Computing with Sequences, Weak Topologies and the Axiom of Choice

Vasco Brattka[1] and Matthias Schröder[2]

[1] Department of Mathematics & Applied Mathematics
University of Cape Town, Rondebosch 7701, South Africa
BrattkaV@maths.uct.ac.za
[2] LFCS, School of Informatics
University of Edinburgh, Edinburgh, UK
mschrode@inf.ed.ac.uk

Abstract. We study computability on sequence spaces, as they are used in functional analysis. It is known that non-separable normed spaces cannot be admissibly represented on Turing machines. We prove that under the Axiom of Choice non-separable normed spaces cannot even be admissibly represented with respect to any compatible topology (a compatible topology is one which makes all bounded linear functionals continuous). Surprisingly, it turns out that when one replaces the Axiom of Choice by the Axiom of Dependent Choice and the Baire Property, then some non-separable normed spaces can be represented admissibly on Turing machines with respect to the weak topology (which is just the weakest compatible topology). Thus the ability to adequately handle sequence spaces on Turing machines sensitively relies on the underlying axiomatic setting.

1 Introduction

In this paper we study computability on certain normed spaces X and their dual spaces X'. The framework for this investigation is computable analysis [2, 3, 8], the Turing machine based theory of computability and complexity on real numbers and other topological spaces. We will, in particular, use the representation based approach to computable analysis [8].

Some of our results depend on the underlying axiomatic setting and we will use the following notations to indicate the axioms:

- **ZF** for Zermelo-Fraenkel set theory.
- **AC** for the Axiom of Choice.
- **DC** for the Axiom of Dependent Choice.
- **BP** for the Baire Property Axiom (which states that any subset of the reals can be represented as a symmetric difference of an open and a meager set).

We will not make any direct use of these axioms but we will use certain results which can either be proved in **ZF+AC** or in **ZF+DC+BP**. It is known that in

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ZF the Hahn-Banach Theorem can be considered as a weak version of the Axiom of Choice **AC** and the Axiom of Dependent Choice **DC** is equivalent to the Baire Category Theorem (see [5] for a general discussion of the role of these axioms in functional analysis). Some counterexamples in functional analysis do only exist in the setting **ZF+AC** whereas **ZF+DC+BP** allows to exclude the existence of the corresponding objects. Such pathological objects are called “intangibles” by Schechter [5] since their existence cannot be proved constructively. Here, it is important to notice that the consistency of **ZF** implies the consistency of **ZF+AC** (proved by Gödel) as well as the consistency of **ZF+DC+BP** (proved by Shelah, see 14.73 and 14.74 in [5]). If not mentioned otherwise, we will work throughout this paper in the setting **ZF+DC**. Only if the full Axiom of Choice is needed, we will explicitly mention that we are working in **ZF+AC** or in case that we need the Baire Property, we will explicitly mention that we are working in **ZF+DC+BP**.

In the following section we will discuss compatible representations of normed spaces and their dual spaces. Such representations are well-behaved in the sense that they make all bounded linear functionals continuous. Our results show that in **ZF+AC** non-separable normed spaces X and their duals X' do not admit compatible representations.

In Section 3 we will consider the sequence spaces ℓ_p, as they are well-known in functional analysis. The space ℓ_∞ is a typical example of a non-separable normed space and we will prove that this spaces admits a compatible representation in **ZF+DC+BP**, but not in **ZF+AC**.

In Section 4 we discuss a canonical representation which is admissible with respect to the so-called weak* topology. Such representations exist at least for dual spaces of spaces with compatible representations. For separable reflexive spaces we obtain a representation which is admissible with respect to the weak topology.

2 Compatible Representations

In this section we will prove that neither non-separable normed spaces nor their dual spaces admit compatible representations. We start with recalling some notions from computable analysis [8]. The basic idea of the representation based approach to computable analysis is to represent infinite objects like real numbers, functions or sets, by infinite strings over some alphabet Σ (which should at least contain the symbols 0 and 1). Thus, a *representation* of a set X is a surjective mapping $\delta :\subseteq \Sigma^\omega \to X$ and in this situation we will call (X, δ) a *represented space*. Here Σ^ω denotes the set of infinite sequences over Σ and the inclusion symbol is used to indicate that the mapping might be partial. If we have two represented spaces, then we can define the notion of a computable function.

Definition 1 (Computable function). Let (X, δ) and (Y, δ') be represented spaces. A function $f :\subseteq X \to Y$ is called (δ, δ')*–computable*, if there exists some computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$.

Of course, we have to define computability of functions $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ to make this definition complete, but this can be done via Turing machines: F is computable if there exists some Turing machine, which computes infinitely long and transforms each sequence p, written on the input tape, into the corresponding sequence $F(p)$, written on the one-way output tape. If the represented spaces are fixed or clear from the context, then we will simply call a function f *computable*.

For the comparison of representations it will be useful to have the notion of *reducibility* of representations. If δ, δ' are both representations of a set X, then δ is called *reducible* to δ', $\delta \leq \delta'$ in symbols, if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta' F(p)$ for all $p \in \mathrm{dom}(\delta)$. Obviously, $\delta \leq \delta'$ holds if and only if the identity $\mathrm{id} : X \to X$ is (δ, δ')–computable. Moreover, δ and δ' are called *equivalent*, $\delta \equiv \delta'$ in symbols, if $\delta \leq \delta'$ and $\delta' \leq \delta$.

