On Geometric Dilation and Halving Chords

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Abstract. Let G be an embedded planar graph whose edges may be curves. The *detour* between two points, p and q (on edges or vertices) of G, is the ratio between the shortest path in G between p and q and their Euclidean distance. The supremum over all pairs of points of all these ratios is called the *geometric dilation* of G. Our research is motivated by the problem of designing graphs of low dilation. We provide a characterization of closed curves of constant halving distance (i.e., curves for which all chords dividing the curve length in half are of constant length) which are useful in this context. We then relate the halving distance of curves to other geometric quantities such as area and width. Among others, this enables us to derive a new upper bound on the geometric dilation of closed curves, as a function of D/w, where D and w are the diameter and width, respectively. We further give lower bounds on the geometric dilation of polygons with n sides as a function of n. Our bounds are tight for centrally symmetric convex polygons.

1 Introduction

Consider a planar graph G embedded in $\mathbb{R}^2$, whose edges are curves that do not intersect. Such graphs arise naturally in the study of transportation networks, like waterways, railroads or streets. For two points, p and q (on edges or vertices) of G, the *detour* between p and q in G is defined as $\delta_G(p,q) = \frac{d_G(p,q)}{|pq|}$ where $d_G(p,q)$ is the shortest path length in G between p and q and $|pq|$ denotes the Euclidean distance. Good transportation networks should have small detour values. To measure the quality of e.g. a network of streets in a city we have to take into account not only the vertices of the graph but all the points on its edges,

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because access to the streets is possible from everywhere. The resulting supremum value is the *geometric dilation* of G. For example the geometric dilation of a square, or that of a square divided into 100 congruent squares is 2.

Ebbers-Baumann et al. [5] recently considered the problem of constructing a graph of lowest possible geometric dilation containing a given finite point set on its edges. They pointed out that even for three points this is a difficult task and proved that there exist point sets which require graphs with dilation at least $\pi/2$. They conjectured that this lower bound is not best possible, which was recently confirmed by Dumitrescu et al. in [4].

Ebbers-Baumann et al. have also shown that for any finite point set there exists a grid-like planar graph that contains the given points and whose geometric dilation is at most 1.678, thereby improving on $\sqrt{3}$, the geometric dilation obtained by embedding the points in a hexagonal grid. Their design uses a certain closed curve of constant halving distance, see Figure 4. Understanding such curves and their properties is a key point in designing networks with a small geometric dilation and is our current focus in this paper. Due to space limitations, some of our proofs are omitted.

2 Basic Definitions and Properties

Throughout this paper we consider finite, simple[1], closed curves in the Euclidean plane. We call them *closed curves* or *cycles* for short. For simplicity, we assume that they are piecewise continuously differentiable, but most of the proofs work for less restrictive differentiability conditions.

By $|C|$ we denote the length of a closed curve. Shortest path distance $d_C(p, q)$, detour $\delta_C(p, q)$ and geometric dilation $\delta(C)$ are defined like in the case of arbitrary graphs.

Ebbers-Baumann, Grüne and Klein [6] introduced halving pairs to facilitate the dilation analysis of closed curves. For a given point $p \in C$, the unique *halving partner* $\hat{p}$ of p is given by $d_C(p, \hat{p}) = |C|/2$. This means that both paths connecting p and $\hat{p}$ on C have equal length. The pair $(p, \hat{p})$ is called *halving pair* and the connecting line segment $p\hat{p}$ is a *halving chord*. The length of a halving chord is the corresponding *halving distance*. By h and H we will denote the *minimum* and *maximum halving distance* of a given closed curve.

Furthermore, we will consider the *diameter* $D := \max\{|pq|, p, q \in C\}$ of a closed curve C and the *width* w of a convex cycle C which is the minimum distance of two parallel lines enclosing C.

The following lemma is the main reason why halving pairs play a crucial role in the dilation analysis of closed curves.

Lemma 1. [6–Lemma 11] *If C is a closed convex curve, its dilation $\delta(C)$ is attained by a halving pair, i.e.* $\delta(C) = |C|/2h$.

[1] A curve is called *simple* if it has no self-intersections.

