begin{bmatrix} 0 & 0 & 0 & 0 & -Z_{\dot{w}}w & Y_{\dot{v}}v \\ 0 & 0 & 0 & Z_{\dot{w}}w & 0 & -X_{\dot{u}}u \\ 0 & 0 & 0 & -Y_{\dot{v}}v & X_{\dot{u}}u & 0 \\ 0 & -Z_{\dot{w}}w & Y_{\dot{v}}v & 0 & -N_{\dot{r}}r & M_{\dot{q}}q \\ Z_{\dot{w}}w & 0 & -X_{\dot{u}}u & N_{\dot{r}}r & 0 & -K_{\dot{p}}p \\ -Y_{\dot{v}}v & X_{\dot{u}}u & 0 & -M_{\dot{q}}q & K_{\dot{p}}p & 0 \end{bmatrix}.$$

The added mass coefficients can be theoretically derived exploiting the geometry of the rigid body and, eventually, its symmetry [127], by applying the strip theory. For a cylindrical rigid body of mass $\overline{m}$, length $\overline{L}$, with circular section of radius $\overline{r}$, the following added mass coefficients can be derived [127]:

$$\begin{aligned} X_{\dot{u}} &= -0.1\overline{m} \\ Y_{\dot{v}} &= -\pi\rho\overline{r}^2\overline{L} \\ Z_{\dot{w}} &= -\pi\rho\overline{r}^2\overline{L} \\ K_{\dot{p}} &= 0 \\ M_{\dot{q}} &= -\frac{1}{12}\pi\rho\overline{r}^2\overline{L}^3 \\ N_{\dot{r}} &= -\frac{1}{12}\pi\rho\overline{r}^2\overline{L}^3. \end{aligned}$$

Notice that, despite (2.40), in this case it is $\boldsymbol{M}_A \geq \boldsymbol{O}$. This result is due to the geometrical approach to the derivation of $\boldsymbol{M}_A$. As a matter of fact, if a sphere submerged in a fluid is considered, it can be observed that a pure rotational motion of the sphere does not involve any fluid movement, i.e., it is not necessary to add an inertia term due to the fluid. This small discrepancy is just an example of the difficulty in representing with a closed set of equations a distributed phenomenon as fluid movement.

In [220] the added mass coefficients for an ellipsoid are derived.

In [145], and in the Appendix, the coefficients for the experimental vehicle NPS AUV Phoenix are reported. These coefficients have been experimentally derived and, since the vehicle can work at a maximum depth of few meters, i.e., it is not submerged in an unbounded fluid, the structure of $\boldsymbol{M}_A$ is not diagonal. To give an order of magnitude of the added mass terms, the vehicle has a mass of about 5000 kg, the term $X_{\dot{u}} \approx -500$ kg.

A detailed theoretical and experimental discussion on the added mass effect of a cylinder moving in a fluid can be found in [203] where it is shown that the added mass matrix is state-dependent and its coefficients are function of the distance traveled by the cylinder.

2.4.2 Damping Effects

The viscosity of the fluid also causes the presence of dissipative drag and lift forces on the body.

A common simplification is to consider only linear and quadratic damping terms and group these terms in a matrix $\boldsymbol{D}_{RB}$ such that:

$$\boldsymbol{D}_{RB}(\boldsymbol{\nu}) > \boldsymbol{O} \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6.$$

The coefficients of this matrix are also considered to be constant. For a completely submerged body, the following further assumption can be made:

$$\begin{aligned}\boldsymbol{D}_{RB}(\boldsymbol{\nu}) = &-\operatorname{diag}\{X_u, Y_v, Z_w, K_p, M_q, N_r\} + \\ &-\operatorname{diag}\left\{X_{u|u|}\,|u|\,, Y_{v|v|}\,|v|\,, Z_{w|w|}\,|w|\,, K_{p|p|}\,|p|\,, M_{q|q|}\,|q|\,, N_{r|r|}\,|r|\right\}.\end{aligned}$$

Assuming a diagonal structure for the damping matrix implies neglecting the coupling dissipative terms.

The detailed analysis of the dissipative forces is beyond the scope of this work. In the following, only the nature of these forces will be briefly discussed. Introductory analysis of this phenomenon can be found in [127, 157, 208, 220, 255], while in depth discussion in [253, 274].

The viscous effects can be considered as the sum of two forces, the *drag* and the *lift* forces. The former are parallel to the relative velocity between the body and the fluid, while the latter are normal to it. Both drag and lift forces are supposed to act on the center of mass of the body. In order to solve the distributed flow problem, an integral over the entire surface is required to compute the net force/moment acting on the body. Moreover, the model of drag and lift forces is not known and, also for some widely accepted models, the coefficients are not known and variables.

For a sphere moving in a fluid, the drag force can be modeled as [157]:

$$F_{drag} = 0.5\rho U^2 S C_d(R_n),$$

where ρ is the fluid density, U is the velocity of the sphere, S is the frontal area of the sphere, C_d is the adimensional drag coefficients and R_n is the Reynolds number. For a generic body, S is the projection of the frontal area along the flow direction. The drag coefficient is then dependent on the Reynolds number, i.e., on the laminar/turbulent fluid motion:

$$R_n = \frac{\rho\,|U|\,D}{\mu}$$

where D is the characteristic dimension of the body perpendicular to the direction of U and μ is the dynamic viscosity of the fluid. In Table 2.2 the drag coefficients in function of the Reynolds number for a cylinder are reported [255]. The drag coefficients can be considered as the sum of two physical effects: a frictional contribution of the surface whose normal is perpendicular to the flow velocity, and a pressure contribution of the surface whose normal is parallel to the flow velocity.

Table 2.2. Lift and Drag Coefficient for a cylinder

Reynolds number	regime motion	C_d	C_l
$R_n < 2 \cdot 10^5$	subcritical flow	1	$3 \div 0.6$
$2 \cdot 10^5 < R_n < 5 \cdot 10^5$	critical flow	$1 \div 0.4$	0.6
$5 \cdot 10^5 < R_n < 3 \cdot 10^5$	transcritical flow	0.4	0.6

The lift forces are perpendicular to the flow direction. For an hydrofoil they can be modeled as [157]:

$$F_{lift} = 0.5\rho U^2 S C_l(R_n, \alpha),$$

where C_l is the adimensional lift coefficient. It can be recognized that it also depends on the angle of attack α. In Table 2.2 the lift coefficients in function of the Reynolds number for a cylinder are reported [255].

Vortex induced forces are an oscillatory effect that affects both drag and lift directions. They are caused by the vortex generated by the body that separates the fluid flow. They then cause a periodic *disturbance* that can be the cause of oscillations in cables and some underwater structures. For underwater vehicles it is reasonable to assume that the vortex induced forces are negligible, this, also in view of the adoption of small design surfaces that can reduce this effect. For underwater manipulators with cylindrical links this effects might be experienced.