Analogously to the notion of computability we can define the notion of (δ, δ')–*continuity* by substituting a continuous function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ for the computable function F in the definition above. On Σ^ω we use the *Cantor topology*, which is simply the product topology of the discrete topology on Σ. The corresponding reducibility will be called *continuous reducibility* and we will use the symbols $\leq_{\mathrm{t}}$ and $\equiv_{\mathrm{t}}$ in this case. Again we will simply say that the corresponding function is *continuous*, if the representations are fixed or clear from the context. The category Rep of represented spaces and of continuous (w.r.t. the ambient representations) functions is cartesian-closed. There is a canonical function space representation $[\delta \to \delta']$ of the set $\mathcal{C}(\delta, \delta')$ of (δ, δ')–continuous functions. It has the property that the represented space $(\mathcal{C}(\delta, \delta'), [\delta \to \delta'])$ is the exponential of (X, δ) and (Y, δ') in the category Rep. Moreover, evaluation and currying are even computable (see [7, 8] for details).

If not mentioned otherwise, we will always assume that a represented space is endowed with the final topology induced by its representation. This will lead to no confusion with the ordinary topological notion of continuity, as long as we are dealing with *admissible* representations. A representation δ of a topological space X is called *admissible*, if δ is maximal among all continuous representations δ' of X, i.e. if $\delta' \leq_{\mathrm{t}} \delta$ holds for all continuous representations δ' of X. If δ_X, δ_Y are admissible representations of topological spaces X, Y, then a function $f : X \to Y$ is (δ_X, δ_Y)–continuous if and only if it is sequentially continuous, cf. [6]. Moreover, $[\delta_X \to \delta_Y]$ is an admissible representation of the space of the sequentially continuous functions between X and Y. Hence the category of sequential topological spaces having an admissible representation and of sequentially continuous functions is cartesian closed as well.

Now we introduce compatible representations of normed spaces. Here we assume that by $\mathbb{F}$ the underlying field is denoted, which might either be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, in each case endowed with the ordinary Euclidean norm and topology.

Definition 2. Let X be a normed space. Then a topology τ on X is called *compatible*, if any bounded linear functional $f : X \to \mathbb{F}$ is continuous with respect to τ. The smallest topology $\tau^{\mathrm{w}} = \sigma(X, X')$ with this property is called the *weak topology* on X.

As usual we will say that a topological space (X, τ) is *separable* if there exists a countable subset $D \subseteq X$ which is dense in X with respect to τ, i.e. such that the closure of D coincides with X.

Lemma 1. *Let* $(X, || \,||)$ *be a normed space and let* τ *be a compatible topology. In* **ZF+AC** *the space* $(X, || \,||)$ *is separable, if* (X, τ) *is separable.*

Proof. Let $(X, || \,||)$ be a normed vector space over $\mathbb{F}$ with a compatible topology τ and let $D = \{d_0, d_1, ...\}$ be a countable dense subset with respect to τ. By $\mathbb{Q}_{\mathbb{F}}$ we denote either $\mathbb{Q}$ or $\mathbb{Q}[\mathrm{i}]$ depending on whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let us assume that $(X, || \,||)$ is not separable. Then the countable set

$$U := \left\{ \sum_{i=0}^{\infty} q_i \cdot d_i : (q_i)_{i \in \mathbb{N}} \in \mathbb{Q}_{\mathbb{F}}^{\mathbb{N}} \text{ and } q_j = 0 \text{ for almost all } j \right\}$$

is not dense in X with respect to the norm $|| \,||$. Hence there is some $y \in X$ which is not in the closure $\overline{U}$ of U with respect to the norm $|| \,||$. Thus $s := \mathrm{dist}(\overline{U}, y) := \inf_{u \in \overline{U}} ||y - u|| > 0$. One easily verifies that $\overline{U}$ and

$$V := \{c \cdot y + u : c \in \mathbb{F}, u \in \overline{U}\}$$

form linear subspaces of X. Since $y \not\in \overline{U}$, we can unambiguously define a linear functional $f : V \to \mathbb{F}$ by $f(c \cdot y + u) := c$ for all $c \in \mathbb{F}$ and $u \in \overline{U}$. Since

$$\frac{|f(c \cdot y + u)|}{||c \cdot y + u||} = \frac{1}{||y - \frac{-u}{c}||} \le \frac{1}{s}$$

for $c \neq 0$ it follows that f is bounded. By the Hahn-Banach Theorem f can be extended to a bounded linear functional $F : X \to \mathbb{F}$. Since τ is compatible, if follows that F is continuous with respect to τ and since $F(y) = 1$, it follows that $F^{-1}(B(1, 1/2)) \in \tau$ is an open set containing y. By density of D there is some $i \in \mathbb{N}$ with $d_i \in F^{-1}(B(1, 1/2))$ which contradicts $F(d_i) = 0$. □

This lemma can also be obtained as a consequence of the result in functional analysis that a convex subset of a locally convex space X is dense if and only if it is dense with respect to the weak topology on X. However, we present a direct proof in order to pinpoint how **AC** is used, namely in the shape of the Hahn-Banach Theorem. In the setting **ZF+DC+BP**, the space ℓ_∞ turns out to be a counterexample to this lemma (cf. Section 3).

Now we extend the notion of compatibility to representations. Therefore, we assume that $\delta_{\mathbb{F}}$ denotes some standard representation of the field $\mathbb{F}$ which is admissible with respect to the Euclidean topology (e.g. its Cauchy representation, see [8]).

Definition 3. A representation δ of a normed space X is called *compatible*, if every bounded linear functional $f : X \to \mathbb{F}$ is $(\delta, \delta_{\mathbb{F}})$–continuous.