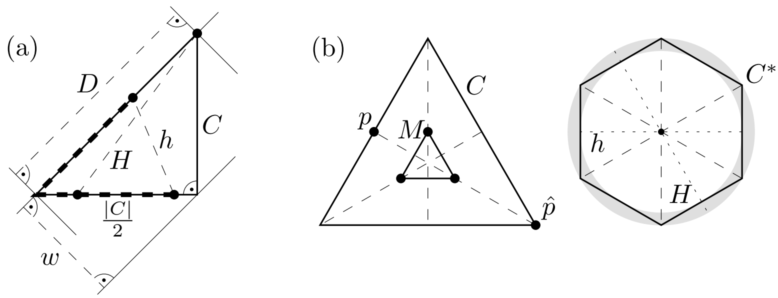

Fig. 1. (a) Diameter D, width w, minimum and maximum halving distance h and H of an isosceles, right-angled triangle (b) An equilateral triangle C and the derived curves M and C^*

In [4] Dumitrescu, Grüne and Rote have introduced a decomposition of a cycle C into two curves C^* and M, see Figure 1(b) for an illustration. Let $c : [0, |C|) \to C$ be an arc-length parameterization of C. Then, the two curves are defined by the parameterizations

$$m(t) := \frac{1}{2}\left(c(t) + c\left(t + \frac{|C|}{2}\right)\right), \quad c^*(t) := \frac{1}{2}\left(c(t) - c\left(t + \frac{|C|}{2}\right)\right) \tag{1}$$

The *midpoint curve* M is formed by the midpoints of the halving chords. It will turn out to be useful in the analysis of curves of constant halving distance. The curve C^* is the result of applying the *halving pair transformation* (defined in [6]) to C. It is obtained by moving the midpoint of every halving chord to the origin.

3 Closed Curves of Constant Halving Distance

Closed curves of constant halving distance turn up naturally if one wants to construct graphs of low dilation (compare to [5]). Lemma 1 shows that the dilation of any convex curve of constant halving distance is attained by all its halving pairs. Hence, it is difficult to improve (decrease) the dilation of such cycles, because local changes decrease h or they increase $|C|$.

Theorem 21 in [6] or the proof of Lemma 1 in [4] show that only curves with constant $m(t)$ and constant halving distance can attain the global dilation minimum of $\pi/2$. It is easy to see that only circles satisfy both conditions (compare to [6–Corollary 23], [1–Corollary 3.3], [9], [7]).

What happens if only one of the conditions is satisfied? Clearly, $m(t)$ is constant if and only if C is centrally symmetric. The class of closed curves of constant halving distance is not as easy to describe. One could guess — incorrectly — that it consists only of circles. The "Rounded Triangle" $C_\triangle$ shown in Figure 2 is a counterexample, and could be seen as an analogy to the Reuleaux triangle [2], the most popular representative of curves of constant

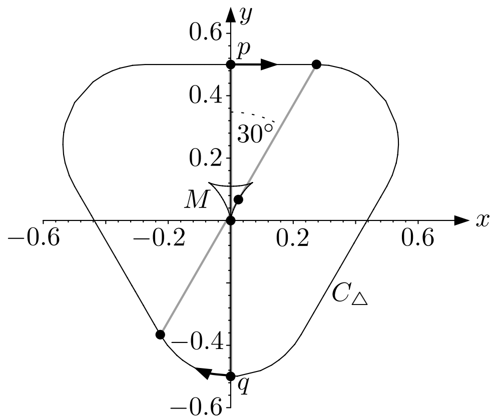

Fig. 2. The "Rounded Triangle", a curve of constant halving distance

width. It seems to be a somehow prominent example, because two groups of the authors of this paper discovered it independently.

We construct $C_\triangle$ by starting with a pair of points $p := (0, 0.5)$ and $q := (0, -0.5)$. Next, we move p to the right along a horizontal line. Simultaneously, q moves to the left such that the distance $|pq| = 1$ is preserved and both points move with equal speed. It can be shown that these conditions lead to a differential equation whose solution defines the path of q uniquely. We move p and q like this until the connecting line segment pq forms an angle of 30° with the y-axis. Next, we swap the roles of p and q. Now, q moves along a line with the direction of its last movement, and p moves with equal speed on the unique curve which guarantees $|pq| = 1$, until pq has rotated with another 30°. In this way we concatenate six straight line and six curved pieces to build the Rounded Triangle $C_\triangle$ depicted in Figure 2.