2.4.3 Current Effects

Control of marine vehicles cannot neglect the effects of specific disturbances such as waves, wind and ocean current. In this book wind and waves phenomena will not be discussed since the attention is focused to autonomous vehicles performing a motion or manipulation task in an underwater environment. However, if this task has to be achieved in very shallow waters, those effects can not be neglected.

Ocean currents are mainly caused by: tidal movement; the atmospheric wind system over the sea earth's surface; the heat exchange at the sea surface; the salinity changes and the Coriolis force due to the earth rotation. Currents can be very different due to local climatic and/or geographic characteristics; as an example, in the fjords, the tidal effect can cause currents of up to 3 m/s [127].

The effect of a small current has to be considered also in structured environments such as a pool. In this case, the refresh of the water is strong enough to affect the vehicle dynamics [34].

Let us assume that the ocean current, expressed in the inertial frame, $\boldsymbol{\nu}_c^I$ is constant and irrotational, i.e.,

$$\boldsymbol{\nu}_c^I = \begin{bmatrix} \nu_{c,x} \\ \nu_{c,y} \\ \nu_{c,z} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\dot{\boldsymbol{\nu}}_c^I = \mathbf{0}$; its effects can be added to the dynamic of a rigid body moving in a fluid simply considering the *relative* velocity in body-fixed frame

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{R}_I^B \boldsymbol{\nu}_c^I \tag{2.41}$$

in the derivation of the Coriolis and centripetal terms and the damping terms.

A simplified modeling of the current effect can be obtained by assuming the current irrotational and constant in the earth-fixed frame, its effect on the vehicle, thus, can be modeled as a constant disturbance in the earth-fixed frame that is further projected onto the vehicle-fixed frame. To this purpose, let define as $\boldsymbol{\theta}_{v,C} \in \mathbb{R}^6$ the vector of constant parameters contributing to the earth-fixed generalized forces due to the current; then, the vehicle-fixed current disturbance can be modelled as

$$\boldsymbol{\tau}_{v,C} = \boldsymbol{\Phi}_{v,C}(\boldsymbol{R}_B^I)\boldsymbol{\theta}_{v,C}, \tag{2.42}$$

where the (6×6) regressor matrix simply expresses the force/moment coordinate transformation between the two frames and it is given by

$$\boldsymbol{\Phi}_{v,C}(\boldsymbol{R}_B^I) = \begin{bmatrix} \boldsymbol{R}_I^B & \boldsymbol{O}_{3\times 3} \\ \boldsymbol{O}_{3\times 3} & \boldsymbol{R}_I^B \end{bmatrix}. \tag{2.43}$$

Notice that in [14, 35] compensation of the ocean current effects is obtained through a quaternion-based velocity/force mapping instead. Moreover, in some papers [34, 35, 125, 255], the effect of the current is simply modeled as a time-varying, vehicle-fixed, disturbance $\boldsymbol{\tau}_{v,C}$ that would lead to the trivial regressor

$$\boldsymbol{\Phi}'_{v,C} = \boldsymbol{I}_6. \tag{2.44}$$

2.5 Gravity and Buoyancy

"Ses deux mains s'accrochaient à mon cou; elles ne se seraient pas accrochées plus furieusement dans un naufrage. Et je ne comprenais pas si elle voulait que je la sauve, ou bien que je me noie avec elle".
Raymond Radiguet, "Le diable au corps" 1923.

When a rigid body is submerged in a fluid under the effect of the gravity two more forces have to be considered: the gravitational force and the buoyancy. The latter is the only hydrostatic effect, i.e., it is not function of a relative movement between body and fluid.

Let us define as

$$\boldsymbol{g}^I = \begin{bmatrix} 0 \\ 0 \\ 9.81 \end{bmatrix} \text{m/s}^2$$

the acceleration of gravity, ∇ the volume of the body and m its mass. The submerged weight of the body is defined as $W = m \left\| \boldsymbol{g}^I \right\|$ while its buoyancy $B = \rho \nabla \left\| \boldsymbol{g}^I \right\|$.

The gravity force, acting in the center of mass $\boldsymbol{r}_C^B$ is represented in body-fixed frame by:

$$\boldsymbol{f}_G(\boldsymbol{R}_I^B) = \boldsymbol{R}_I^B \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix},$$

while the buoyancy force, acting in the center of buoyancy $\boldsymbol{r}_B^B$ is represented in body-fixed frame by:

$$\boldsymbol{f}_B(\boldsymbol{R}_I^B) = -\boldsymbol{R}_I^B \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}.$$

The (6×1) vector of force/moment due to gravity and buoyancy in body-fixed frame, included in the left hand-side of the equations of motion, is represented by:

$$\boldsymbol{g}_{RB}(\boldsymbol{R}_I^B) = - \begin{bmatrix} \boldsymbol{f}_G(\boldsymbol{R}_I^B) + \boldsymbol{f}_B(\boldsymbol{R}_I^B) \\ \boldsymbol{r}_G^B \times \boldsymbol{f}_G(\boldsymbol{R}_I^B) + \boldsymbol{r}_B^B \times \boldsymbol{f}_G(\boldsymbol{R}_I^B) \end{bmatrix}.$$

In the following, the symbol $\boldsymbol{r}_G^B = \begin{bmatrix} x_G & y_G & z_G \end{bmatrix}^{\mathrm{T}}$ (with $\boldsymbol{r}_G^B = \boldsymbol{r}_C^B$) will be used for the center of gravity.

The expression of $\boldsymbol{g}_{RB}$ in terms of Euler angles is represented by:

$$\boldsymbol{g}_{RB}(\boldsymbol{\eta}_2) = \begin{bmatrix} (W-B)s_\theta \\ -(W-B)c_\theta s_\phi \\ -(W-B)c_\theta c_\phi \\ -(y_G W - y_B B)c_\theta c_\phi + (z_G W - z_B B)c_\theta s_\phi \\ (z_G W - z_B B)s_\theta + (x_G W - x_B B)c_\theta c_\phi \\ -(x_G W - x_B B)c_\theta s_\phi - (y_G W - y_B B)s_\theta \end{bmatrix}, \tag{2.45}$$

while in terms of quaternion is represented by:

$$\boldsymbol{g}_{RB}(\mathcal{Q}) = \begin{bmatrix} 2(\eta\varepsilon_2 - \varepsilon_1\varepsilon_3)(W - B) \\ -2(\eta\varepsilon_1 + \varepsilon_2\varepsilon_3)(W - B) \\ (-\eta^2 + \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2)(W - B) \\ (-\eta^2 + \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2)(y_G W - y_B B) + 2(\eta\varepsilon_1 + \varepsilon_2\varepsilon_3)(z_G W - z_B B) \\ -(-\eta^2 + \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2)(x_G W - x_B B) + 2(\eta\varepsilon_2 - \varepsilon_1\varepsilon_3)(z_G W - z_B B) \\ -2(\eta\varepsilon_1 + \varepsilon_2\varepsilon_3)(x_G W - x_B B) - 2(\eta\varepsilon_2 - \varepsilon_1\varepsilon_3)(y_G W - y_B B) \end{bmatrix}.$$