If δ is a compatible representation of X, then the function space representation $[\delta \to \delta_{\mathbb{F}}]$ can be considered as a representation of the dual space X' (which is the set of bounded linear functionals $f : X \to \mathbb{F}$ endowed with the operator norm $||f|| := \sup_{x \in B(0,1)} |f(x)|$). Here and in the following we will use for every $x \in X$ the canonical linear bounded evaluation functional

$$\iota_x : X' \to \mathbb{F}, f \mapsto f(x),$$

defined on the dual space X' of X. The maps ι_x induce a linear bounded map

$$\iota : X \to X'', x \mapsto \iota_x$$

and with the help of the Hahn-Banach Theorem one can prove that ι is injective and even an isometry (see Corollaries III.1.6 and III.1.7 in [9]). Those spaces for which ι is even bijective and thus an isometric isomorphism, are called *reflexive*. For the moment we will use the embedding ι in order to transfer compatible representations of X' to compatible representations of X.

Proposition 1. *In* **ZF+AC** *a normed space X admits a compatible representation, if its dual space X' admits a compatible representation.*

Proof. Let δ' be a compatible representation of the dual space X'. In **ZF+AC** one can prove that ι is injective and since δ' is compatible, we can define a representation δ of X by

$$\delta(p) = x :\iff [\delta' \to \delta_{\mathbb{F}}](p) = \iota_x.$$

Since the evaluation

$$\text{ev} : X' \times X \to \mathbb{F}, (f, x) \to f(x) = \iota_x(f)$$

is $([\delta', \delta], \delta_{\mathbb{F}})$–continuous, it follows that each bounded linear functional

$$f : X \to \mathbb{F}, x \mapsto f(x) = \text{ev}(f, x)$$

is $(\delta, \delta_{\mathbb{F}})$–continuous. This means that δ is compatible. □

Now we are prepared to prove the main result of this section from which we can conclude that the possibilities to introduce a computability theory on non-separable normed spaces which is well-behaved with respect to dual spaces are very limited (given the Axiom of Choice).

Theorem 1. *Let X be a non-separable normed space. In* **ZF+AC** *neither X nor its dual space X' admit a compatible representation.*

Proof. Assume δ is a compatible representation of X and let τ be the final topology of δ, viewed as a total function from the domain of δ endowed with the countably based subspace topology inherited from the Cantor space. Then every linear bounded functional $f : X \to \mathbb{F}$ is $(\delta, \delta_{\mathbb{F}})$–continuous and hence continuous with respect to τ. Therefore, τ is a compatible topology. But since (X, τ) is a quotient of a countably based space, it admits a countable dense subset. This contradicts Lemma 1. The statement on the dual space follows from Proposition 1. □

3 Sequence Spaces

In this section we will study the sequence spaces

$$\ell_p := \{x \in \mathbb{F}^{\mathbb{N}} : ||x||_p < \infty\}$$

with the norms

$$||x||_p := \sqrt[p]{\sum_{i=0}^{\infty} |x_i|^p}$$

in case of $1 \leq p < \infty$ and

$$||x||_\infty := \sup_{i \in \mathbb{N}} |x_i|$$

in case of $p = \infty$ for all $x = (x_i)_{i \in \mathbb{N}}$, as they are known in functional analysis. One important duality property of these spaces is expressed by the following theorem (see, for instance, Theorem II.2.3 in [9]):

Theorem 2 (Landau). *Let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$. Then the map $\lambda : \ell_q \to \ell_p', a \mapsto \lambda_a$ with $\lambda_a : \ell_p \to \mathbb{F}$, $(x_k)_{k \in \mathbb{N}} \mapsto \sum_{k=0}^{\infty} a_k x_k$ is an isometric isomorphism. The map λ is also isometric in case of $p = \infty$ and $q = 1$.*

The proof is mainly based on Hölder's Inequality. It is known that the fact that λ is an isomorphism cannot be generalized to the case $p = \infty$ and $q = 1$ straightforwardly, since the result depends on the underlying axiomatic setting in this case. On the one hand, using the Hahn-Banach Theorem one can extend the limit functional $\lim : c \to \mathbb{F}$ on the space of convergent sequences c to a functional $L : \ell_\infty \to \mathbb{F}$ with the same norm and it is easy to see that this functional cannot be represented as λ_a with some $a \in \ell_1$. Thus we obtain the following classical property of the map λ defined in Landau's Theorem (see, for instance, Theorem II.1.11 in [9]):

Theorem 3. *In* **ZF+AC** *the map $\lambda : \ell_1 \to \ell_\infty'$ is not surjective.*

Thus, one could say that ℓ_∞' is a proper superset of ℓ_1. On the other hand, Pincus proved a result, first stated by Solovay, which shows that the situation changes if we replace the Axiom of Choice by Dependent Choice and the Baire Property (see 29.37 in [5]):

Theorem 4 (Solovay, Pincus). *In* **ZF+DC+BP** *the map $\lambda : \ell_1 \to \ell_\infty'$ is an isometric isomorphism.*

The Theorem of Landau and its counterpart for the case $p = \infty$ and $q = 1$ have certain consequences concerning the existence of compatible representations of the sequence spaces. As a preparation we prove a characterization of weak convergence for these spaces. We recall that in functional analysis weak convergence means convergence with respect to the weak topology, i.e. a sequence $(x_n)_{n \in \mathbb{N}}$ in a normed space X is said to *converge weakly* to x, if $(f(x_n))_{n \in \mathbb{N}}$ converges to

$f(x)$ for any linear bounded functional $f : X \to \mathbb{F}$. The first part of the following lemma is a known fact. We include the proof in order to indicate how the second part follows from the previous theorem.

