# Entropy and Subsethood for General Interval-Valued Intuitionistic Fuzzy Sets

Xiao-dong Liu, Su-hua Zheng, and Feng-lan Xiong

Department of Mathematics, Ocean University of China,
Qingdao 266071, P.R. China
yingmu1229@sina.com, su-hua-2003@163.com, yingmu1229@hotmail.com

**Abstract.** In this paper, we mainly extend entropy and subsethood from intuitionistic fuzzy sets to general interval-valued intuitionistic fuzzy sets, propose a definition of entropy and subsethood , offer a function of entropy and construct a class of subsethood function. Then from discussing the relationship between entropy and subsethood, we know that while choosing the subsethood, we can get some kinds of function of entropy based on subsethood. Our work is also applicable to practical fields such as: neural networks, expert systems, and other.

## 1 Introduction

Since L.A.Zadeh introduced fuzzy sets [1] in 1965, a lot of new theories treating imprecision and uncertainty have been introduced. Some of them are extensions of fuzzy set theory. K.T.Atanassov extended this theory, proposed the definition of intuitionistic fuzzy sets ($IFS$, for short) ([3] [4]) and interval-valued intuitionistic fuzzy sets ($IVIFS$, for short) [5], which have been found to be highly useful to deal with vagueness out of several higher order fuzzy sets. And then, in the year 1999, Atanassov defined a Lattice-intuitionistic fuzzy set [6].

A measure of fuzziness often used and cited in the literature is entropy first mentioned in 1965 by L.A.Zadeh [2]. The name entropy was chosen due to an intrinsic similarity of equations to the ones in the Shannon entropy [7]. But they are different in types of uncertainty and Shannon entropy basically measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment. This theory was extended and a non-probabilistic-type entropy measure for $IFS$ was proposed by Eulalia Szmidt, Janusz Kacprzyk [8]. We extended it again onto general interval-valued intuitionistic fuzzy sets ($VIFS$, for short) in this paper.

We organize this paper as follows: in section 2, at first, we define the definition of $VIFS$ , then discuss the relation between $VIFS$ and some other kinds of intuitionistic fuzzy sets. From the discussion, we make sure that all these sets are subset of $VIFS$ in the view of isomorphic imbedding mapping, and this ensures that our work about the theory of entropy and subsethood is a reasonable extension of $FS$, $IFS$ and $IVIFS$. So according to section 2, theories in this paper are feasible on all these kinds of intuitionistic fuzzy sets. In section 3, we

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mainly extend entropy [7] onto $VIFS(X)$. Firstly, we give a reasonable definition of $A$ is less fuzzy than $B$ on $VIFS$. For there is not always a comparable relation between any two elements of $L$, which is a lattice. So we premise the definition of A refines B with the discussion of the incomparable condition. Then we definite entropy by giving the properties of $A$ refines $B$ in theorem 3.1. Finally from theorem 3.2, we get a idiographic entropy function. In section 4, we definite a class of subsethood function by three real functions. Then, from discussing the relation between entropy and subsethood, we get a class of entropy function which is useful in different conditions and eg 2 ensures that is reasonable. Thus, while choosing the subsethood, we can get some kinds of function of entropy based on subsethood.

## 2 Definitions and Quantities of Some Kinds of IFS

Throughout this paper, let $X$ is a nonempty definite set,$|X| = n$.

**Definition 2.1.** An interval number over $[0, 1]$ is defined as an object of the form: $\underline{a} = [a^-, a^+]$ with the property: $0 \leq a^- \leq a^+ \leq 1$ .

Let $L$ denote all interval numbers over $[0, 1]$. We define the definition of relation (it is specified by " $\leq$ "):

$$\underline{a} \leq \underline{b} \Leftrightarrow a^- \leq b^- \text{ and } a^+ \leq b^+ \ .$$

We can easily prove that " $\leq$ " is a partially ordered relation on $L$. So $< L,$ " $\leq$ " $>$ is a complete lattice where

$$\underline{a} \vee \underline{b} = [a^- \vee b^-, a^+ \vee b^+]\ ,\ \ \underline{a} \wedge \underline{b} = [a^- \wedge b^-, a^+ \wedge b^+]\ ,$$
$$\bigvee_{i\in I} \underline{a}_i = [\bigvee_{i\in I} a_i^-\ \bigvee_{i\in I} a_i^+] \quad ,\ \bigwedge_{i\in I} \underline{a}_i = [\bigwedge_{i\in I} a_i^-,\ \bigwedge_{i\in I} a_i^+] \quad .$$

Let $\underline{1} = [1, 1]$ denotes the maximum of $< L,$ " $\leq$ " $>$ and $\underline{0} = [0, 0]$ denotes the minimum of $< L,$ " $\leq$ " $>$.

We define the "$C$" operation, where $\quad \underline{a}^c = [1 - a^+, 1 - a^-]$ .

We can easily prove that the complement operation "$C$" has the following properties:

1. $(\underline{a}^c)^c = \underline{a}$ , 2. $(\underline{a} \vee \underline{b})^c = (\underline{a}^c \wedge \underline{b}^c), (\underline{a} \wedge \underline{b})^c = (\underline{a}^c \vee \underline{b}^c)$ ,
3. $(\bigvee_{i\in I} \underline{a}_i)^c = \bigwedge_{i\in I} \underline{a}_i^c, (\bigwedge_{i\in I} \underline{a}_i)^c = \bigvee_{i\in I} \underline{a}_i^c$ , 4. If $\underline{a} \leq \underline{b}$ , then $(\underline{a}^c \geq \underline{b}^c)$ $\quad \forall \underline{a}, \underline{b} \in L$ .

Then we will give the definition of intuitionistic fuzzy sets on $L$ .