We have to omit the details of the differential equation and its solution. Here, we mention only that the perimeter of $C_\triangle$ equals $3 \ln 3$. By Lemma 1 this results in

$$\delta(C_\triangle) = |C_\triangle|/(2h(C_\triangle)) = \frac{3}{2} \ln 3 \approx 1.6479\ .$$

The midpoint curve of $C_\triangle$ is built from six congruent pieces that are arcs of a tractrix, which we will discuss in the end of this section. First, we give a necessary and sufficient condition for curves of constant halving distance.

Theorem 1. *Let C be a planar closed curve, and let $c : [0, |C|) \to C$ be an arc-length parameterization. Then, the following two statements are equivalent:*

1. *If c is differentiable in t and in $t + |C|/2$, $\dot{m}(t) \neq 0$, and $\dot{c}^*(t) \neq 0$, then the halving chord $c(t)c(t + |C|/2)$ is tangent to the midpoint curve at $m(t)$. And if the midpoint stays at $m \in \mathbb{R}^2$ on a whole interval (t_1, t_2), the halving pairs are located on the circle with radius $h(C)/2$ and center point m.*
2. *The closed curve C is a cycle of constant halving distance.*

Proof. "*2.* ⇒ *1.*"

Let C have constant halving distance. If c is differentiable in t and $t+|C|/2$, c^* and m are differentiable in t. And due to $|c^*| \equiv h(C)/2$ it follows that $\dot{c}^*(t)$ must be orthogonal to $c^*(t)$ which can be shown by

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}|c^*(t)|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle c^*(t), c^*(t)\rangle = 2\langle c^*(t), \dot{c}^*(t)\rangle \ . \tag{2}$$

On the other hand, by using the linearity of the scalar product and $|\dot{c}(t)| = 1$, we obtain

$$\begin{aligned} \langle \dot{m}(t), \dot{c}^*(t)\rangle &\overset{(1)}{=} \frac{1}{4}\left\langle \dot{c}(t) + \dot{c}\left(t + \frac{|C|}{2}\right), \dot{c}(t) - \dot{c}\left(t + \frac{|C|}{2}\right)\right\rangle \\ &= \frac{1}{4}\left(|\dot{c}(t)|^2 - \left|\dot{c}\left(t + \frac{|C|}{2}\right)\right|^2\right) = \frac{1}{4}(1-1) = 0. \end{aligned} \tag{3}$$

The derivative vectors $\dot{m}(t)$ and $\dot{c}^*(t)$ are orthogonal. Hence, $\dot{m}(t) \neq 0 \neq \dot{c}^*(t)$ implies $\dot{m}(t) \parallel c^*(t)$ and the first condition of 1. is proven. The second condition follows trivially from $c(t) = m(t) + c^*(t)$.

"*1.* ⇒ *2.*"

Let us assume that both conditions of 1. hold. We have to show that $|c^*(t)|$ is constant.

First, we consider an interval $(t_1, t_2) \subseteq [0, |C|)$, where $m(t)$ is constant $(= m)$ and the halving pairs are located on a circle with radius $h(C)/2$ and center m. This immediately implies that $|c^*|$ is constant on (t_1, t_2).

If $(t_1, t_2) \subseteq [0, |C|)$ denotes an interval where $|c^*(t)| = 0$, then obviously $|c^*|$ is constant.

Now, let $(t_1, t_2) \subseteq [0, |C|)$ be an open interval where $c(t)$ and $c\left(t + \frac{|C|}{2}\right)$ are differentiable and $\dot{m}(t) \neq 0$ and $\dot{c}^*(t) \neq 0$ for every $t \in (t_1, t_2)$. We follow the proof of "*2.* ⇒ *1.*" in the opposite direction. Equation (3) shows that $\dot{c}^*(t) \perp \dot{m}(t)$ and the first condition of 1. gives $c^*(t) \parallel \dot{m}(t)$. Combining both statements results in $\dot{c}^*(t) \perp c^*(t)$ which by (2) yields that $|c^*(t)|$ is constant.