Lemma 2. *Let $1 < p < \infty$. A sequence $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ in ℓ_p converges weakly to $(x_j)_{j\in\mathbb{N}}$, if and only if the sequence converges with respect to the product topology on $\mathbb{F}^{\mathbb{N}}$ to $(x_j)_{j\in\mathbb{N}}$ and if it is bounded in $|| \; ||_p$. For $p = \infty$, the equivalence holds in* **ZF+DC+BP**, *whereas in* **ZF+AC** *merely the only-if-part is true.*

Proof. Let $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ be a sequence in ℓ_p which converges weakly to $(x_j)_{j\in\mathbb{N}} \in \ell_p$, i.e. $(f((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ converges for any linear bounded functional $f : \ell_p \to \mathbb{F}$ to $f((x_j)_{j\in\mathbb{N}})$. Since the canonical projections

$$\mathrm{pr}_j : \ell_p \to \mathbb{F}, (y_j)_{j\in\mathbb{N}} \to y_j$$

are linear bounded functionals, it follows that $(\mathrm{pr}_j((x_{ij})_{j\in\mathbb{N}}))_{i\in\mathbb{N}} = (x_{ij})_{i\in\mathbb{N}}$ converges for any fixed $j \in \mathbb{N}$ to $\mathrm{pr}_j((x_j)_{j\in\mathbb{N}}) = x_j$ and hence $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ converges with respect to the product topology on $\mathbb{F}^{\mathbb{N}}$ to $(x_j)_{j\in\mathbb{N}}$. Moreover, it is known that any weakly convergent sequence in ℓ_p is bounded. (This is a consequence of the Uniform Boundedness Theorem, see for instance Korollar IV.2.3 in [9], and can be proven in **ZF+DC**.)

Now let us assume that $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ is a sequence in ℓ_p which converges to $(x_j)_{j\in\mathbb{N}} \in \ell_p$ with respect to the product topology on $\mathbb{F}^{\mathbb{N}}$ and which is bounded in $|| \; ||_p$. We have to prove that the sequence $(f((x_{ij})_{j\in\mathbb{N}}))_{i\in\mathbb{N}}$ converges for any functional $f : \ell_p \to \mathbb{F}$ to $f((x_j)_{j\in\mathbb{N}})$. Let q be such that $1/p + 1/q = 1$ or $q = 1$ in case of $p = \infty$. If the map $\lambda : \ell_q \to \ell_p'$ from Landau's Theorem 2 is an isometric isomorphism, then it suffices to prove that $(\lambda_a((x_{ij})_{j\in\mathbb{N}}))_{i\in\mathbb{N}} = (\sum_{j=0}^{\infty} a_j x_{ij})_{i\in\mathbb{N}}$ converges for any $a = (a_j)_{j\in\mathbb{N}} \in \ell_q$ to $\lambda_a((x_j)_{j\in\mathbb{N}}) = \sum_{j=0}^{\infty} a_j x_j$. Therefore, let $a = (a_j)_{j\in\mathbb{N}} \in \ell_q$, i.e. $||a||_q = (\sum_{j=0}^{\infty} |a_j|^q)^{1/q} < \infty$. Since $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ is bounded in ℓ_p, it follows that $S := \sup_{i\in\mathbb{N}} ||(x_{ij})_{j\in\mathbb{N}} - (x_j)_{j\in\mathbb{N}}||_p + 1$ exists. Let $\varepsilon > 0$. There is some $J \in \mathbb{N}$ such that $(\sum_{j=J+1}^{\infty} |a_j|^q)^{1/q} < \varepsilon/(2S)$. Let $M := \max\{|a_0|, |a_1|, ..., |a_J|\}$. Since $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ converges to $(x_j)_{j\in\mathbb{N}}$ with respect to the product topology on $\mathbb{F}^{\mathbb{N}}$ there is some $I \in \mathbb{N}$ such that $|x_{ij} - x_j| < \varepsilon/(2M(J+1))$ for all $i \geq I$ and $j = 0, ..., J$. Now we obtain by Hölder's Inequality for all $i \geq I$

$$\begin{aligned}
&\left| \sum_{j=0}^{\infty} a_j x_{ij} - \sum_{j=0}^{\infty} a_j x_j \right| \\
&\leq \sum_{j=0}^{J} |a_j \cdot (x_{ij} - x_j)| + \sum_{j=J+1}^{\infty} |a_j \cdot (x_{ij} - x_j)| \\
&\leq \sum_{j=0}^{J} |a_j| \cdot |x_{ij} - x_j| + ||(0, ..., 0, a_{J+1}, a_{J+2}, ...)||_q \cdot ||(x_{ij} - x_j)_{j\in\mathbb{N}}||_p \\
&\leq (J+1)M \cdot \frac{\varepsilon}{2M(J+1)} + \frac{\varepsilon}{2S} \cdot S = \varepsilon.
\end{aligned}$$

This proves the desired convergence.

It remains to recall that by Landau's Theorem 2 $\lambda : \ell_q \to \ell_p'$ is an isometric isomorphism for $1 < p < \infty$ and by the Theorem of Solovay and Pincus 4 this also holds in **ZF+DC+BP** for the case $p = \infty$. □

The reader should notice that the previous result does not capture the case $p = 1$. This is not an accidental omission, but the result cannot be extended to this case. This is due to the following well-known result (see 28.20 in [5]):

Lemma 3 (Schur). *A sequence in ℓ_1 converges weakly to a certain limit if and only if it converges to the same limit with respect to the norm $|| \; ||_1$.*

We recall that a topology τ is called *sequential*, if any sequentially open set is open. A set U is called *sequentially open*, if any sequence with limit in U is eventually in U. The *sequentialization* $\mathrm{seq}(\tau)$ is the set of all sequentially open sets or, in other words, the smallest sequential topology which contains τ. Two sequential topologies coincide, if their convergence relations on sequences are identical. Any topology induced by a norm is sequential.

By the Lemma of Schur, the sequentialization of the weak topology of ℓ_1 is just the norm topology induced by the norm $|| \; ||_1$. Since it is known that for infinite dimensional normed spaces $(X, || \; ||)$ the norm $|| \; || : X \to \mathbb{R}$ itself is not continuous with respect to the weak topology (see 28.18 in [5]) and, in particular, the norm topology is different from the weak topology, it follows that the weak topology on ℓ_1 is not a sequential topology.