**Definition 2.2.** A general interval valued intuitionistic fuzzy sets $A$ on $X$ is an object of the form: $\quad A = \{< x, \mu_A(x), \nu_A(x), x \in X >\}$ ,
where $\quad \mu_A : X \to L$ and $\nu_A : X \to L$ ,
with the property: $\quad \underline{0} \leq \mu_A(x) \leq \nu_A(x)^c \leq \underline{1} \quad (\forall x \in x)$.
For briefly, we denote $A = \{< x, \mu_A(x), \nu_A(x) >, x \in X\}$ by $A =< \mu_A(x), \nu_A(x) >$ and denote $\quad \mu_A(x) = [\mu_A^-(x), \mu_A^+(x)] \in L, \nu_A(x) = [\nu_A^-(x), \nu_A^+(x)] \in L$ .
Let $VIFS(X)$ denote all set of general interval valued intuitionistic fuzzy sets on $L$ . We define the relation " $\leq$ ": $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ .
We can easily prove that $\leq$ is also a partially ordered relation on $VIFS(X)$. So $< VIFS(X),$ " $\leq$ " $>$ is a complete lattice where

$$A \vee B = [\mu_A(x) \vee \mu_B(x),\ \nu_A(x) \wedge \nu_B(x)]\ , \quad \bigvee_{i\in I} A_i = [\bigvee_{i\in I} \mu_{A_i},\ \bigwedge_{i\in I} \nu_{A_i}]\ ,$$
$$A \wedge B = [\mu_A(x) \wedge \mu_B(x),\ \nu_A(x) \vee \nu_B(x)]\ , \quad \bigwedge_{i\in I} A_i = [\bigwedge_{i\in I} \mu_{A_i},\ \bigvee_{i\in I} \nu_{A_i}]\ .$$

We define the "$C$" operation, where $A^c =< \nu_A(x)\ ,\ \mu_A(x) >$ . We can easily prove that the complement operation "$C$" has the following properties:
1. $(A^c)^c = A$ , 2. $(A \vee B)^c = (A^c \wedge B^c)$, 3. $(\bigvee_{i\in I} A_i)^c = \bigwedge_{i\in I} A_i^c, (\bigwedge_{i\in I} A_i)^c = \bigvee_{i\in I} A_i^c$,
4. If $A \leq B$, then $(A^c \geq B^c)\quad \forall A, B \in VIFS(X)$.
So we know $< VIFS(X), \text{“} \leq \text{”} >$ is a complete lattice with complement . Let $I$ denotes the maximum of $VIFS(X)$ and $\theta$ denotes the minimum of $VIFS(X)$, Where $\mu_I(x) = \underline{1}$ , $\nu_I(x) = \underline{0}$ , $\mu_\theta(x) = \underline{0}$ , $\nu_\theta(x) = \underline{1}$. Then we will discuss the relationship between $IVIFS(X)$[5] and $VIFS(X)$ .

**Definition 2.3.** An interval-valued intuitionistic fuzzy set on $X$ ( $IVIFS(X)$, for short) is an object of the form $B = \{< x, M_B(x), N_B(x), x \in X >\}$ .
Where $M_B : X \to L$ and $N_B : X \to L$ $(\forall x \in X)$ ,
With the condition $0 \leq M_B^+(x) + N_B^+(x) \leq 1 \qquad (\forall x \in X)$.

**Proposition 2.1.** $IVIFS(X)$[5] is a subset of $VIFS(X)$ .

**Proof.** To any $B$ belongs to $IVIFS(X)$[5], we have
$B = \{< x, M_B(x), N_B(x), x \in X >\}$ , Where $M_B : X \to L$ and $N_B : X \to L$
With the condition $0 \leq M_B^+(x) + N_B^+(x) \leq 1 \quad (\forall x \in X)$ ,
that is $0 \leq M_B^+(x) \leq 1 - N_B^+(x) \quad (\forall x \in X)$. from definition 2.2 we can easily prove $0 \leq M_B(x) \leq N_B(x)^C \leq 1 \quad (\forall x \in X)$. That is $B$ belongs to $VIFS(X)$ . So $IVIFS(X)$[5] is a subset of $VIFS(X)$ . We have an example here:

**Eg1.** Let $A = \{< x, [0, 0.3], [0, 0.8] >, x \in X\}$ for $[0, 0.8]^c = [0.2, 1], [0, 0.3] \leq [0.2, 1]$, we have $A \in VIFS(X)$. But from $0.3 + 0.8 \geq 1$, we get $A \notin VIFS(X)$. Then we get $IVIFS(X) \subset VIFS(X)$ and $IVIFS(X) \neq VIFS(X)$. It is shown in the Theorem 2.1 that in the view of isomorphism insertion, we consider $P(X)$ is a subset of $VIFS(X)$.

**Definition 2.4.** A function $P(X) \to VIFS(X)$ ,where
$$\mu_{\delta(A)}(x) = \begin{cases} \underline{1}\ , & \text{if } x \in A, \\ \underline{0}\ , & \text{if } x \notin A \end{cases} ;\ \nu_{\delta(A)}(x) = \begin{cases} \underline{0}\ , & \text{if } x \in A, \\ \underline{1}\ , & \text{if } x \notin A \end{cases} \quad (\forall A \in P(X))\ .$$

**Theorem 2.1.** The function $\delta$ is a injection map reserving union, intersection and complement.

**Proof.** Let $A_i \in I$ ,
1. It is easy to prove that $\delta$ is a injection .
2. $\mu_{\delta(\bigvee_{i\in I} A_i)}(x) = \underline{1} \Leftrightarrow x \in \bigvee_{i\in I} A_i \Leftrightarrow \exists i_0 \in I$ satisfying $x \in A_{i_0}$ ,
$\bigvee_{i\in I} \mu_{\delta(A_i)}(x) = \underline{1} \Leftrightarrow \exists i_0 \in I,\ \mu_{\delta(A_{i_0})}(x) = \underline{1} \Leftrightarrow \exists i_0 \in I$ satisfying $x \in A_{i_0}$ ,
That is $\mu_{\delta(\bigvee_{i\in I} A_i)}(x) = \bigvee_{i\in I} \mu_{\delta(A_i)}(x) \quad (\forall x \in X)$ ,

$$\nu_{\delta(\bigvee_{i\in I} A_i)}(x) = \underline{1} \Leftrightarrow x \notin \bigvee_{i\in I} A_i \Leftrightarrow \forall i \in I \quad \text{there is} \quad x \notin A_i \ .$$

$$\bigwedge_{i\in I} \nu_{\delta(A_i)}(x) = \underline{1} \Leftrightarrow \forall i \in I, \ \nu_{\delta(A_i)}(x) = \underline{1} \Leftrightarrow \forall i \in I \ , \ \text{there is} \ \ x \notin A_i \ .$$

That is $\nu_{\delta(\bigvee_{i\in I} A_i)}(x) = \bigwedge_{i\in I} \nu_{\delta(A_i)}(x)$ $(\forall x \in X)$. So we have $\delta_{(\bigvee_{i\in I} A_i)} = \bigvee_{i\in I} \delta(A_i)$.