The range $[0, |C|/2)$ can be divided into countably many disjoint intervals $[t_i, t_{i+1})$ where m and c^* are differentiable on the open interval (t_i, t_{i+1}), and one of the three conditions $\dot{m}(t) = 0$, $\dot{c}^*(t) = 0$ or $\dot{m}(t) \neq 0 \neq \dot{c}^*(t)$ holds for the whole interval (t_i, t_{i+1}). We have shown that $|c^*|$ must be constant on all these open intervals. Thus, due to c^* being continuous on $[0, |C|/2)$, $|c^*|$ must be globally constant. □

The theorem shows that curves of constant halving distance can consist of three types of parts; parts where the halving chords lie tangentially to the midpoint curve, circular arcs of radius $h(C)/2$, and parts where $\dot{c}^*(t) = 0$ and the halving pairs are only moved by the translation due to m, i.e., for every τ_1 and τ_2 within such a part we have $c(\tau_2) - c(\tau_1) = c(\tau_2 + |C|/2) - c(\tau_1 + |C|/2) = m(\tau_2) - m(\tau_1)$. For convex cycles of constant halving distance, the translation parts cannot occur:

Lemma 2. *Let C be a closed convex curve of constant halving distance. Then there exists no non-empty interval $(t_1, t_2) \subset [0, |C|)$ such that c^* is constant on (t_1, t_2).*

Proof. Assume that c^* is constant on (t_1, t_2) and choose s_1, s_2 with $t_1 < s_1 < s_2 < t_2$ and $s_2 < s_1 + |C|/2$. If the four points $p_1 = C(s_1)$, $p_2 = C(s_2)$, $p_3 = C(s_2 + |C|/2)$, $p_4 = C(s_1 + |C|/2)$ don't lie on a line, they form a parallelogram in which p_1p_4 and p_2p_3 are parallel sides. However, these points appear on C in the cyclic order $p_1p_2p_4p_3$, which is different from their convex hull order $p_1p_2p_3p_4$ (or its reverse), a contradiction. The case when the four points lie on a line ℓ can be dismissed easily (convexity of C implies that the whole curve C would have to lie on ℓ, but then C could not be a curve of constant halving distance). □

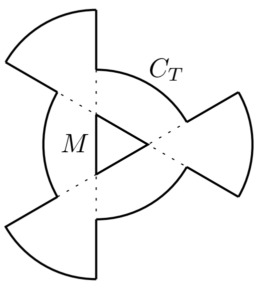

Fig. 3. C_T consists of translation parts and circular arcs

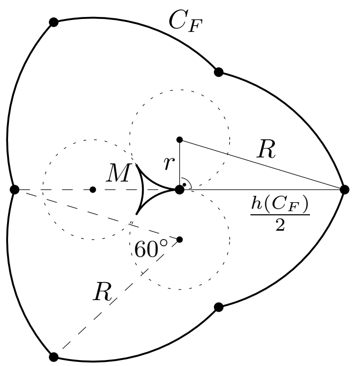

Fig. 4. The "Flower" from [5] is a non-convex cycle of constant halving distance

Figures 3 and 4 show examples of non-convex cycles of constant halving distance. The first one, C_T, demonstrates that such closed curves can indeed include translation parts. The second one, C_F, was used in [5] to build a grid of low dilation. It turned out that the non-convex parts were useful for this purpose although the dilation of this "Flower" is[2] $\delta(C_F) = 1.6787\ldots$, which is somewhat larger than the dilation of the Rounded Triangle.

Now we show that the midpoint curve of the Rounded Triangle $C_\triangle$ is built from six tractrix pieces. The tractrix is illustrated in Figure 5. A watch is placed on a table, say at the origin $(0, 0)$ and the end of its watchchain of length 1 is pulled along the horizontal edge of the table starting at $(0, 1)$, either to the left or to the right. As the watch is towed in the direction of the chain, the chain is always tangential to the path of the watch, the tractrix. There are several known

[2] Of course, the dilation of the "Flower" depends on the values of r and R. The radii chosen in [5] result in the cited dilation value.