Now we will discuss compatible representations of sequence spaces. In particular, we will exploit the characterization of weak convergence to show that under certain assumptions such representations exist. In particular, we are interested in the following representations (which have been introduced in the more general context of general computable normed spaces [1]; here $\delta_{\mathbb{N}}$ denotes some canonical representation of the natural numbers $\mathbb{N}$):

Definition 4. Let $1 \le p \le \infty$. We define three representations $\delta_p, \delta_p^{=}, \delta_p^{\geq}$ of ℓ_p as follows:

- $\delta_p(r) = x :\iff [\delta_{\mathbb{N}} \to \delta_{\mathbb{F}}](r) = x$,
- $\delta_p^{=}\langle r, s\rangle = x :\iff \delta_p(r) = x$ and $\delta_{\mathbb{R}}(s) = ||x||_p$,
- $\delta_p^{\geq}\langle r, s\rangle = x :\iff \delta_p(r) = x$ and $\delta_{\mathbb{R}}(s) \ge ||x||_p$,

for all $r, s \in \Sigma^\omega$.

The representation δ_p is nothing but the standard representation of $\mathbb{F}^{\mathbb{N}}$ restricted to ℓ_p and it is admissible with respect to the subtopology τ_p on ℓ_p of the product topology on $\mathbb{F}^{\mathbb{N}}$. In [1] it has been shown that $\delta_p^{=}$ is admissible with respect to the weakest topology $\tau_p^{=}$ on ℓ_p which contains the topology τ_p and which makes the norm $|| \; ||_p$ continuous. Finally, $\delta_p^{\geq}$ is admissible with respect to the inductive limit topology $\tau_p^{\geq} = \underrightarrow{\lim}\, \sigma_k$ of the subtopologies σ_k of τ_p on $X_k := \{x \in \ell_p : ||x||_p \le k\}$. These results mainly rely on closure properties provided in [6]. Moreover, the product topology on $\mathbb{F}^{\mathbb{N}}$ is a sequential topology with a countable basis and thus it follows that τ_p is a sequential topology with a countable basis. The topology $\tau_p^=$ is obtained by an initial construction from sequential topologies with countable bases and $\tau_p^\geqslant$ is obtained by a final construction from sequential topologies. Using these properties, one can conclude that all three topologies τ_p, $\tau_p^=$ and $\tau_p^\geqslant$ are sequential topologies as well (see [6, 7, 10]).

Now the question occurs how these topologies are related to topologies considered in functional analysis. Firstly, we will characterize the topology $\tau_p^=$ for $1 \leq p < \infty$ which turns out to be just the norm topology $\tau_{||\ ||_p}$ induced by the norm $||\ ||_p$. This does not hold true in case of $p = \infty$, where the sequence $(e_1 + e_{2+i})_{i\in\mathbb{N}}$ built from the unit vectors e_i (which are zero except for the i–th position where they are one) is an obvious counterexample. The following lemma expresses a fact which is folklore in functional analysis. For completeness we include the proof.

Lemma 4. *Let $1 \leq p < \infty$. A sequence $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ in ℓ_p converges to $(x_j)_{j\in\mathbb{N}}$ with respect to the norm $||\ ||_p$, if and only if the sequence converges with respect to the product topology on $\mathbb{F}^{\mathbb{N}}$ to $(x_j)_{j\in\mathbb{N}}$ and if $(||(x_{ij})_{j\in\mathbb{N}}||_p)_{i\in\mathbb{N}}$ converges to $||(x_j)_{j\in\mathbb{N}}||_p$.*

Proof. If $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ converges to $(x_j)_{j\in\mathbb{N}}$ with respect to the norm $||\ ||_p$, then it converges weakly to the same limit. Literally the same proof as for the first part of Lemma 2 shows that it also converges to the same limit with respect to the product topology. Moreover, the norm $||\ ||_p : \ell_p \to \mathbb{R}$ is continuous with respect to the norm topology, hence it is sequentially continuous which proves that the norm of the sequence converges to the norm of the limit.

For the other direction let us assume that $((x_{ij})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ converges to $(x_j)_{j\in\mathbb{N}}$ with respect to the product topology and that $(||(x_{ij})_{j\in\mathbb{N}}||_p)_{i\in\mathbb{N}}$ converges to $||(x_j)_{j\in\mathbb{N}}||_p$. Let $\varepsilon > 0$. There is some $J \in \mathbb{N}$ such that $\sum_{j=J+1}^{\infty} |x_j|^p < \varepsilon/8$ and there is some $I \in \mathbb{N}$ such that

$$\max\{|x_{ij} - x_j|^p, |x_j|^p - |x_{ij}|^p\} < \frac{\varepsilon}{4(J+1)}$$

for all $j = 0, ..., J$ and $i > I$ and such that

$$||(x_{ij})_{j\in\mathbb{N}}||_p^p - ||(x_j)_{j\in\mathbb{N}}||_p^p < \frac{\varepsilon}{4}$$

for all $i > I$. We obtain

$$\begin{aligned} & ||(x_{ij})_{j\in\mathbb{N}} - (x_j)_{j\in\mathbb{N}}||_p^p \\ &= \sum_{j=0}^{J} |x_{ij} - x_j|^p + \sum_{j=J+1}^{\infty} |x_{ij} - x_j|^p \\ &\leq (J+1)\frac{\varepsilon}{4(J+1)} + \sum_{j=J+1}^{\infty} (|x_{ij}|^p + |x_j|^p) \end{aligned}$$

$$= \frac{\varepsilon}{4} + \sum_{j=0}^{\infty}|x_{ij}|^p - \sum_{j=0}^{J}|x_{ij}|^p + 2\sum_{j=J+1}^{\infty}|x_j|^p - \sum_{j=0}^{\infty}|x_j|^p + \sum_{j=0}^{J}|x_j|^p$$

$$< \frac{\varepsilon}{4} + ||(x_{ij})_{j\in\mathbb{N}}||_p^p - ||(x_j)_{j\in\mathbb{N}}||_p^p + 2\frac{\varepsilon}{8} + \sum_{j=0}^{J}(|x_j|^p - |x_{ij}|^p)$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + (J+1)\frac{\varepsilon}{4(J+1)} = \varepsilon.$$

This proves the desired convergence. □

Using the same estimations one can prove the following slightly more general result, where δ_{ℓ_p} denotes the so-called Cauchy representation of the space $(\ell_p, ||\ ||_p)$ (which is a standard representation that is admissible with respect to the norm topology, see [8]).