3. Similarly, we can prove $\delta_{(\bigwedge_{i\in I} A_i)} = \bigwedge_{i\in I} \delta(A_i)$ ,

4.
$$\mu_{\delta(A^c)}(x) = \underline{1} \Leftrightarrow x \in A^c \Leftrightarrow x \notin A \ , \tag{1}$$
$$\nu_{\delta(A^c)} = \underline{1} \Leftrightarrow x \notin A^c \Leftrightarrow x \in A \ , \tag{2}$$
$$\mu_{(\delta(A))^c}(x) = \nu_{(\delta(A))}(x) = \underline{1} \Leftrightarrow x \notin A \ , \tag{3}$$
$$\nu_{(\delta(A))^c}(x) = \mu_{(\delta(A))}(x) = \underline{1} \Leftrightarrow x \in A \ . \tag{4}$$

From (1) and (3) we have

$$\mu_{\delta(A^c)}(x) = \mu_{(\delta(A))^c}(x) \quad (\forall x \in X) \ , \ \ \nu_{\delta(A^c)} = \nu_{(\delta(A))^c}(x) \quad (\forall x \in X) \ .$$

Then we get $\delta(A^c) = (\delta(A))^c$. So $\delta$ is a injection map reserving union, intersection and complement. In the view of imbedding mapping, we may consider $P(X)$ as the subset of $VIFS(X)$. Let $F(X)$ is the class of all fuzzy sets proposed by L.A.Zadeh, let $\xi$: $F(x) \to VIFS(X)$ ,$\xi(F) = A =< \mu_A(x), \ \nu_A(x) >$ , $\mu_A(x) = [F(x), F(x)]$ , $\nu_A(x) = [1 - F(x), 1 - F(x)]$ .

So that the injection $\xi$ holds union, intersection and complement.

In the same way, we may also consider $F(X)$ as the subset of $VIFS(X)$in the view of imbedding mapping.

**Definition 2.5.** An intuitionistic fuzzy set on $X$ ($IFS(X)$ , for short) is an object of the form: $A = \{< x, \mu_A(x), \nu_A(x), x \in X >\}$ ,
where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ ,
with the property: $0 \le \nu_A(x) + \mu_A(x) \le 1 \quad (\forall x \in x)$.

Let's discuss the relationship between $IFS(X)$[4] and $VIFS(X)$ .
Let $\eta$: $IFS(X) \to VIFS(X)$ Where any $C$ belongs to $IFS(X)$[4],

$$C = \{< x, \alpha(x), \beta(x) >, x \in X\} \ \ (\alpha(x) + \beta(x) \le 1),$$

$$\eta(C) = A =< \mu_A(x), \nu_A(x) >, \ \mu_A(x) = [\alpha(x), \alpha(x)] \ , \ \nu_A(x) = [\beta(x), \beta(x)] \ .$$

Similarly we have $IFS(X)$ is the subset of $VIFS(X)$ in the view of imbedding mapping. All the above ensure that what we get in this paper are feasible on $P(X), F(X), IFS(X)$ and $IVIFS(X)$.

A method of judging whether set $A \in VIFS(X)$ is a subset of $P(X)$, is given by the following theorem .

**Theorem 2.2.** Let $A \in VIFS(X)$ , so we have $A \vee A^c = I \Leftrightarrow A \in \delta(P(X))$ .

**Proof.** " $\Rightarrow$ " Let $A =< \mu_A(x), \nu_A(x) >$ , from definition 1) we have:

$$A^c =< \nu_A(x), \mu_A(x) >, \ \ \mu_{A\vee A^c} = \mu_A(x) \vee \nu_A(x), \ \ \nu_{A\vee A^c} = \mu_A(x) \wedge \nu_A(x) \ .$$

From $\mu_A(x) \vee \nu_A(x) = \underline{1}$ , we have $\mu_A^-(x) \vee \nu_A^-(x) = 1$ ; (1)
From $\mu_A(x) \wedge \nu_A(x) = \underline{0}$ , we have $\mu_A^+(x) \wedge \nu_A^+(x) = 0$ . (2)
There are two different cases:

1. If $\nu_A^+(x) = 0$ , then we have $\nu_A^-(x) = 0$ , that is $\nu_A(x) = \underline{0}$ , and from (1) we have $\mu_A^-(x) = 1$ . So we get $\mu_A^+(x) = 1$, that is $\mu_A(x) = \underline{1}$ ;
2. If $\mu_A^+(x) = 0$ then we have $\mu_A^-(x) = 0$ , that is$\mu_A(x) = \underline{0}$ .

And from (1) we have $\nu_A^-(x) = 1$ . So we get $\nu_A^+(x) = 1$ that is $\nu_A(x) = \underline{1}$ . So we know that for any $x$ belongs to $X$, $\mu_A(x)$ and $\nu_A(x)$ can only be $\underline{0}$ or $\underline{1}$ with the following property: $\mu_A(x) = \underline{1} \Leftrightarrow \nu_A(x) = \underline{0}$ . Let $K = \{x | x \in X, \mu_A(x) = \underline{1}\}$ , from definition we have: $\mu_{\delta(K)}(x) = \underline{1} \Leftrightarrow x \in K \Leftrightarrow \mu_A(x) = \underline{1}$ ; $\nu_{\delta(K)}(x) = \underline{0} \Leftrightarrow x \in K \Leftrightarrow \mu_A(x) = \underline{1} \Leftrightarrow \nu_A(x) = \underline{0}$ $(\forall x \in X)$ . So we get $\mu_{\delta(K)}(x) = \mu_A(x), \nu_{\delta(K)}(x) = \nu_A(x)$ $(\forall x \in X)$ . That is $\delta(K) = A$ .
" $\Leftarrow$ " It is straight-forward .

**Corollary 2.1.** $A \wedge A^c = \theta \Leftrightarrow A \in \delta(P(X))$ .
From proposition 2.2 to proposition 2.4, we give some qualities of $VIFS$.

**Proposition 2.2.** Let $A \in VIFS(X)$,then we have:
1. $\mu_A(x) = \underline{1} \Rightarrow \nu_A(x) = \underline{0}$ $(\forall x \in X)$, 2. $\nu_A(x) = \underline{1} \Rightarrow \mu_A(x) = \underline{0}$ $(\forall x \in X)$.

**Proof.** 1. For $\nu_A(x) \leq (\mu_A(x))^c = (\underline{1})^c = \underline{0}$ , we have $\nu_A = \underline{0}$ . Similarly we can prove 2.