By looking at (2.45), it can be recognized that the difference between gravity and buoyancy $(W - B)$ only affects the linear force acting on the vehicle; it is also clear that the restoring linear force is constant in the earth-fixed frame. On the other hand, the two vectors of the first moment of inertia $W\boldsymbol{r}_G^B$ and $B\boldsymbol{r}_B^B$ affect the moment acting on the vehicle and are constant in the vehicle-fixed frame. In summary, the expression of the restoring vector is linear with respect to the vector of four constant parameters

$$\boldsymbol{\theta}_{v,R} = [\, W - B \quad x_G W - x_B B \quad y_G W - y_B B \quad z_G W - z_B B \,]^{\mathrm{T}} \tag{2.46}$$

through the (6×4) regressor

$$\boldsymbol{\Phi}_{v,R}(\boldsymbol{R}_B^I) = \begin{bmatrix} \boldsymbol{R}_I^B \boldsymbol{z} & \boldsymbol{O}_{3\times 3} \\ \boldsymbol{0}_{3\times 1} & \boldsymbol{S}\left(\boldsymbol{R}_I^B \boldsymbol{z}\right) \end{bmatrix}, \tag{2.47}$$

i.e.,

$$\boldsymbol{g}_{RB}(\boldsymbol{R}_B^I) = \boldsymbol{\Phi}_{v,R}(\boldsymbol{R}_B^I)\boldsymbol{\theta}_{v,R}.$$

In (2.47) $\boldsymbol{S}(\cdot)$ is the operator performing the cross product. Notice that, alternatively to (2.45), the restoring vector can be written in terms of quaternions; however, this would lead again to the regressor (2.47) and to the vector of dynamic parameters (2.46).

2.6 Thrusters' Dynamics

Underwater vehicles are usually controlled by thrusters (Figure 2.2) and/or control surfaces.

Control surfaces, such as rudders and sterns, are common in cruise vehicles; those are torpedo-shaped and usually used in cable/pipeline inspection. Since the force/moment provided by the control surfaces is function of the velocity and it is null in hovering, they are not useful to manipulation missions in which, due to the manipulator interaction, full control of the vehicle is required.

The relationship between the force/moment acting on the vehicle $\boldsymbol{\tau}_v \in \mathrm{I\!R}^6$ and the control input of the thrusters $\boldsymbol{u}_v \in \mathrm{I\!R}^{p_v}$ is highly nonlinear. It is function of some structural variables such as: the density of the water; the tunnel cross-sectional area; the tunnel length; the volumetric flowrate between input-output of the thrusters and the propeller diameter. The state

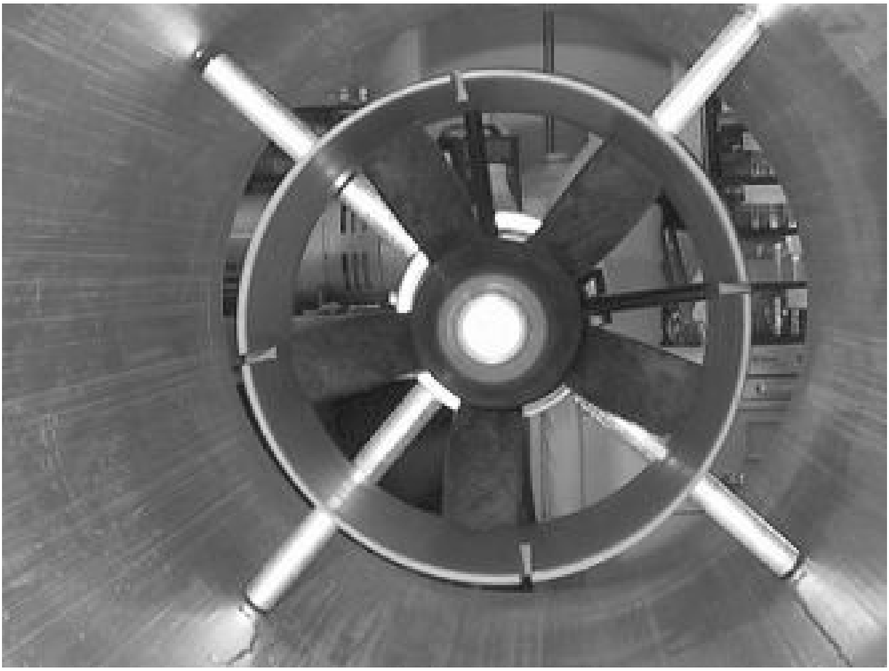

Fig. 2.2. Thruster of SAUVIM (courtesy of J. Yuh, Autonomous Systems Laboratory, University of Hawaii)

of the dynamic system describing the thrusters is constituted by the propeller revolution, the speed of the fluid going into the propeller and the input torque.

A detailed theoretical and experimental analysis of thrusters' behavior can be found in [40, 147, 176, 178, 220, 270, 300, 309]. Roughly speaking, thrusters are the main cause of limit cycle in vehicle positioning and bandwidth constraint.

A common simplification is to consider a linear relationship between $\boldsymbol{\tau}_v$ and $\boldsymbol{u}_v$:

$$\boldsymbol{\tau}_v = \boldsymbol{B}_v \boldsymbol{u}_v , \qquad (2.48)$$

where $\boldsymbol{B}_v \in \mathbb{R}^{6 \times p_v}$ is a known constant matrix known as the Thruster Control Matrix (TCM). Along the book, the matrix $\boldsymbol{B}_v$ will be considered square or low rectangular, i.e., $p_v \geq 6$. This means full control of force/moments of the vehicle.

As an example, ODIN has the following TCM:

$$\boldsymbol{B}_v = \begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ * & * & * & * & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.49}$$

where $*$ means a non-zero constant factor depending on the thruster allocation. Different TCM can be observed as in, e.g., the vehicle Phantom S3 manufactured by Deep Ocean Engineering that has 4 thrusters:

$$\boldsymbol{B}_v = \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix} \tag{2.50}$$

in which it can be recognized that not all the directions are independently actuated.

On the other hand, if the vehicle is controlled by thrusters, each of which is locally fed back, the effects of the nonlinearities discussed above is very limited and a linear input-output relation between desired force/moment and thruster's torque is experienced. This is the case, e.g., of ODIN [35, 83, 215, 216] where the experimental results show that the linear approximation is reliable.