Proposition 2. *Let $1 \leq p < \infty$. Then $\delta_{\ell_p} \equiv \delta_p^{=}$.*

In [1] it has also been proved that in general we obtain $\tau_{||\ ||_p} \supseteq \tau_p^{=} \supseteq \tau_p^{\geqslant} \supseteq \tau_p$ for the corresponding topologies, where $\tau_{||\ ||_p}$ denotes the norm topology again. This raises the question whether the weak topology $\tau_p^{\mathrm{w}} = \sigma(\ell_p, \ell_p')$ can be included in this inclusion chain. Using Lemma 2 and Lemma 4 we can directly conclude the following corollary.

Corollary 1. *For the spaces ℓ_p with $1 < p < \infty$ we obtain $\tau_{||\ ||_p} = \tau_p^{=} \supsetneqq \tau_p^{\geqslant} = \mathrm{seq}(\tau_p^{\mathrm{w}})$.*

For the space ℓ_1 the situation is different and we can conclude from the Lemma of Schur 3 and Lemma 4 the following result.

Corollary 2. *For the space ℓ_1 we obtain $\tau_{||\ ||_1} = \tau_1^{=} = \mathrm{seq}(\tau_1^{\mathrm{w}}) \supsetneqq \tau_1^{\geqslant}$.*

For the non-separable space ℓ_∞ the situation is yet different again and it depends on the underlying axiomatic setting.

Theorem 5. *For the space ℓ_∞ we obtain $\tau_{||\ ||_\infty} \supsetneqq \tau_\infty^{=} \supsetneqq \tau_\infty^{\geqslant}$. Additionally,*

- *in* **ZF+AC**, *$\tau_{||\ ||_\infty} \supsetneqq \mathrm{seq}(\tau_\infty^{\mathrm{w}}) \supsetneqq \tau_\infty^{\geqslant}$ and $\mathrm{seq}(\tau_\infty^{\mathrm{w}})$ is incomparable with $\tau_\infty^{=}$, whereas*
- *in* **ZF+DC+BP**, *$\mathrm{seq}(\tau_\infty^{\mathrm{w}}) = \tau_\infty^{\geqslant}$.*

Proof. The first two strict inclusions have been proved in [1] (and they hold for non-separable general computable normed spaces in general). The fact that $\mathrm{seq}(\tau_\infty^{\mathrm{w}}) \supseteq \tau_\infty^{\geqslant}$ holds follows from the only-if-part of Lemma 2 (which does not require the Axiom of Choice). The inclusion has to be strict and $\tau_\infty^{=} \not\supseteq \mathrm{seq}(\tau_\infty^{\mathrm{w}})$, both in **ZF+AC**, since the contrary would contradict Theorem 1 (this is because $\delta_\infty^{=}$ is admissible with respect to $\tau_\infty^{=}$ whereas by Theorem 1 no representation is admissible with respect to $\mathrm{seq}(\tau_\infty^{\mathrm{w}})$).

Next we prove $\mathrm{seq}(\tau_\infty^{\mathrm{w}}) \not\supseteq \tau_\infty^{=}$. We consider the sequence $(e_i)_{i\in\mathbb{N}}$ of unit vectors (which are zero except for the i–th position where they are one; for simplicity we assume $e_0 = 0$). Let $f : \ell_\infty \to \mathbb{F}$ be some arbitrary linear bounded functional with $s := ||f||$. Let us assume that $(f(e_i))_{i\in\mathbb{N}}$ does not converge to 0. Then there is some $k \in \mathbb{N}$ and some strictly increasing $\varphi : \mathbb{N} \to \mathbb{N}$ with $|f(e_{\varphi(i)})| > 1/k$ for all $i \geq 1$ (in particular, $\varphi(i) \geq 1$ for all $i \geq 1$). Now we consider

$$z := \sum_{i=1}^{ks} e_{\varphi(i)} \frac{|f(e_{\varphi(i)})|}{f(e_{\varphi(i)})}$$

and we obtain $||z||_\infty = 1$ and $|f(z)| > s$ which is a contradiction! Thus, $(f(e_i))_{i\in\mathbb{N}}$ does converge to $f(0) = 0$ and hence $(e_i)_{i\in\mathbb{N}}$ converges to 0 with respect to τ_∞^{w} and hence also with respect to $\mathrm{seq}(\tau_\infty^{\mathrm{w}})$. On the other hand, it is obvious that $(e_i)_{i\in\mathbb{N}}$ does not converge to 0 with respect to $\tau_\infty^{=}$.

Since the weak topology is always contained in the norm topology and the norm topology is sequential, we obtain $\tau_{||\ ||_\infty} \supseteq \mathrm{seq}(\tau_\infty^{\mathrm{w}})$. The inclusion is strict, since otherwise $\mathrm{seq}(\tau_\infty^{\mathrm{w}}) = \tau_{||\ ||_\infty} \supseteq \tau_\infty^{=}$ would follow.