**Proposition 2.3.** $A \vee A^c \neq \theta$ $(\forall A \in VIFS(X))$

**Proof.** Let $A =< \mu_A(x), \nu_A(x) >$
then $A^c =< \nu_A(x), \mu_A(x) >$, $A \vee A^c =< \mu_A(x) \vee \nu_A(x), \nu_A(x) \wedge \mu_A(x) >$.
If $A \vee A^c = \theta$ , then we have $\mu_A(x) \vee \nu_A(x) = \underline{0}$, $\nu_A(x) \wedge \mu_A(x) = \underline{1}$ .
It is contradiction. Hence $A \vee A^c \neq \theta$ $(\forall A \in VIFS(X))$.

**Proposition 2.4.** Let $A \in VIFS(x)$ ,Then we have $\nu_{A \vee A^c}(x) = \nu_{A \wedge A^c}(x) \Leftrightarrow \mu_A(x) = \nu_A(x)$ $(\forall x \in X)$ .
**Proof.** " $\Rightarrow$ " From the definition2.2,

$$\nu_{A \vee A^c}(x) = \nu_A(x) \wedge \mu_A(x) = [\nu_A^-(x) \wedge \mu_A^-(x), \nu_A^+(x) \wedge \mu_A^+(x)] ,$$
$$\nu_{A \wedge A^c}(x) = \nu_A(x) \vee \mu_A(x) = [\nu_A^-(x) \vee \mu_A^-(x), \nu_A^+(x) \vee \mu_A^+(x)] .$$

From $\nu_{A \vee A^c}(x) = \nu_{A \wedge A^c}(x)$ ,
we have $\nu_A^-(x) \wedge \mu_A^-(x) = \nu_A^-(x) \vee \mu_A^-(x)$ , $\nu_A^+(x) \wedge \mu_A^+(x) = \nu_A^+(x) \vee \mu_A^+(x)$ .
So we get $\nu_A^-(x) = \mu_A^-(x), \nu_A^+(x) = \mu_A^+(x)$ . That is $\nu_A(x) = \mu_A(x)$ .
" $\Leftarrow$ " It is straight-forward .

## 3 Entropy on VIFS

De Luca and Termini [12] first axiomatized non-probabilistic entropy. The De Luca-Termini axioms formulated for intuitionistic fuzzy sets are intuitive and have been wildly employed in the fuzzy literature. To extend this theory, firstly we definite the function as follows:

**Definition 3.1.** $M : VIFS(X) \to [0, 1]$, where $\forall A \in VIFS(x)$ ,

$$M(A) = \frac{1}{2n} \sum_{x \in X} [2 - \nu_A^-(x) - \nu_A^+(x)] \quad (\forall x \in X) .$$

**Proposition 3.1.** To any $A \in VIFS(X)$, $M(A) = 0 \Leftrightarrow \nu_A^-(x) = \nu_A^+(x) = 1 \Leftrightarrow A = \theta$ $(\forall x \in X)$.

**Proof.** $M(A)=0 \Leftrightarrow 2 - \nu_A^-(x) - \nu_A^+(x) = 0$ $(\forall x \in X)$ ,
for $\nu_A^-(x) \le 1, \nu_A^+(x) \le 1$ ,we have $M(A)=0 \Leftrightarrow \nu_A^-(x)=\nu_A^+(x)=1$ $(\forall x \in X)$ . From proposition 2.2, we have $\mu_A(x) = \underline{0}$ . So we have $\mu_A(x)=\mu_\theta(x)$, $\nu_A(x)=\nu_\theta(x)$ . That is $A = \theta$ , hence $M(A)=0 \Leftrightarrow \nu_A^-(x)=\nu_A^+(x) = 1 \Leftrightarrow A = \theta$ $(\forall x \in X)$.

**Proposition 3.2.** Let $A, B \in VIFS(X)$ , then we have $A \ge B \Rightarrow M(A) \ge M(B)$ .

**Proof.** For $A \ge B$ , then we have $\nu_A(x) \le \nu_B(x)$ $(\forall x \in X)$ . So we have

$$\nu_A^-(x) \le \nu_B^-(x),\ \nu_A^+(x) \le \nu_B^+(x) .$$

Hence $\sum_{x\in X} [2-\nu_A^-(x)-\nu_A^+(x)] \ge \sum_{x\in X} [2 - \nu_A^-(x) - \nu_A^+(x)]$ . That is $M(A) \ge M(B)$ .

**Proposition 3.3.** Let $A, B \in VIFS(X)$ , then we have

$$\nu_A(x) \le \nu_B(x) \text{ and } M(A) \ge M(B) \Leftrightarrow \nu_A(x) = \nu_B(x) \quad (\forall x \in X) .$$

**Proof.** " $\Leftarrow$ "It is straight-forward.
" $\Rightarrow$ " For $\nu_A(x) \le \nu_B(x)$ , we have $\nu_A^-(x) \le \nu_B^-(x)$ and $\nu_A^+(x) \le \nu_B^+(x)$ .
So we get $2 - \nu_A^-(x) - \nu_A^+(x) \ge \nu_B^-(x) - \nu_B^+(x)$ $(\forall x \in X)$. If there is a $x_0 \in X$, let one of the following two inequations $\nu_A^-(x) < \nu_B^-(x)$, $\nu_A^+(x) < \nu_B^+(x)$ establish, then we have

$$\begin{aligned} M(A) = & \sum_{x\neq x_o,\ x\in X} [2 - \nu_A^-(x) - \nu_A^+(x)] + [2 - \nu_A^-(x_0) - \nu_A^+(x_0)] \\ > & \sum_{x\neq x_o,\ x\in X} [2 - \nu_B^-(x) - \nu_B^+(x)] + [2 - \nu_B^-(x_0) - \nu_B^+(x_0)] \\ = & \quad M(B) . \end{aligned}$$

This is contrary to $M(A) = M(B)$ . So for any $x \in X$ ,we have $\nu_A^-(x) = \nu_B^-(x)$ , $\nu_A^+(x) = \nu_B^+(x)$ . That is $\nu_A(x) = \nu_B(x)$ $(\forall \in X)$ .