2.7 Underwater Vehicles' Dynamics in Matrix Form

By taking into account the inertial generalized forces, the hydrodynamic effects, the gravity and buoyancy contribution and the thrusters' presence, it is possible to write the equations of motion of an underwater vehicle in matrix form:

$$\boldsymbol{M}_v \dot{\boldsymbol{\nu}} + \boldsymbol{C}_v(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}_{RB}(\boldsymbol{R}_B^I) = \boldsymbol{B}_v \boldsymbol{u}_v, \tag{2.51}$$

where $\boldsymbol{M}_v = \boldsymbol{M}_{RB} + \boldsymbol{M}_A$ and $\boldsymbol{C}_v = \boldsymbol{C}_{RB} + \boldsymbol{C}_A$ include also the added mass terms. Taking into account the current, a possible, approximated, model is given by:

$$\boldsymbol{M}_v \dot{\boldsymbol{\nu}} + \boldsymbol{C}_v(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}_{RB}(\boldsymbol{R}_B^I) = \boldsymbol{\tau}_v - \boldsymbol{\tau}_{v,C}. \tag{2.52}$$

The following properties hold:

- the inertia matrix is symmetric and positive definite, i.e., $\boldsymbol{M}_v = \boldsymbol{M}_v^{\mathrm{T}} > \boldsymbol{O}$;
- the damping matrix is positive definite, i.e., $\boldsymbol{D}_{RB}(\boldsymbol{\nu}) > \boldsymbol{O}$;

- the matrix $\boldsymbol{C}_v(\boldsymbol{\nu})$ is skew-symmetric, i.e., $\boldsymbol{C}_v(\boldsymbol{\nu}) = -\boldsymbol{C}_v^{\mathrm{T}}(\boldsymbol{\nu}), \forall \boldsymbol{\nu} \in \mathbb{R}^6$.

It is possible to rewrite the dynamic model (2.51) in terms of earth-fixed coordinates; in this case, the state variables are the (6×1) vectors $\boldsymbol{\eta}$, $\dot{\boldsymbol{\eta}}$ and $\ddot{\boldsymbol{\eta}}$. The equations of motion are then obtained, through the kinematic relations (2.1)–(2.2) as

$$\boldsymbol{M}_v^\star(\boldsymbol{R}_B^I)\ddot{\boldsymbol{\eta}} + \boldsymbol{C}_v^\star(\boldsymbol{R}_B^I, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} + \boldsymbol{D}_{RB}^\star(\boldsymbol{R}_B^I, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} + \boldsymbol{g}_{RB}^\star(\boldsymbol{R}_B^I) = \boldsymbol{\tau}_v^\star, \quad (2.53)$$

where [127]

$$\begin{aligned}
\boldsymbol{M}_v^\star &= \boldsymbol{J}_e^{-\mathrm{T}}(\boldsymbol{R}_B^I)\boldsymbol{M}_v\boldsymbol{J}_e^{-1}(\boldsymbol{R}_B^I)\\
\boldsymbol{C}_v^\star &= \boldsymbol{J}_e^{-\mathrm{T}}(\boldsymbol{R}_B^I)\left(\boldsymbol{C}_v(\boldsymbol{\nu}) - \boldsymbol{M}_v\boldsymbol{J}_e^{-1}(\boldsymbol{R}_B^I)\dot{\boldsymbol{J}}(\boldsymbol{R}_B^I)\right)\boldsymbol{J}_e^{-1}(\boldsymbol{R}_B^I)\\
\boldsymbol{D}_{RB}^\star &= \boldsymbol{J}_e^{-\mathrm{T}}(\boldsymbol{R}_B^I)\boldsymbol{D}_{RB}(\boldsymbol{\nu})\boldsymbol{J}_e^{-1}(\boldsymbol{R}_B^I)\\
\boldsymbol{g}_{RB}^\star &= \boldsymbol{J}_e^{-\mathrm{T}}(\boldsymbol{R}_B^I)\boldsymbol{g}_{RB}(\boldsymbol{R}_B^I)\\
\boldsymbol{\tau}_v^\star &= \boldsymbol{J}_e^{-\mathrm{T}}(\boldsymbol{R}_B^I)\boldsymbol{\tau}_v.
\end{aligned}$$

Again, the current can be taken into account by resorting to the relative velocity or, introducing an approximation, considering the following equations of motion:

$$\boldsymbol{M}_v^\star(\boldsymbol{R}_B^I)\ddot{\boldsymbol{\eta}} + \boldsymbol{C}_v^\star(\boldsymbol{R}_B^I, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} + \boldsymbol{D}_{RB}^\star(\boldsymbol{R}_B^I, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} + \boldsymbol{g}_{RB}^\star(\boldsymbol{R}_B^I) = \boldsymbol{\tau}_v^\star - \boldsymbol{\tau}_{v,C}^\star,$$

where $\boldsymbol{\tau}_{v,C}^\star \in \mathbb{R}^6$ is the disturbance introduced by the current. It is worth noticing that the earth-fixed and the body-fixed models with the introduction of the current as a simple external disturbance implies different dynamic properties. In particular, this is true if, in case of the design of a control action, the disturbance is considered as constant or slowly varying.

2.7.1 Linearity in the Parameters

Relation (2.51) can be written by exploiting the linearity in the parameters property. It must be noted that, while this property is proved for rigid bodies moving in the space [254], for underwater rigid bodies it depends on a suitable representations of the hydrodynamics terms. With a vector of parameters $\boldsymbol{\theta}_v$ of proper dimension it is possible to write the following:

$$\boldsymbol{\Phi}_v(\boldsymbol{R}_B^I, \boldsymbol{\nu}, \dot{\boldsymbol{\nu}})\boldsymbol{\theta}_v = \boldsymbol{\tau}_v. \quad (2.54)$$

The inclusion of the ocean current is straightforward by using the relative velocity as shown in Subsection 2.4.3. However, it might be useful to consider also the regressor form of the two approximations given by considering the current as an external disturbance. In particular, it is of interest to isolate the contribution of the restoring forces and current effects, those are the sole terms giving a non-null contribution to the dynamic with the vehicle still and for this reason will be defined as *persistent dynamic terms*.

Starting from the equation (2.52) let first consider the current as an external disturbance $\boldsymbol{\tau}_{v,C}$ constant in the body-fixed frame, it is possible to write:

$$\boldsymbol{M}_v\dot{\boldsymbol{\nu}} + \boldsymbol{C}_v(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}_{v,R}(\boldsymbol{R}_B^I)\boldsymbol{\theta}_{v,R} + \boldsymbol{\Phi}'_{v,C}\boldsymbol{\theta}_{v,C} = \boldsymbol{\tau}_v$$

that can be rewritten as:

$$\boxed{\boldsymbol{M}_v\dot{\boldsymbol{\nu}} + \boldsymbol{C}_v(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}_{v,P'}(\boldsymbol{R}_B^I)\boldsymbol{\theta}_{v,P} = \boldsymbol{\tau}_v} \tag{2.55}$$

with the use of the (6×10) regressor:

$$\boldsymbol{\Phi}_{v,P'}(\boldsymbol{R}_B^I) = \begin{bmatrix} \boldsymbol{R}_I^B\boldsymbol{z} & \boldsymbol{O}_{3\times3} & \boldsymbol{I}_3 & \boldsymbol{O}_{3\times3} \\ \boldsymbol{0}_{3\times1} & \boldsymbol{S}\left(\boldsymbol{R}_I^B\boldsymbol{z}\right) & \boldsymbol{O}_{3\times3} & \boldsymbol{I}_3 \end{bmatrix}.$$