Finally, in **ZF+DC+BP** $\mathrm{seq}(\tau_\infty^{\mathrm{w}}) = \tau_\infty^{\gtrless}$ by Lemma 2. □

We could also prove the previous theorem without reference to Theorem 1 by a direct usage of the following example.

Example 1. In **ZF+AC** one can apply the Hahn-Banach Theorem in order to prove that the limit functional $\lim : c \to \mathbb{F}$ has a linear bounded extension $L : \ell_\infty \to \mathbb{F}$. Any such extension L is not continuous with respect to $\tau_\infty^{=}$ and hence not with respect to $\tau_\infty^{\gtrless}$: the sequence $(x_n)_{n\in\mathbb{N}}$ with elements $x_n = (1, 0, ..., 0, 1, 1, ...) \in \ell_\infty$ (with n zeros) converges to $x = (1, 0, 0, ...) \in \ell_\infty$ with respect to $\tau_\infty^{=}$, but $L(x_n) = 1 \neq 0 = L(x)$ for all n. In particular, $\tau_\infty^{=} \not\supseteq \mathrm{seq}(\tau_\infty^{\mathrm{w}})$.

We can also combine our results on compatible representations of ℓ_∞ as follows.

Corollary 3. *In* **ZF+AC** *neither ℓ_∞ nor its dual space admit compatible representations, whereas in* **ZF+DC+BP** *the space ℓ_∞ as well as its dual space ℓ_1 admit compatible representations.*

In fact, in **ZF+DC+BP** the representation $\delta_\infty^{\gtrless}$ is a compatible representation of ℓ_∞ which is admissible with respect to the weak topology on ℓ_∞. In **ZF+AC**, $\delta_\infty^{\gtrless}$ has at least the property that a linear function $f : \ell_\infty \to \mathbb{F}$ is $(\delta_\infty^{\gtrless}, \delta_\mathbb{F})$–continuous if and only if there is some $a \in \ell_1$ such that $f = \lambda_a$. Moreover, $\delta_\infty^{\gtrless}$ is admissible with respect to the weakest topology on ℓ_∞ for which every function λ_a, $a \in \ell_1$, is continuous.

4 The Weak-Star Topology

In this section we investigate the so-called weak* topology in our context. This will also allow us to generalize some of the positive results of the previous section to a more general setting.

Definition 5. Let X be a normed space. The *weak* topology* on the dual space X' is the smallest topology $\tau^{w^*} = \sigma(X', X)$ which makes for all $x \in X$ the functionals $\iota_x : X' \to \mathbb{F}$, $f \mapsto f(x)$ continuous.

It is obvious that the weak* topology $\sigma(X', X)$ on X' is weaker or equal to the weak topology $\sigma(X', X'')$ on X', i.e. $\sigma(X', X'') \supseteq \sigma(X', X)$. By a Theorem of Banach, Smulian, James and others (see 28.41 in [5]) the topologies coincide exactly for reflexive spaces, i.e. such spaces for which the canonical embedding $\iota : X \to X''$, $x \mapsto \iota_x$ is bijective (however, this result requires the Axiom of Choice).

The next lemma shows that a sequence of functionals converges with respect to the weak* topology if and only if it converges with respect to the compact-open topology. Readers familar with topological vector spaces might derive this fact from Theorem 4.6 in Paragraph 5 of Chapter 3 in [4]. For completeness we include a direct proof.

Lemma 5. *Let X be a Banach space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of linear bounded functionals $f_n : X \to \mathbb{F}$ and let $f : X \to \mathbb{F}$ be another such functional. Then $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to the weak* topology on X' if and only if it converges to f with respect to the compact-open topology on $\mathcal{C}(X, \mathbb{F})$.*

Proof. First of all, by definition of the weak* topology the sequence $(f_n)_{n\in\mathbb{N}}$ converges with respect to the weak* topology to f if and only if it converges pointwise to f.

Now if the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f, then $\sup_{n\in\mathbb{N}} |f_n(x)|$ exists for each $x \in X$ and by the Uniform Boundedness Theorem $M := \sup_{n\in\mathbb{N}}||f_n||$ also exists. Let us assume that $(f_n)_{n\in\mathbb{N}}$ does not converge to f with respect to the compact-open topology. Then there exists a non-empty compact subset $K \subseteq X$ and some $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there is some $k_n > n$ with $\sup_{x\in K} |f_{k_n}(x) - f(x)| > \varepsilon$. We can assume that $(k_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence. Then for any $n \in \mathbb{N}$ there is some $x_n \in K$ such that $|f_{k_n}(x_n) - f(x_n)| > \varepsilon$ and since K is compact the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_i})_{i\in\mathbb{N}}$ which converges to some $x \in K$. Since $(f_{k_{n_i}}(x))_{i\in\mathbb{N}}$ converges to $f(x)$, there is some $i \in \mathbb{N}$ such that $|f_{k_{n_i}}(x) - f(x)| < \varepsilon/2$ and $||x_{n_i} - x|| < \varepsilon/((M + ||f|| + 1)2)$. Now we obtain

$$\begin{aligned}
\varepsilon &< |f_{k_{n_i}}(x_{n_i}) - f(x_{n_i})| \\
&\leq |f_{k_{n_i}}(x_{n_i}) - f_{k_{n_i}}(x)| + |f_{k_{n_i}}(x) - f(x)| + |f(x) - f(x_{n_i})| \\
&\leq (||f_{k_{n_i}}|| + ||f||) \cdot ||x_{n_i} - x|| + |f_{k_{n_i}}(x) - f(x)| \\
&< (M + ||f||) \cdot \frac{\varepsilon}{(M + ||f|| + 1)2} + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

which is a contradiction. Thus, the assumption was wrong and $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to the compact-open topology.