**Definition 3.2.** Let $A, B \in VIFS(X)$ with the following properties:
1. If $\mu_B(x) \ge \nu_B(x)$, then $\mu_A(x) \ge \mu_B(x)$ and $\nu_A(x) \le \nu_B(x)$ ,
2. If $\mu_B(x) < \nu_B(x)$, then $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ ,
3. If $\mu_B(x)$ and $\nu_B(x)$ is incomparable, there are two conditions:

1). If $\mu_B^-(x) \ge \nu_B^-(x)$ , then there are four inequations as follow:
(1)$\mu_A^-(x) \ge \mu_B^-(x)$ , (2)$\mu_A^+(x) \le \mu_B^+(x)$ , (3)$\nu_A^-(x) \le \nu_B^-(x)$ , (4)$\nu_A^+(x) \ge \nu_B^+(x)$ .
2).If $\mu_B^-(x) < \nu_B^-(x)$ then there are four inequations as follow:
(5)$\mu_A^-(x) \le \mu_B^-(x)$ , (6)$\mu_A^+(x) \ge \mu_B^+(x)$ , (7)$\nu_A^-(x) \ge \nu_B^-(x)$ , (8)$\nu_A^+(x) \le \nu_B^+(x)$ .
Thus we call that $A$ refines $B$ (that is $A$ is less fuzzy than $B$).

**Theorem 3.1.** Let $A, B \in VIFS(X)$ , and $A$ refines $B$, then we have

$$1.\ (A \wedge A^c) \le (B \wedge B^c) , \quad 2.\ (A \vee A^c) \ge (B \vee B^c) .$$

**Proof.** We prove inequation 1 first:
1. If $\mu_B(x) \ge \nu_B(x)$ from definition 3.2, we have $\mu_A(x) \ge \mu_B(x)$ and $\nu_A(x) \le \nu_B(x)$ . Then we have $\mu_A(x) \ge \mu_B(x) \ge \nu_B(x) \ge \nu_A(x)$ .

Hence $\mu_{A\wedge A^c}(x)=\mu_A(x)\wedge\nu_A(x)=\nu_A(x)$, $\mu_{B\wedge B^c}(x)=\mu_B(x)\wedge\nu_B(x)=\nu_B(x)$ ,
$\nu_{A\wedge A^c}(x)=\mu_A(x)\vee\nu_A(x)=\mu_A(x)$ , $\nu_{B\wedge B^c}(x)=\mu_B(x)\vee\nu_B(x)=\mu_B(x)$ .
So $\mu_{A\wedge A^c}(x)\le\mu_{B\wedge B^c}(x)$ , $\nu_{A\wedge A^c}(x)\ge\nu_{B\wedge B^c}(x)$ .
2. If $\mu_B(x)<\nu_B(x)$ , from definition 3.2, we have $\mu_A(x)\le\mu_B(x)$ and $\nu_A(x)\ge\nu_B(x)$ . Then we have $\mu_A(x)\le\mu_B(x)\le\nu_B(x)\le\nu_A(x)$ .
Hence $\mu_{A\wedge A^c}(x)=\mu_A(x)\wedge\nu_A(x)=\mu_A(x)$, $\mu_{B\wedge B^c}(x)=\mu_B(x)\wedge\nu_B(x)=\mu_B(x)$ ,
$\nu_{A\wedge A^c}(x)=\mu_A(x)\vee\nu_A(x)=\nu_A(x)$, $\nu_{B\wedge B^c}(x)=\mu_B(x)\vee\nu_B(x)=\nu_B(x)$ .
So $\mu_{A\wedge A^c}(x)\le\mu_{B\wedge B^c}(x)$ , $\nu_{A\wedge A^c}(x)\ge\nu_{B\wedge B^c}(x)$ .
3. In the case of $\mu_B(x)$ and $\nu_B(x)$ is incomparable, there are still another two different cases:
1). If $\mu_B^-(x)\ge\nu_B^-(x)$ (9), for $\mu_B(x)$ is not larger than $\nu_B(x)$, we have $\mu_B^+(x)\le\nu_B^+(x)$(10). From (1), (3) and (9) we get $\mu_A^-(x)\ge\mu_B^-(x)\ge\nu_B^-(x)\ge\nu_A^-(x)$. So we have $\mu_A^-(x)\ge\nu_A^-(x)$ (11) and from (2), (4) and (10) we get $\mu_A^+(x)\le\mu_B^+(x)\le\nu_B^+(x)\le\nu_A^+(x)$. So we have $\mu_A^+(x)\le\nu_A^+(x)$ (12). Then from (9) and (10), we have
$\mu_{B\wedge B^c}(x)=\mu_B(x)\wedge\nu_B(x)=[\mu_B^-(x)\wedge\nu_B^-(x),\mu_B^+(x)\wedge\nu_B^+(x)]=[\nu_B^-(x),\mu_B^+(x)]$ ,
$\nu_{B\wedge B^c}(x)=\mu_B(x)\vee\nu_B(x)=[\mu_B^-(x)\vee\nu_B^-(x),\mu_B^+(x)\vee\nu_B^+(x)]=[\mu_B^-(x),\nu_B^+(x)]$ .
In the same way, from (11) and (12), we get $\mu_{A\wedge A^c}(x)=[\nu_A^-(x),\mu_A^+(x)]$ and $\nu_{A\wedge A^c}(x)=[\mu_A^-(x),\nu_A^+(x)]$ . Then from (2) and (3), we get $\mu_{A\wedge A^c}(x)\le\mu_{B\wedge B^c}(x)$ . From(1) and (4) we get $\nu_{A\wedge A^c}(x)\ge\nu_{B\wedge B^c}(x)$ .
2). If $\mu_B^-(x)<\nu_B^-(x)$ (13), for $\mu_B(x)$ is not less than $\nu_B(x)$, we have $\mu_B^+(x)\ge\nu_B^+(x)$ (14). From (5), (13) and (7), we get $\mu_A^-(x)\le\mu_B^-(x)<\nu_B^-(x)\le\nu_A^-(x)$. Hence $\mu_A^-(x)<\nu_A^-(x)$ (15), and from (6),(14) and (8), we get $\mu_A^+(x)\ge\mu_B^+(x)\ge\nu_B^+(x)\ge\nu_A^+(x)$. Hence$\mu_A^+(x)\ge\nu_A^+(x)$ (16). Then from (13)and (14), we have
$\mu_{B\wedge B^c}(x)=\mu_B(x)\wedge\nu_B(x)=[\mu_B^-(x)\wedge\nu_B^-(x),\mu_B^+(x)\wedge\nu_B^+(x)]=[\mu_B^-(x),\nu_B^+(x)]$,
$\nu_{B\wedge B^c}(x)=\mu_B(x)\vee\nu_B(x)=[\mu_B^-(x)\vee\nu_B^-(x),\mu_B^+(x)\vee\nu_B^+(x)]=[\nu_B^-(x),\mu_B^+(x)]$.
In the same way, from (15) and (16), we get $\mu_{A\wedge A^c}(x)=[\mu_A^-(x),\nu_A^+(x)]$ and $\nu_{A\wedge A^c}(x)=[\nu_A^-(x),\mu_A^+(x)]$. Then from (5) and (8), we get $\mu_{A\wedge A^c}(x)\le\mu_{B\wedge B^c}(x)$. From(6) and (7), we get $\nu_{A\wedge A^c}(x)\ge\nu_{B\wedge B^c}(x)$.