On the other side the current can be modeled as constant in the earth-fixed frame and, merged again with the restoring forces contribution, gives the following

$$\boxed{\boldsymbol{M}_v\dot{\boldsymbol{\nu}} + \boldsymbol{C}_v(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}_{v,P}(\boldsymbol{R}_B^I)\boldsymbol{\theta}_{v,P} = \boldsymbol{\tau}_v} \tag{2.56}$$

with the use of the (6×9) regressor:

$$\boldsymbol{\Phi}_{v,P}(\boldsymbol{R}_B^I) = \begin{bmatrix} \boldsymbol{O}_{3\times3} & \boldsymbol{R}_I^B & \boldsymbol{O}_{3\times3} \\ \boldsymbol{S}\left(\boldsymbol{R}_I^B\boldsymbol{z}\right) & \boldsymbol{O}_{3\times3} & \boldsymbol{R}_I^B \end{bmatrix}.$$

It is worth noticing that the two regressors have different dimensions. In order to extrapolate the minimum number of independent parameters, i.e., the number of columns of the regressor, it is possible to resort to the numerical method proposed by Gautier [135] based on the Singular Value Decomposition.

Model (2.56) can by rewritten in a sole regressor of proper dimension yielding:

$$\boldsymbol{\Phi}_{v,T}(\boldsymbol{R}_B^I, \boldsymbol{\nu}, \dot{\boldsymbol{\nu}})\boldsymbol{\theta}_{v,T} = \boldsymbol{\tau}_v. \tag{2.57}$$

2.8 Kinematics of Manipulators with Mobile Base

In Figure 2.3 a sketch of an Underwater Vehicle-Manipulator System with relevant frames is shown. The frames are assumed to satisfy the Denavit-Hartenberg convention [254]. The position and orientation of the end effector, thus, is easily obtained by the use of homogeneous transformation matrices.

Let $\boldsymbol{q} \in \mathbb{R}^n$ be the vector of joint positions where n is the number of joints. The vector $\dot{\boldsymbol{q}} \in \mathbb{R}^n$ is the corresponding time derivative. Let define $\boldsymbol{\zeta} \in \mathbb{R}^{6+n}$ as

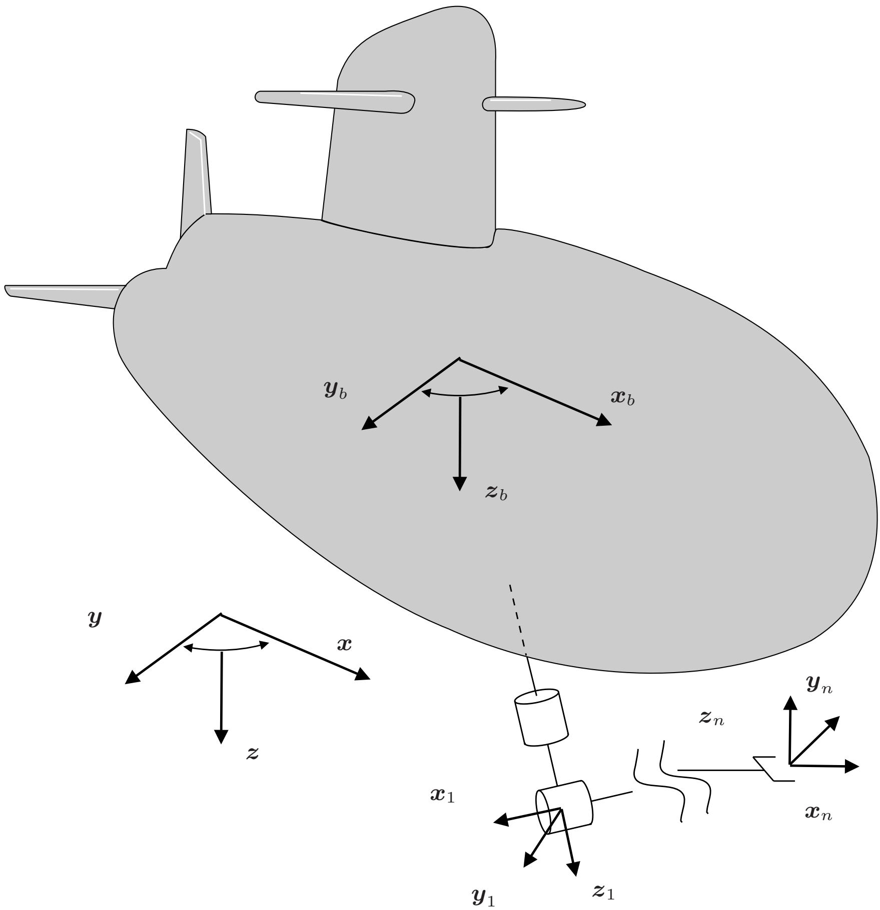

Fig. 2.3. Sketch of an Underwater Vehicle-Manipulator System with relevant frames

$$\zeta = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \dot{q} \end{bmatrix};$$

It is useful to rewrite the relationship between body-fixed and earth-fixed velocities given in equations (2.1)-(2.2) in a more compact form:

$$\zeta = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \dot{q} \end{bmatrix} = \begin{bmatrix} R_I^B & O_{3\times3} & O_{3\times n} \\ O_{3\times3} & J_{k,o}(R_I^B) & O_{3\times n} \\ O_{n\times3} & O_{n\times3} & I_n \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{q} \end{bmatrix} = J_k \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{q} \end{bmatrix}, \quad (2.58)$$

where $O_{n_1\times n_2}$ is the null $(n_1 \times n_2)$ matrix and the matrix $J_{k,o}(R_I^B)$ in terms of Euler angles has been introduced in (2.3).

Knowing $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$, $\dot{\boldsymbol{\nu}}_1$, $\dot{\boldsymbol{\nu}}_2$, (vehicle linear and angular velocities and acceleration in body fixed frame), $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$, (joint velocities and acceleration) it is possible to calculate, for every link, the following variables:

$\boldsymbol{\omega}_i^i$, angular velocity of the frame i,

$\dot{\boldsymbol{\omega}}_i^i$, angular acceleration of the frame i,

$\boldsymbol{v}_i^i$, linear velocity of the origin of the frame i,

$\boldsymbol{v}_{ic}^i$, linear velocity of the center of mass of link i,

$\boldsymbol{a}_i^i$, linear acceleration of the origin of frame i,

by resorting to the following relationships:

$$\boldsymbol{\omega}_i^i = \boldsymbol{R}_{i-1}^i \left(\boldsymbol{\omega}_{i-1}^{i-1} + \dot{q}_i \boldsymbol{z}_{i-1} \right) \tag{2.59}$$

$$\dot{\boldsymbol{\omega}}_i^i = \boldsymbol{R}_{i-1}^i \left(\dot{\boldsymbol{\omega}}_{i-1}^{i-1} + \boldsymbol{\omega}_{i-1}^{i-1} \times \dot{q}_i \boldsymbol{z}_{i-1} + \ddot{q}_i \boldsymbol{z}_{i-1} \right) \tag{2.60}$$