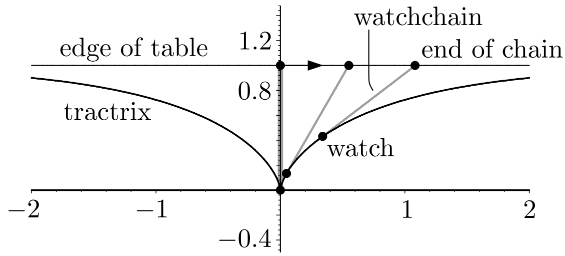

Fig. 5. The tractrix, the curve of a watch on a table town with its watchchain (the curve is symmetric about the y-axis)

parameterizations. We will use one of them in the end of section 4.1 to calculate the area of $C_\triangle$.

From the definition it is clear that the midpoint curve of the cycle $C_\triangle$ consists of such tractrix pieces, scaled by 1/2, because by definition and Theorem 1 its halving chords are always tangential to the midpoint curve, always one of the points of these pairs is moving on a straight line, and its distance to the midpoint curve stays 1/2.

4 Relating Halving Distance to Other Geometric Quantities

One of the most important topics in convex geometry is the relation between different geometric quantities of convex bodies like area A and diameter D. Scott and Awyong [10] give a short survey of basic inequalities in $\mathbb{R}^2$. For example, it is known that $4A \leq \pi D^2$, and equality is attained only by circles, the so-called *extremal set* of this inequality.

In this context the minimum and maximum halving distance h and H give rise to some new interesting questions, namely the relation to other basic quantities. As the inequality $h \leq w$ is immediate from definition, the known upper bounds on w hold for h as well. However, not all of them are tight for h. One counter-example ($A \geq w^2/\sqrt{3} \geq h^2/\sqrt{3}$) will be discussed in the following subsection.

4.1 Minimum Halving Distance and Area

Here, we consider the relation between the minimum halving distance h and the area A (for convex cycles). Clearly, the area can get arbitrarily big while h stays constant. For instance this is the case for a rectangle of smaller side length h where the bigger side length tends to infinity.

How small the area A can get for a given minimum halving distance h? A first answer $A \geq h^2/\sqrt{3}$ is easy to prove, because it is known [11–ex. 6.4, p.221] that $A \geq w^2/\sqrt{3}$, and we combine this with $w \geq h$. This bound is not tight since the equilateral triangle is the only closed curve attaining $A = w^2/\sqrt{3}$ and its

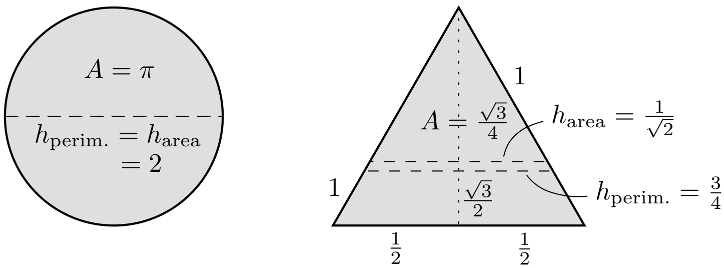

Fig. 6. The equilateral triangle has a smaller ratio $A/h^2_{\text{perim.}} = 4/(3\sqrt{3}) \approx 0.770$ and a bigger ratio $A/h^2_{\text{area}} = \sqrt{3}/2 \approx 0.866$ than the circle ($\pi/4 \approx 0.785$)

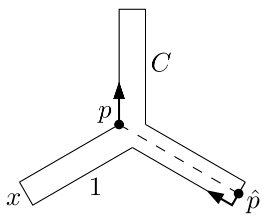

Fig. 7. By decreasing x we can make the area of this closed curve arbitrarily small while h stays bounded

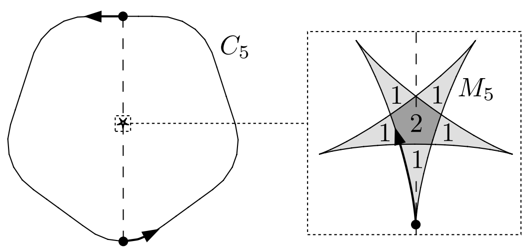

Fig. 8. The midpoint curve of a rounded pentagon, constructed analogously to $C_\triangle$ of Figure 2, contains regions with winding number 2 and regions with winding number 1

width $w = \sqrt{3}/2 \approx 0.866$ (for side length 1) is strictly bigger than its minimum halving distance $h = 3/4 = 0.75$.