Summary, we get inequation 1. $(A\wedge A^c)\le(B\wedge B^c)$. From inequation 1. , we can easily get inequation 2. $(A\vee A^c)\ge(B\vee B^c)$.

Then we can define the definition of entropy on $VIFS$.

The De Luca-Termini axioms[12] were formulated in the following way. Let $E$ be a set-to-point mapping $E:F(X)\to[0,1]$ . Hence $E$ is a fuzzy set defined on fuzzy sets. $E$ is an entropy measure if it satisfies the four De Luca and Termini axioms:
1. $E(A)=0$ iff $A\in X(A$ non-fuzzy) , 2. $E(A)=1$ iff $\mu_A(x)=0.5$ for $\forall x\in X$,
3. $E(A)\le E(B)$ if $A$ is less fuzzy than $B$, *i.e.*, if $\mu_A\le\mu_B$ when $\mu_B\le0.5$ and $\mu_A\le\mu_B$ when $\mu_B\ge0.5$ ,
4.$E(A)=E(A^c)$ .
Since the De Luca and Termini axioms were formulated for fuzzy sets, we extend them for $VIFS$.

**Definition 3.3.** A real function $E:VIFS(X)\to[0,1]$ is an entropy measure if $E$ has the following properties:

1.$E(A) = 0 \Leftrightarrow A \in \delta(P(X))$ , 2.$E(A) = 1 \Leftrightarrow \nu_A(x) = \mu_A(x) \quad (\forall x \in X)$ ,
3.$E(A) \leq E(B)$ if $A$refines $B$ , 4.$E(A) = E(A^c)$ .

**Definition 3.4.** Let $\sigma$: $VIFS(X) \to [0,1]$, where any $A \in VIFS(X)$, we have

$$\sigma(A) = \frac{M(A \wedge A^c)}{M(A \vee A^c)} \quad (\forall x \in X).$$

From proposition 2.3, we know that $A \vee A^c \neq \theta(\forall A \in VIFS(X))$ , then from proposition 3.1, we know that $M(A \vee A^c) \neq 0$, and then from $A \vee A^c \geq A \wedge A^c$ and proposition 3.2, we know $M(A \vee A^c) \geq M(A \wedge A^c)$, that is
$0 \leq \sigma(A) = \frac{M(A \wedge A^c)}{M(A \vee A^c)} \leq 1$ . So the definition of is reasonable.

**Theorem 3.2.** $\sigma$ is entropy.

**Proof.**
1. $\sigma(A) = 0 \Leftrightarrow M(A \wedge A^C) = 0 \Leftrightarrow A \wedge A^c = \theta \Leftrightarrow A \in \delta(P(X))$ .
2. $\sigma(A) = 1 \Leftrightarrow M(A \wedge A^c) = M(A \vee A^c)$ .
For $A \vee A^c \geq A \wedge A^c$, we get $\nu_{A \vee A^c}(x) \leq \nu_{A \wedge A^c}(x)$ , $(\forall x \in X)$ . So from proposition 3.3, we have $\nu_{A \vee A^c}(x) = \nu_{A \wedge A^c}(x)$ $(\forall x \in X)$. And from proposition 2.4, we get $\mu_A(x) = \nu_A(x)$ $(\forall x \in X)$.
3. If $A$ refines $B$, we have $A \wedge A^c \leq B \wedge B^c$. From proposition 3.2, we have $M(A \wedge A^c) \leq M(B \wedge B^c)$. And from $A \vee A^c \geq B \vee B^c$, we have $M(A \vee A^c)$ $\geq M(B \vee B^c)$. So we get $\frac{M(A \wedge A^c)}{M(A \vee A^c)} \leq \frac{M(B \wedge B^c)}{M(B \vee B^c)}$, that is $\sigma(A) \leq \sigma(B)$ .
4. It is straight-forward.
So we constructed a class of entropy function on $VIFS$.

## 4 Subsethood on VIFS

**Definition 4.1.** A real function $Q : IFS(X) \times IFS(X) \to [0,1]$ is called subsethood, if $Q$ has the following properties:
1. $Q(A,B) = 0 \Leftrightarrow A = I, B = \theta$, 2. If $A \leq B \Rightarrow Q(A,B) = 1$,
3. If $A \geq B$ and $Q(A,B) = 1$, then $A = B$,
4. If $A \leq B \leq C$, then $Q(C,A) \geq Q(C,B), Q(C,A) \leq (Q(B,A))$.

**Definition 4.2.** We define the function $f : VIFS(X) \times VIFS(X) \to [0,1]$ by

$$f(A,B) = \frac{1}{2n}\{\sum_{x \in X} min[1, g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), \psi(\nu_A^-(x) - \nu_B^-(x) + 1))] + \sum_{x \in X} min[1, g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1), \psi(\nu_A^+(x) - \nu_B^+(x) + 1))]\} .$$

where $\varphi$: $[0,2] \to [0,2]$ and $\psi$: $[0,2] \to [0,2]$ with the following properties:
1. $\alpha > \beta \Rightarrow \varphi(\alpha) > \varphi(\beta), \psi(\alpha) > \psi(\beta) \quad (\alpha, \beta \in [0,2])$,
2. $\varphi(\alpha) = 2 \Leftrightarrow \alpha = 2; \psi(\beta) = 0 \Leftrightarrow \beta = 0$ ,
3. $\varphi(1) = \psi(1) = 1$ .
And the function $g : [0,2] \times [0,2] \to [0,2]$ with the following properties:
1. $\alpha > \beta \Rightarrow g(\alpha,\gamma) < g(\beta,\gamma), \quad g(\gamma,\alpha) > g(\gamma,\beta) \qquad (\alpha,\beta,\gamma \in [0,2])$ ,
2. $g(\alpha,\beta) = 0 \Leftrightarrow \alpha = 2, \beta = 0$ ,
3. $g(1,1) = 1$ .