$$\boldsymbol{v}_i^i = \boldsymbol{R}_{i-1}^i \boldsymbol{v}_{i-1}^{i-1} + \boldsymbol{\omega}_i^i \times \boldsymbol{r}_{i-1,i}^i \tag{2.61}$$

$$\boldsymbol{v}_{ic}^i = \boldsymbol{R}_{i-1}^i \boldsymbol{v}_{i-1}^{i-1} + \boldsymbol{\omega}_i^i \times \boldsymbol{r}_{i-1,c}^i \tag{2.62}$$

$$\boldsymbol{a}_i^i = \boldsymbol{R}_{i-1}^i \boldsymbol{a}_{i-1}^{i-1} + \dot{\boldsymbol{\omega}}_i^i \times \boldsymbol{r}_{i-1,i}^i + \boldsymbol{\omega}_i^i \times \left(\boldsymbol{\omega}_i^i \times \boldsymbol{r}_{i-1,i}^i \right) \tag{2.63}$$

where $\boldsymbol{z}_i$ is the versor of frame i, $\boldsymbol{r}_{i-1,i}^i$ is the constant vector from the origin of frame $i-1$ toward the origin of frame i expressed in frame i.

Since the task of UVMS missions is usually force/position control of the end effector frame, it is necessary to consider the position of the end effector in the inertial frame, $\boldsymbol{\eta}_{ee1} \in \mathrm{I\!R}^3$; this is a function of the system configuration, i.e., $\boldsymbol{\eta}_{ee1}(\boldsymbol{\eta}_1, \boldsymbol{R}_I^B, \boldsymbol{q})$. The vector $\dot{\boldsymbol{\eta}}_{ee1} \in \mathrm{I\!R}^3$ is the corresponding time derivative.

Let us further define $\boldsymbol{\eta}_{ee2} \in \mathrm{I\!R}^3$ as the orientation of the end effector in the inertial frame expressed by Euler angles: also $\boldsymbol{\eta}_{ee2}$ is a function of the system configuration, i.e., $\boldsymbol{\eta}_{ee2}(\boldsymbol{R}_I^B, \boldsymbol{q})$. Again, the vector $\dot{\boldsymbol{\eta}}_{ee2} \in \mathrm{I\!R}^3$ is the corresponding time derivative.

The relation between the end-effector posture $\boldsymbol{\eta}_{ee} = [\, \boldsymbol{\eta}_{ee1}^{\mathrm{T}} \quad \boldsymbol{\eta}_{ee2}^{\mathrm{T}} \,]^{\mathrm{T}}$ and the system configuration can be expressed by the following nonlinear equation:

$$\boldsymbol{\eta}_{ee} = \boldsymbol{k}(\boldsymbol{\eta}, \boldsymbol{q}). \tag{2.64}$$

The vectors $\dot{\boldsymbol{\eta}}_{ee1}$ and $\dot{\boldsymbol{\eta}}_{ee2}$ are related to the body-fixed velocities $\boldsymbol{\nu}_{ee}$ via relations analogous to (2.1) and (2.2), i.e.,

$$\boldsymbol{\nu}_{ee1} = \boldsymbol{R}_I^n \dot{\boldsymbol{\eta}}_{ee1} \tag{2.65}$$

$$\boldsymbol{\nu}_{ee2} = \boldsymbol{J}_{k,o}(\boldsymbol{\eta}_{ee2}) \dot{\boldsymbol{\eta}}_{ee2} \tag{2.66}$$

where $\boldsymbol{R}_I^n$ is the rotation matrix from the inertial frame to the end-effector frame (i.e., frame n) and $\boldsymbol{J}_{k,o}$ is the matrix defined as in (2.3) with the use of the Euler angles of the end-effector frame. If the end-effector orientation is expressed via quaternion the relation between end-effector angular velocity and time derivative of the quaternion can be easily obtained by the *quaternion propagation equation* (2.10).

The end-effector velocities (expressed in the inertial frame) are related to the body-fixed system velocity by a suitable Jacobian matrix, i.e.,

$$\begin{bmatrix} \dot{\boldsymbol{\eta}}_{ee1} \\ \dot{\boldsymbol{\eta}}_{ee2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_{pos}(\boldsymbol{R}_B^I, \boldsymbol{q}) \\ \boldsymbol{J}_{or}(\boldsymbol{R}_B^I, \boldsymbol{q}) \end{bmatrix} \boldsymbol{\zeta} = \boldsymbol{J}_w(\boldsymbol{R}_B^I, \boldsymbol{q})\boldsymbol{\zeta}. \tag{2.67}$$

In Chapter 6, a different version of the (2.67) will be considered. To have a compact expression to the representation of the attitude error via quaternions, the end-effector velocities (expressed in the earth-fixed frame) are related to the body-fixed system velocity by the following Jacobian matrix:

$$\dot{\boldsymbol{x}}_E = \begin{bmatrix} \dot{\boldsymbol{\eta}}_{ee1} \\ \boldsymbol{R}_n^I \boldsymbol{\nu}_{ee2} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{R}_B^I, \boldsymbol{q})\boldsymbol{\zeta}. \tag{2.68}$$

Notice that the Jacobian has been derived with respect to the angular velocity of the end effector expressed in the earth-fixed frame (the matrix $\boldsymbol{R}_n^I = \boldsymbol{R}_I^{n^{\mathrm{T}}}$ is the rotation from the frame n the the earth-fixed frame).

2.9 Dynamics of Underwater Vehicle-Manipulator Systems

By knowing the forces acting on a body moving in a fluid it is possible to easily obtain the dynamics of a serial chain of rigid bodies moving in a fluid.

The inertial forces and moments acting on the generic body are represented by:

$$\begin{aligned} \boldsymbol{F}_i^i &= \boldsymbol{M}_i[\boldsymbol{a}_i^i + \dot{\boldsymbol{\omega}}_i^i \times \boldsymbol{r}_{i,c}^i + \boldsymbol{\omega}_i^i \times (\boldsymbol{\omega}_i^i \times \boldsymbol{r}_{i,c}^i)] \\ \boldsymbol{T}_i^i &= \boldsymbol{I}_i^i \dot{\boldsymbol{\omega}}_i^i + \boldsymbol{\omega}_i^i \times (\boldsymbol{I}_i^i \boldsymbol{\omega}_i^i), \end{aligned}$$

where $\boldsymbol{M}_i$ is the (3×3) mass matrix comprehensive of the added mass, $\boldsymbol{I}_i^i$ is the (3×3) inertia matrix plus added inertia with respect to the center of mass, $\boldsymbol{r}_{i,c}^i$ is the vector from the origin of frame i toward the center of mass of link i expressed in frame i.