For the analogous problem considering chords bisecting the *area* instead of chords halving the *perimeter*, Santaló conjectured[3]that $A \geq (\pi/4)h^2_{\text{area}}$ (see [3–A26, p.37]). Note that equality is attained by a circular disk. As pointed out earlier, in the case of perimeter halving distance this inequality does not hold: the equilateral triangle gives a counterexample, $A/h^2 = \frac{\sqrt{3}}{4}/\frac{9}{16} \approx 0.770 < 0.785 \approx \frac{\pi}{4}$, see Figure 6. But we do not know if the equilateral triangle is the convex cycle minimizing A/h^2. On the other hand A/h^2 can become arbitrarily small if we drop the convexity condition, see Figure 7.

Not only the equilateral triangle attains a smaller ratio A/h^2 than the circle, so does every curve of constant halving distance.

Lemma 3. *If C is a convex cycle of constant halving distance h, its area satisfies $A = (\pi/4)h^2 - 2A(M)$ where $A(M)$ denotes the area bounded by the midpoint curve M. In $A(M)$, the area of any region encircled several times by M is counted*

[3] We would like to thank Salvador Segura Gomis for pointing this out.

with the multiplicity of the corresponding winding number, see Figure 8 for an example. In particular, $A \leq (\pi/4)h^2$.

The idea (proof omitted here) is: assuming $h = 2$, we consider parameterizations $c^*(\alpha) = (\cos\alpha, \sin\alpha)$ and $\dot{m}(\alpha) = v(\alpha)(\cos\alpha, \sin\alpha)$ which exist by Theorem 1 and Lemma 2. Then, we calculate $A = \int x\mathrm{d}y$ for both curves, C and M, and take advantage of the periodicity of v.

The theorem shows that the circle is the cycle of constant halving distance attaining maximum area. But which cycle of constant halving distance attains minimum area? We conjecture that the answer is the Rounded Triangle $C_\triangle$. Lemma 3 helps us to calculate its area $A(C_\triangle)$. The tractrix-construction of the midpoint curve M makes it possible to get a closed form for $A(M)$. It results in

$$A(C_\triangle) = \pi\frac{h^2}{4} - 2A(M) = (\pi - 2\cdot 0.01976\ldots)\frac{h^2}{4} = 0.7755\ldots\cdot h^2.$$

4.2 Minimum Halving Distance and Width

In order to achieve a lower bound to h in terms of w, we examine the relation of both quantities to the area A and the diameter D. The following inequality was first proved by Kubota [8] in 1923 and is listed in [10].

Lemma 4 (Kubota [8]). *If C is a convex curve, then* $A \geq Dw/2$.

We will combine this known inequality with the following new result.

Lemma 5. *If C is a convex curve, then* $A \leq hD$.

Proof. Without loss of generality we assume that a halving chord pq of minimum length h lies on the y-axis, p on top and q at the bottom (see Figure 9). Let C_- be the part of C with negative x-coordinate and let $C_+ := C \setminus C_-$ be the remainder. We have $|C_-| = |C_+| = |C|/2$ because pq is a halving chord.