**Theorem 4.1.** $f$ is subsethood.

**Proof.** Let $A, B \in VIFS(X)$
1. For each $x \in X$, we have $\mu_I(x) = \underline{1}, \nu_I(x) = \underline{0}, \mu_\theta(x) = \underline{0}$ and $\nu_\theta(x) = 1$. Then
$$g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) = 0 ,$$
$$g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1), \psi(\nu_A^+(x) - \nu_B^+(x) + 1)) = 0 .$$
So we get $f(I, \theta) = 0$ .

On the contrary, If $f(A, B) = 0$ , Then for each $x \in X$ , We have
$$g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), \ \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) = 0 .$$
And then $\varphi(\mu_A^-(x) - \mu_B^-(x) + 1) = 2$ , $\psi(\nu_A^-(x) - \nu_B^-(x) + 1) = 0$ .
So we can get $(\mu_A^-(x) - \mu_B^-(x) + 1) = 2$ (1), $(\nu_A^-(x) - \nu_B^-(x) + 1) = 0$ (2) .
From (1), we have $\mu_A^-(x) = \mu_B^-(x) + 1$ . for $\mu_A^-(x) \leq 1, \mu_B^-(x) \geq 0$ , We have
$$\mu_A^-(x) = 1 , \quad \mu_B^-(x) = 0 .$$
From (2), we have $\nu_B^-(x) = \nu_A^-(x) + 1$ . Similarly, we have $\nu_A^-(x) = 0, \nu_B^-(x) = 1$ . By the same way, we can prove $\mu_A^+(x) = 1, \nu_A^+(x) = 0, \mu_B^+(x) = 0$, and $\nu_B^+(x) = 1$ . Thus we have $\mu_A(x) = \underline{1}$ , $\nu_A(x) = \underline{0}$ , $\mu_B(x) = \underline{0}$ , and $\nu_B(x) = \underline{1}$ , that is $A = I$ and $B = \theta$ .
2. If $A \leq B$ , then for each $x \in X$, we have $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$. So we have the following four inequations:
$$\mu_A^-(x) - \mu_B^-(x) + 1 \leq 1 , \quad \mu_A^+(x) - \mu_B^+(x) + 1 \leq 1 ,$$
$$\nu_A^-(x) - \nu_B^-(x) + 1 \geq 1 , \quad \nu_A^+(x) - \nu_B^+(x) + 1 \geq 1 .$$
Then we have $\varphi(\mu_A^-(x) - \mu_B^-(x) + 1) \leq 1$ , $\psi(\mu_A^+(x) - \mu_B^+(x) + 1) \leq 1$ ,
$$\varphi(\nu_A^-(x) - \nu_B^-(x) + 1) \geq 1 , \ \psi(\nu_A^+(x) - \nu_B^+(x) + 1) \geq 1 .$$
And then we have
$$g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) \geq g(1, 1) = 1 ,$$
$$g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1), \psi(\nu_A^+(x) - \nu_B^+(x) + 1)) \geq g(1, 1) = 1 .$$
So we get
$$\begin{aligned} f(A, B) = \tfrac{1}{2n}\{&\sum min[1, g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) \\ &+ \sum min[1, g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1), \psi(\nu_A^+(x) - \nu_B^+(x) + 1))\} \\ =\ & 1 . \end{aligned}$$
3. If $f(A, B) = 1$, then for each $x \in X$, we have
$$g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1) \ , \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) \geq 1 ,$$
$$g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1) \ , \psi(\nu_A^+(x) - \nu_B^+(x) + 1)) \geq 1 .$$
And from $A \geq B$, we have $\mu_A(x) \geq \mu_B(x), \mu_A(x) \leq \mu_B(x)$ $(\forall x \in X)$. So we have the following four inequations: $\mu_A^-(x) - \mu_B^-(x) + 1 \geq 1, \mu_A^+(x) - \mu_B^+(x) + 1 \geq 1, \nu_A^-(x) - \nu_B^-(x) + 1 \leq 1, \nu_A^+(x) - \nu_B^+(x) + 1 \leq 1$. If $A \neq B$ , then at least one of the four inequations given above would be never equal:
1) If $\mu_A^-(x) - \mu_B^-(x) + 1 > 1$, then we have
$$\begin{aligned} &g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) \\ &< g(1, \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) \\ &\leq g(1, 1) \leq 1. \quad \text{It is contradiction.} \end{aligned}$$
2)If $\mu_A^+(x) - \mu_B^+(x) + 1 > 1$, then we have
$$\begin{aligned} &g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1), \psi(\nu_A^+(x) - \nu_B^+(x) + 1)) \\ &< g(1, \psi(\nu_A^+(x) - \nu_B^+(x) + 1)) \\ &\leq g(1, 1) = 1. \quad \text{It is contradiction.} \end{aligned}$$

3) If $\nu_A^-(x) - \nu_B^-(x) + 1 < 1$, then we have

$$g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), \psi(\nu_A^-(x) - \nu_B^-(x) + 1)) < g(\varphi(\mu_A^-(x) - \mu_B^-(x) + 1), 1) \leq g(1,1) = 1. \quad \text{It is contradiction.}$$

4) If $\nu_A^+(x) - \nu_B^+(x) + 1 < 1$, then we have

$$g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1), \psi(\nu_A^+(x) - \nu_B^+(x) + 1)) < g(\varphi(\mu_A^+(x) - \mu_B^+(x) + 1), 1) \leq g(1,1) = 1. \quad \text{It is contradiction.}$$

So we have $A = B$.