Finally, if $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to the compact-open topology, then it also converges to f pointwise. □

For the compact-open topology on $X' \subseteq \mathcal{C}(X, \mathbb{F})$, we can conclude that $\mathrm{seq}(\tau^{\mathrm{co}}) = \mathrm{seq}(\tau^{\mathrm{w}^*})$. We do not know whether the two topologies agree themselves or whether they are sequential.

Now let δ be a compatible representation of X and let τ_δ be the final topology of δ. Analogously to the previous proof, one can show that a sequence of linear bounded functionals $(f_n)_n$ converges to a linear bounded functional f with respect to the weak* topology if and only if it converges with respect to sequentially-compact-open topology[1] on $\mathcal{C}((X, \tau_\delta), \mathbb{F})$. From [7], we know that the *dual representation* δ' of X', defined by

$$\delta'(p) = f :\Longleftrightarrow [\delta \to \delta_{\mathbb{F}}](p) = f,$$

is admissible with respect to the sequentially-compact-open topology on X'. Thus we obtain the following corollary.

Corollary 4. *Let X be a Banach space with some compatible representation δ. Then the dual representation δ' of X' is admissible with respect to the weak* topology $\tau^{\mathrm{w}^*} = \sigma(X', X)$ on X'.*

Let us denote by $\tau_p^{\mathrm{w}^*} = \sigma(\ell_p, \ell_q)$ the weak* topology on ℓ_p induced by the corresponding conjugate space ℓ_q with $1/p + 1/q = 1$. Then we can formulate our results on the ℓ_p spaces as follows.

Theorem 6. *For the spaces ℓ_p with $1 < p \leq \infty$ we obtain $\tau_p^{\gtrless} = \mathrm{seq}(\tau_p^{\mathrm{w}^*})$.*

Proof. In case of $1 < p \leq \infty$ it follows from the effective Theorem of Landau (see Theorem 7.2 in [1]) that $\delta_p^{\gtrless} \equiv_{\mathrm{t}} \delta_q' = [\delta_{\ell_q} \to \delta_{\mathbb{F}}]$ for the conjugate q, but the latter representation is admissible with respect to the weak* topology $\tau_p^{\mathrm{w}^*}$ by the previous corollary. □

Note that in case of $1 < p < \infty$ we could also conclude the claim from Corollary 1 and the fact that for these p the spaces ℓ_p are reflexive. For reflexive spaces the weak and the weak* topologies coincide.

In general the previous corollary opens a possibility to define a canonical representation of a separable reflexive normed space which is admissible with respect to the weak topology.

Corollary 5. *Let X be a reflexive normed space with some compatible representation δ. Define a representation δ^{w} of X by*

$$\delta^{\mathrm{w}}(p) = x :\Longleftrightarrow \delta''(p) = \iota_x$$

Then δ^{w} is admissible with respect to the weak topology on X.

Similarly as in Proposition 1 one can prove that for a compatible representation δ of X, the representation δ' is a compatible representation of X' (now using reflexivity instead of **ZF+AC**). Since X' always is complete, we can now apply Corollary 4 to δ' in order to derive the previous corollary.

[1] A subbase is given by the sets $\{f \in \mathcal{C}((X, \tau_\delta), \mathbb{F}) \mid f[K] \subseteq O\}$, where $K \subseteq X$ is sequentially compact and $O \subseteq \mathbb{F}$ is open.

p	**ZF+AC**	**ZF+DC+BP**
$p=\infty$	$\tau_{\|\|\ \|\|_\infty} \supsetneqq \begin{matrix}\mathrm{seq}(\tau_\infty^{\mathrm{w}}) \\ \\ \tau_\infty^{=}\end{matrix} \supsetneqq \mathrm{seq}(\tau_\infty^{\mathrm{w}^*}) = \tau_\infty^{\gtrless}$	$\tau_{\|\|\ \|\|_\infty} \supsetneqq \tau_\infty^{=} \supsetneqq \mathrm{seq}(\tau_\infty^{\mathrm{w}}) = \mathrm{seq}(\tau_\infty^{\mathrm{w}^*}) = \tau_\infty^{\gtrless}$
$1<p<\infty$	$\tau_{\|\|\ \|\|_p} = \tau_p^{=} \supsetneqq \mathrm{seq}(\tau_p^{\mathrm{w}}) = \mathrm{seq}(\tau_p^{\mathrm{w}^*}) = \tau_p^{\gtrless}$	
$p=1$	$\tau_{\|\|\ \|\|_1} = \tau_1^{=} = \mathrm{seq}(\tau_1^{\mathrm{w}}) \supsetneqq \tau_1^{\gtrless}$	

Fig. 1. Weak topologies on the ℓ_p spaces

5 Conclusions

In this paper we have proved that the possibilities to handle non-separable spaces on Turing machines sensitively rely on the underlying axiomatic setting. In **ZF+AC** non-separable normed spaces do not admit compatible representations whereas in **ZF+DC+BP** such representations do exist for certain spaces.

In particular we have studied the sequence spaces ℓ_p which can be handled in a uniform way and which include ℓ_∞ as a typical example of a non-separable normed space. The results for these spaces turned out to be surprisingly diverse and the table in Figure 1 summarizes the inclusions which we have obtained comparing different weak topologies for these spaces. The last two rows contain the results for $1 \le p < \infty$ that do not depend on the axiomatic setting.

Our results suggest that the setting **ZF+DC+BP** is more natural from the point of view of computable analysis. However, functional analysis is classically developed in **ZF+AC** and a **ZF+DC+BP** version would be substantially different, even classically. Nevertheless, even in **ZF+DC+BP** the separable version of the Hahn-Banach Theorem is available (see [5]). Hence for a computable version of functional analysis the setting **ZF+DC+BP** might be sufficient.

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