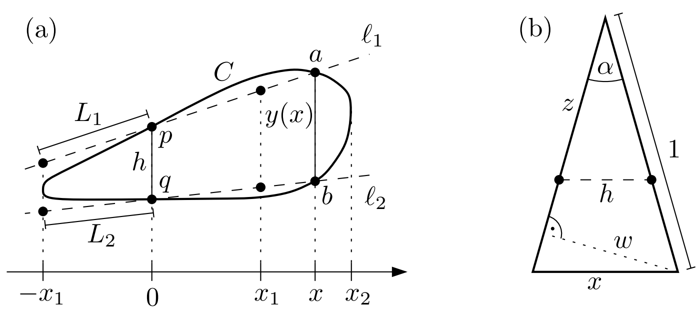

Fig. 9. (a) Proving by contradiction that $y(x) \leq h$ for every x in $[x_1, x_2]$. (b) In a thin isosceles triangle $h/w \searrow 1/2$ if $\alpha \to 0$

Let $-x_1$ and x_2 denote the minimum and maximum x-coordinate of C. Note that x_1 has a positive value. We assume that $x_2 > x_1$. Otherwise we could mirror the situation at the y-axis. Let $y(x)$ be the length of the vertical line segment of x-coordinate x inside C, for every $x \in [-x_1, x_2]$. These definitions result in $x_1 + x_2 \leq D$ and $A = \int_{-x_1}^{x_2} y(x)\mathrm{d}x$. Furthermore, the convexity of C implies

$$\forall x \in [0, x_1] : \quad y(-x) + y(x) \leq 2h \ . \tag{4}$$

As a next step, we want to show that

$$\forall x \in [x_1, x_2] : \quad y(x) \leq h \ . \tag{5}$$

We assume that $y(x) > h$. Let ab be the vertical segment of x-coordinate x inside C, a on top and b at the bottom. Then, we consider the lines ℓ_1 through p and a and ℓ_2 through q and b. Let L_1 (L_2) be the length of the piece of ℓ_1 (ℓ_2) in the x-interval $[0, x_1]$. By construction the corresponding lengths in the x-interval $[-x_1, 0]$ are equal. Then, by the convexity of C, we have $|C_-| \leq L_1 + L_2 + h < L_1 + L_2 + y(x) \leq |C_+|$. This contradicts to pq being a halving chord, and the proof of (5) is completed.

Now we can plug everything together and get

$$\begin{aligned} A &= \int_{-x_1}^{x_2} y(x)\mathrm{d}x = \int_{0}^{x_1} y(-x) + y(x)\mathrm{d}x \ + \int_{x_1}^{x_2} y(x)\mathrm{d}x \\ &\overset{(4),(5)}{\leq} x_1 \cdot 2h + (x_2 - x_1)h = (x_1 + x_2)h \leq Dh \ . \end{aligned}$$

□

Finally, we obtain the desired inequality relating h and w.

Lemma 6. *If C is a convex curve, then $h \geq w/2$. This bound cannot be improved.*

Proof. The inequality follows directly from Lemma 4 and Lemma 5. To see that the bound is tight, consider a thin isosceles triangle like that depicted in Figure 9(b) and let α tend to 0. □

5 Dilation Bounds

5.1 Upper Bound on Geometric Dilation

Our Lemma 6 leads to a new upper bound depending only on the ratio D/w. This complements the lower bound

$$\delta(C) \geq \arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{D}{w}\right)^2 - 1} \tag{6}$$

of Ebbers-Baumann et al.[6–Theorem 22]. The new upper bound is stated in the following theorem.

Theorem 2. *If C is a convex curve, then*

$$\delta(C) \leq 2\left(\frac{D}{w}\arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{D}{w}\right)^2 - 1}\right) .$$

Proof. Kubota [8] (see also [10]) showed that

$$|C| \leq 2D\arcsin\left(\frac{w}{D}\right) + 2\sqrt{D^2 - w^2} . \tag{7}$$

Combining this with Lemma 6 and Lemma 1 yields

$$\delta(C) \overset{\text{Lem.1}}{=} \frac{|C|}{2h} \overset{\text{Lem.6}}{\leq} \frac{|C|}{w} \overset{(7)}{\leq} 2\left(\frac{D}{w}\arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{D}{w}\right)^2 - 1}\right) . \qquad \square$$

5.2 Lower Bounds on the Geometric Dilation of Polygons

In this subsection we apply the lower bound (6) of Ebbers-Baumann et al.[6] to deduce lower bounds on the dilation of polygons with n sides (in special cases we proceed directly). We start with the case of a triangle (and skip the easy proof):

Lemma 7. *For any triangle C, $\delta(C) \geq 2$. This bound cannot be improved.*

Note that plugging the well-known inequality $D/w \geq 2/\sqrt{3}$ into (6) would only give $\delta(C) \geq \pi/3 + 1/\sqrt{3} \approx 1.624$. We continue with the case of centrally symmetric convex polygons, for which we obtain a tight bound.