4. Let $A \leq B \leq C$ $(A, B, C \in VIFS(X))$ ,then for each $x$ belongs to $X$, we have $\mu_A(x) \leq \mu_B(x) \leq \mu_C(x)$ and $\nu_A(x) \geq \nu_B(x) \geq \nu_C(x)$. So we have

$$\mu_C^-(x) - \mu_A^-(x) \geq \mu_C^-(x) - \mu_B^-(x)\ ,\ \nu_C^-(x) - \nu_A^-(x) \leq \nu_C^-(x) - \nu_B^-(x)\ ,$$
$$\mu_C^+(x) - \mu_A^+(x) \geq \mu_C^+(x) - \mu_B^+(x)\ ,\ \nu_C^+(x) - \nu_A^+(x) \leq \nu_C^+(x) - \nu_B^+(x)\ .$$

Then we get

$$\begin{aligned} f(C,A) &= \tfrac{1}{2n}\{\textstyle\sum min[1, g(\varphi(\mu_C^-(x) - \mu_A^-(x) + 1), \psi(\nu_c^-(x) - \nu_A^-(x) + 1))] \\ &\quad + \textstyle\sum min[1, g(\varphi(\mu_C^+(x) - \mu_A^+(x) + 1), \psi(\nu_C^+(x) - \nu_A^+(x) + 1))\} \\ &\leq \tfrac{1}{2n}\{\textstyle\sum min[1, g(\varphi(\mu_C^-(x) - \mu_B^-(x) + 1), \psi(\nu_c^-(x) - \nu_B^-(x) + 1))] \\ &\quad + \textstyle\sum min[1, g(\varphi(\mu_C^+(x) - \mu_B^+(x) + 1), \psi(\nu_C^+(x) - \nu_B^+(x) + 1))]\} \\ &= f(C,B)\ . \end{aligned}$$

In the same way, we can prove $f(C,A) = f(B,A)$ .

**Eg2.** For $\varphi, \psi : [0,2] \to [0,2], g : [0,2] \times [0,2] \to [0,2]$, $\varphi(x) = x, \psi(y) = y$, $g(x,y) = [(2-x)+y] \times \frac{1}{2}$. It is straight-forward to prove that $\varphi$ and $\psi$ has the following qualities: 1). $\alpha > \beta \Rightarrow \varphi(\alpha) > \varphi(\beta), \psi(\alpha) > \psi(\beta),\ (\alpha, \beta \in [0,2])$ . 2). $\varphi(\alpha) = 2 \Leftrightarrow \alpha = 2$ ; $\psi(\beta) = 0 \Leftrightarrow \beta = 0$ . 3). $\varphi(1) = \psi(1) = 1$ .
And to $g$ , it also has qualities as follows:
1). $\alpha > \beta \Rightarrow g(\alpha, \gamma) < g(\beta, \gamma), g(\gamma, \alpha) > g(\gamma, \beta)\ (\alpha, \beta, \gamma \in [0,2])$ ,
2). $g(\alpha, \beta) = 0 \Leftrightarrow \alpha = 2, \beta = 0$ , 3).$g(1,1) = 1$ .
It is shown in theorem 4.2. the relationship between entropy and subsethood on $VIFS$.

**Theorem 4.2.** Let $Q$ is subsethood, $\rho$: $VIFS(X) \to [0,1]$, where $\rho(A) = Q(A \vee A^c, A \wedge A^c)$ $(\forall A \in VIFS(X))$, then we have $\rho$ is entropy.

**Proof.** 1. $\forall A \in \delta(P(X))$ for $A \vee A^c = I, A \wedge A^c = \theta$ , we have

$$\rho(A) = Q(A \vee A^c, A \wedge A^c) = Q(I, \theta) = 0\ .$$

On the contrary, if $\rho(A) = Q(A \vee A^c, A \wedge A^c) = 0$ . Then we have $A \vee A^c = I, A \wedge A^c = \theta$ , so we get $\forall A \in \delta(P(X))$ .
2. If $E(A) = 1$ that is $Q(A \vee A^c, A \wedge A^c) = 1$ and $A \vee A^c \geq A \wedge A^c$. Then we have $A \vee A^c = A \wedge A^c$. So we get $\mu_A(x) = \nu_A(x)$ ;
On the contrary, if $\mu_A(x) = \nu_A(x)$, we have $A \vee A^c = A \wedge A^c$, So we may get

$$Q(A \vee A^c, A \wedge A^c) = E(A) = 1.$$

3. If A refines B, then we have $A \wedge A^c \leq B \wedge B^c \leq B \vee B^c \leq A \vee A^c$, so we can get $E(A) = Q(A \vee A^c, A \wedge A^c) \leq Q(B \vee B^c, A \wedge A^c) \leq Q(B \vee B^c, B \wedge B^c) \leq E(B)$.
4. $E(A) = Q(A \vee A^c, A \wedge A^c) = Q(A^c \vee A, A^c \wedge A) = E(A^c)$ .

From theorem 4.1 and theorem 4.2, we can construct a class of reasonable subsethood and entropy function, which are useful in different practical conditions.

## 5 Conclusion

In this paper, we offered different kinds of entropy function and subsethood function and they would be practical in different experiments. would be practical in different experiments.

## References

1. Zadeh, L.A.: Fuzzy sets. Inform.and Control 8 (1965) 338–353
2. Zadeh, L.A.: Fuzzy sets and Systems. In: Proc. Systems Theory, Polytechnic Institute of Brooklyn, New York (1965) 29–67
3. Atanassov, K.T.: Intuitionistic Fuzzy Sets. In: V.Sgurev(Ed.), VII ITKR's session, Sofia, June (1983), Central Sci.and Techn.Library, Bulgaria Academy of Sciences(1984)
4. Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20 (1986) 87–97
5. Atanassov, K.T., Gargov, G.: Interval valued intuitionistic fuzzy sets. Fuzzy Sets and Systems 31 (1989) 343–349
6. Atanassov, K.T.: Intuitionistic Fuzzy Sets. Physica-Verlag, Heidelberg, New York(1999)
7. Jaynes, E.T.: Where do We Stand on Maximum Entropy? In: Levine, Tribus (Eds.), The Maximum Entropy Formalism, MIT Press, Cambridge, MA.
8. Szmidt, E., Kacprzyk, J.: Entropy for Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems 118 (2001) 467–477
9. Yu-hai Liu, Feng-lan Xiong, Subsethood on Intuitionistic Fuzzy Sets. International Conference on Machine Learning and Cybernetics V.3 (2002) 1336–1339
10. Glad Deschrijver, Etienne E.Kerre, On her Relationship between some Extensions of Fuzzy Set Theory. Fuzzy Sets and Systems 133 (2003) 277–235
11. Guo-jun Wang, Ying-ming He: Intuitionistic Sets and Fuzzy Sets. Fuzzy Sets and Systems 110 (2000) 271–274
12. Luca, A.Ed., Termini, S.: A Definition of a Non-probabilistic Entropy in the Setting of Fuzzy Sets Theory. Inform. and Control 20 (1972) 301–312