Let us define $\boldsymbol{d}_i^i$ the drag and lift forces acting on the center of mass of link i, $\boldsymbol{r}_{i-1,i}^i$ the vector from the origin of frame $i-1$ to the origin of frame i expressed in frame i, $\boldsymbol{r}_{i-1,c}^i$ the vector from the origin of frame $i-1$ to the center of mass of link i expressed in frame i and $\boldsymbol{r}_{i-1,b}^i$ the vector from the origin of frame $i-1$ to the center of buoyancy of link i expressed in frame i,

$$\boldsymbol{g}^i = \boldsymbol{R}_I^i \boldsymbol{g}^I = \boldsymbol{R}_I^i \begin{bmatrix} 0 \\ 0 \\ 9.81 \end{bmatrix} \quad \mathrm{m/s}^2 .$$

The total forces and moments acting on the generic body of the serial chain are given by:

$$\begin{aligned}\boldsymbol{f}_i^i &= \boldsymbol{R}_{i+1}^i \boldsymbol{f}_{i+1}^{i+1} + \boldsymbol{F}_i^i - m_i \boldsymbol{g}^i + \rho \nabla_i \boldsymbol{g}^i + \boldsymbol{p}_i \\ \boldsymbol{\mu}_i^i &= \boldsymbol{R}_{i+1}^i \boldsymbol{\mu}_{i+1}^{i+1} + \boldsymbol{R}_{i+1}^i \boldsymbol{r}_{i-1,i}^{i+1} \times \boldsymbol{R}_{i+1}^i \boldsymbol{f}_{i+1}^{i+1} + \boldsymbol{r}_{i-1,c}^i \times \boldsymbol{F}_i^i + \boldsymbol{T}_i^i + \\ &\quad + \boldsymbol{r}_{i-1,c}^i \times (-m_i \boldsymbol{g}^i + \boldsymbol{d}_i) + \boldsymbol{r}_{i-1,b}^i \times \rho \nabla_i \boldsymbol{g}^i\end{aligned}$$

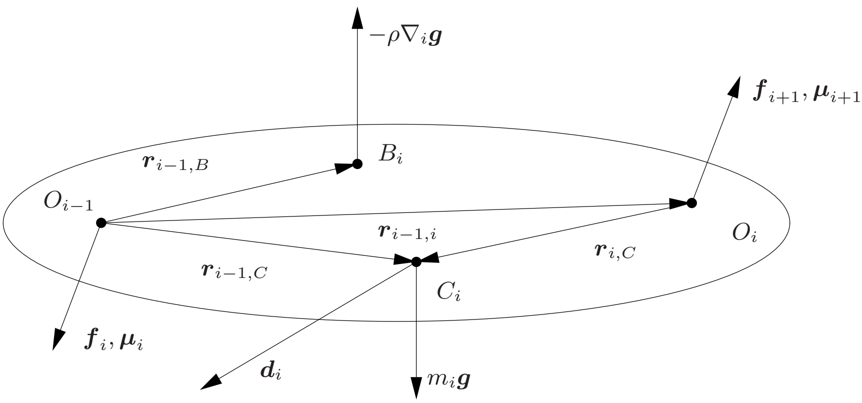

Fig. 2.4. Force/moment acting on link i

The torque acting on joint i is finally given by:

$$\tau_{q,i} = \boldsymbol{\mu}_i^{i\,\mathrm{T}} \boldsymbol{z}_{i-1}^i + f_{di} \mathrm{sign}(\dot{q}_i) + f_{vi} \dot{q}_i \tag{2.69}$$

with f_{di} and f_{vi} the motor dry and viscous friction coefficients.

Let us define $\boldsymbol{\tau}_q = \begin{bmatrix} \tau_{q,1} & \dots & \tau_{q,n} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ the vector of joint torques and $\boldsymbol{\tau} \in \mathbb{R}^{6+n}$

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_v \\ \boldsymbol{\tau}_q \end{bmatrix} \tag{2.70}$$

the vector of force/moment acting on the vehicle as well as joint torques. It is possible to write the equations of motions of an UVMS in a matrix form:

$$\boxed{\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\zeta}} + \boldsymbol{C}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} + \boldsymbol{D}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} + \boldsymbol{g}(\boldsymbol{q}, \boldsymbol{R}_B^I) = \boldsymbol{\tau}} \tag{2.71}$$

where $\boldsymbol{M} \in \mathbb{R}^{(6+n)\times(6+n)}$ is the inertia matrix including added mass terms, $\boldsymbol{C}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} \in \mathbb{R}^{6+n}$ is the vector of Coriolis and centripetal terms, $\boldsymbol{D}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} \in \mathbb{R}^{6+n}$ is the vector of dissipative effects, $\boldsymbol{g}(\boldsymbol{q}, \boldsymbol{R}_I^B) \in \mathbb{R}^{6+n}$ is the vector of gravity and buoyancy effects. The relationship between the generalized forces $\boldsymbol{\tau}$ and the control input is given by:

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_v \\ \boldsymbol{\tau}_q \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_v & \boldsymbol{O}_{6\times n} \\ \boldsymbol{O}_{n\times 6} & \boldsymbol{I}_n \end{bmatrix} \boldsymbol{u} = \boldsymbol{B}\boldsymbol{u}, \tag{2.72}$$

where $\boldsymbol{u} \in \mathbb{R}^{p_v+n}$ is the vector of the control input. Notice that, while for the vehicle a generic number $p_v \geq 6$ of control inputs is assumed, for the manipulator it is supposed that n joint motors are available.

It can be proven that:

- The inertia matrix $\boldsymbol{M}$ of the system is symmetric and positive definite:

  $$\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} > \boldsymbol{O}$$

  moreover, it satisfies the inequality

  $$\lambda_{\min}(\boldsymbol{M}) \leq \|\boldsymbol{M}\| \leq \lambda_{\max}(\boldsymbol{M}),$$

  where $\lambda_{\min}(\boldsymbol{M})$ ($\lambda_{\max}(\boldsymbol{M})$) is the minimum (maximum) eigenvalue of $\boldsymbol{M}$.
- For a suitable choice of the parametrization of $\boldsymbol{C}$ and if all the single bodies of the system are symmetric, $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is skew-symmetric [67]

  $$\boldsymbol{\zeta}^{\mathrm{T}} \left(\dot{\boldsymbol{M}} - 2\boldsymbol{C} \right) \boldsymbol{\zeta} = 0$$

  which implies

  $$\dot{\boldsymbol{M}} = \boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}$$

  moreover, the inequality

  $$\|\boldsymbol{C}(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{c}\| \leq C_M \|\boldsymbol{b}\| \|\boldsymbol{c}\|$$

  and the equality

  $$\boldsymbol{C}(\boldsymbol{a}, \alpha_1 \boldsymbol{b} + \alpha_2 \boldsymbol{c}) = \alpha_1 \boldsymbol{C}(\boldsymbol{a}, \boldsymbol{b}) + \alpha_2 \boldsymbol{C}(\boldsymbol{a}, \boldsymbol{c})$$

  hold.
- The matrix $\boldsymbol{D}$ is positive definite

  $$\boldsymbol{D} > \boldsymbol{O}$$

  and satisfies

  $$\|\boldsymbol{D}(\boldsymbol{q}, \boldsymbol{a}) - \boldsymbol{D}(\boldsymbol{q}, \boldsymbol{b})\| \leq D_M \|\boldsymbol{a} - \boldsymbol{b}\| .$$

In [255], it can be found the mathematical model written with respect to the earth-fixed-frame-based vehicle position and the manipulator end-effector. However, it must be noted that, in that case, a 6-dimensional manipulator is considered in order to have square Jacobian to work with; moreover, kinematic singularities need to be avoided.