Theorem 3. *If C is a centrally symmetric convex n-gon (n even), then*

$$\delta(C) \geq \frac{n}{2}\tan\frac{\pi}{n}.$$

This bound cannot be improved.

Proof. We adapt the proof of Theorem 22 in [6], which proves inequality (6) for closed curves. Since C is centrally symmetric, it must contain a circle of radius $r = h/2$. It can easily be shown (using the convexity of the tangent function) that the shortest n-gon containing such a circle is a regular n-gon. Its length equals $2rn\tan\pi/n$ which further implies that

$$\delta(C) \overset{\text{Lemma 1}}{=} \frac{|C|}{2h} \geq \frac{2rn\tan\frac{\pi}{n}}{2r} = \frac{n}{2}\tan\frac{\pi}{n}. \tag{8}$$

The bound is tight for a regular n-gon. $\square$

In the last part of this section we address the case of arbitrary (not necessarily convex) polygons. Let C be a polygon with n vertices, and let $C' = conv(C)$. Clearly C' has at most n vertices. By Lemma 9 in [6], $\delta(C) \geq \delta(C')$. Further on, consider $C'' = \frac{C'+(-C')}{2}$, the convex curve obtained by *central symmetrization* from C' (see [11, 6]). It is easy to check that C'' is a convex polygon, whose number of vertices is at most twice that of C', therefore at most $2n$. One can now replace n by $2n$ in (8) to obtain a lower bound on the geometric dilation for any polygon with n sides.

Corollary 1. *The geometric dilation of any polygon C with n sides satisfies*

$$\delta(C) \geq n \tan \frac{\pi}{2n}.$$

References

1. A. Abrams, J. Cantarella, J. Fu, M. Ghomi, and R. Howard. Circles minimize most knot energies. *Topology*, 42(2):381–394, 2002.
2. G.D. Chakerian and H. Groemer. Convex bodies of constant width. *Convexity and its Applications*, pages 49–96, 1983.
3. H.P. Croft, K. J. Falconer, and R.K. Guy. *Unsolved Problems in Geometry.* Springer-Verlag, 1991.
4. A. Dumitrescu, A. Grüne, and G. Rote. On the geometric dilation of curves and point sets. preprint, July 2004.
5. A. Ebbers-Baumann, A. Grüne, and R. Klein. On the geometric dilation of finite point sets. In *14th Annual International Symposium on Algorithms and Computation*, volume 2906 of *LNCS*, pages 250–259. Springer, 2003.
6. A. Ebbers-Baumann, A. Grüne, and R. Klein. Geometric dilation of closed planar curves: New lower bounds. *to appear in special issue of CGTA dedicated to Euro-CG '04*, 2004.
7. M. Gromov, J. Lafontaine, and P. Pansu. Structures métriques pour les variétés riemanniennes. volume 1 of *Textes math.* CEDIC / Fernand Nathan, Paris, 1981.
8. T. Kubota. Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen. *Sci. Rep. Tŏhoku Univ.*, 12:45–65, 1923.
9. R.B. Kusner and J.M. Sullivan. On distortion and thickness of knots. In S. G. Whittington, D. W. Sumners, and T. Lodge, editors, *Topology and Geometry in Polymer Science*, volume 103 of *IMA Volumes in Math. and its Applications*, pages 67–78. Springer, 1998.
10. P.R. Scott and P.W. Awyong. Inequalities for convex sets. *Journal of Inequalities in Pure and Applied Mathematics*, 1(1), 2000. http://jipam.vu.edu.au/article.php?sid=99.
11. I.M. Yaglom and V.G. Boltyanski. *Convex Figures.* English Translation, Holt, Rinehart and Winston, New York, NY, 1961.