Reference [174] reports some interesting dynamic considerations about the interaction between the vehicle and the manipulator. The analysis performed allows to divide the dynamics in separate meaningful terms.

2.9.1 Linearity in the Parameters

UVMS have a property that is common to most mechanical systems, e.g., serial chain manipulators: linearity in the dynamic parameters. Using a suitable mathematical model for the hydrodynamic forces, (2.71) can be rewritten in a matrix form that exploits this property:

$$\boldsymbol{\Phi}(\boldsymbol{q}, \boldsymbol{R}_B^I, \boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}})\boldsymbol{\theta} = \boldsymbol{\tau} \tag{2.73}$$

with $\boldsymbol{\Phi} \in R^{(6+n)\times n_\theta}$, being n_θ the total number of parameters. Notice that n_θ depends on the model used for the hydrodynamic generalized forces and joint friction terms. For a single rigid body the number of dynamic parameter $n_{\theta,v}$ is a number greater than 100 [127]. For an UVMS it is $n_\theta = (n+1) \cdot n_{\theta,v}$, that gives an idea of the complexity of such systems.

Differently from ground fixed manipulators, in this case the number of parameters can not be reduced because, due to the 6 degrees of freedom (DOFs) of the sole vehicle, all the dynamic parameters provide an individual contribution to the motion.

2.10 Contact with the Environment

If the end effector of a robotic system is in contact with the environment, the force/moment at the tip of the manipulator acts on the whole system according to the equation ([254])

$$\boxed{\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\zeta}} + \boldsymbol{C}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} + \boldsymbol{D}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} + \boldsymbol{g}(\boldsymbol{q}, \boldsymbol{R}_B^I) = \boldsymbol{\tau} + \boldsymbol{J}_w^{\mathrm{T}}(\boldsymbol{q}, \boldsymbol{R}_B^I)\boldsymbol{h}_e,} \tag{2.74}$$

where $\boldsymbol{J}_w$ is the Jacobian matrix defined in (2.67) and the vector $\boldsymbol{h}_e \in \mathbb{R}^6$ is defined as

$$\boldsymbol{h}_e = \begin{bmatrix} \boldsymbol{f}_e \\ \boldsymbol{\mu}_e \end{bmatrix}$$

i.e., the vector of force/moments at the end effector expressed in the inertial frame. If it is assumed that only linear forces act on the end effector equation (2.74) becomes

$$\boxed{\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\zeta}} + \boldsymbol{C}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} + \boldsymbol{D}(\boldsymbol{q}, \boldsymbol{\zeta})\boldsymbol{\zeta} + \boldsymbol{g}(\boldsymbol{q}, \boldsymbol{R}_B^I) = \boldsymbol{\tau} + \boldsymbol{J}_{pos}^{\mathrm{T}}(\boldsymbol{q}, \boldsymbol{R}_B^I)\boldsymbol{f}_e} \tag{2.75}$$

Contact between the manipulator and the environment is usually difficult to model. In the following the simple model constituted by a frictionless and elastically compliant plane will be considered. The force at the end effector is then related to the deformation of the environment by the following simplified model [79] (see Figure 2.5)

$$\boldsymbol{f}_e = \boldsymbol{K}(\boldsymbol{x} - \boldsymbol{x}_e), \tag{2.76}$$

where $\boldsymbol{x}$ is the position of the end effector expressed in the inertial frame, $\boldsymbol{x}_e$ characterizes the constant position of the unperturbed environment expressed in the inertial frame and

$$\boldsymbol{K} = k\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}, \tag{2.77}$$

with $k > 0$, is the stiffness matrix being $\boldsymbol{n}$ the vector normal to the plane [194].

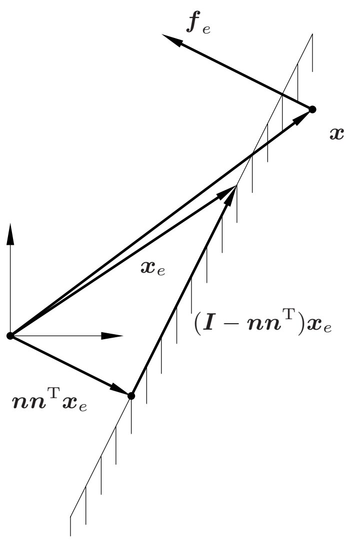

Fig. 2.5. Planar view of the chosen model for the contact force

In our case it is $\boldsymbol{x} = \boldsymbol{\eta}_{ee1}$; however, in the force control chapter, the notation $\boldsymbol{x}$ will be maintained.

2.11 Identification

Identification of the dynamic parameters of underwater robotic structures is a very challenging task. The mathematical model shares its main characteristics with the model of a ground-fixed industrial manipulator, e.g., it is non-linear and coupled. In case of underwater structure, however, the hydrodynamic terms are approximation of the physical effects. The actuation system of the vehicle is achieved mainly by the thrusters the models of which are still object of research. Finally, accurate measurement of the whole configuration is not easy. For these reasons, while from the mathematical aspect the problem is not new, from the practical point of view it is very difficult to set-up a systematic and reliable identification procedure for UVMSs. At the best of our knowledge, there is no significant results in the identification of full

UVMSs model. Few experimental results, moreover, concern the sole vehicle; often driven in few DOFs.

Since most of the fault detection algorithms rely on the accuracy of the mathematical model, this is also the reason why in this domain too, there are few experimental results (see Chapter 4).

In [2] the hydrodynamic damping terms of the vehicle Roby 2 developed at the Naval Automation Institute, National Research Council, Italy, (now CNR-ISSIA) have been experimentally estimated and further used to develop fault detection/tolerance strategies. The vehicle is stable in roll and pitch, hence, considering a constant depth, the sole planar model is identified.

In [227] some sea trials have been set-up in order to estimate the hydrodynamic derivatives of an 1/3-scale PAP-104 mine countermeasures ROV. The paper assumes that the added mass is already known, moreover, the identification concerns the planar motion for the 6-DOFs model. The position of the vehicle is measured by means of a redundant acoustic system; all the measurements are fused in an EKF in order to obtain the optimum state estimation. Experimental results are given.

The work [271] reports some experimental results on the single DOF models for the ROV developed at the John Hopkins University (JHUROV). First, the mathematical model is written so as to underline the Input-to-State-Stability; then a stable, on-line, adaptive identification technique is derived. The latter method is compared with a classic, off-line, Least-Squares approach. The approximation required by the proposed technique is that the equations of motion are decoupled, diagonal, there is no tether disturbance and the added mass is constant. Interesting experimental results are reported.

The interested reader can refer also to [62, 111].