# States, Transitions, and Life Tracks in Temporal Concept Analysis

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**Abstract.** Based on Formal Concept Analysis, we introduce Temporal Concept Analysis as a temporal conceptual granularity theory for movements of general objects in abstract or "real" space and time such that the notions of states, situations, transitions and life tracks of objects in conceptual time systems are defined mathematically. The life track lemma is a first approach to granularity reasoning. Applications of Temporal Concept Analysis in medicine and in chemical industry are demonstrated as well as recent developments of computer programs for graphical representations of temporal systems. Basic relations between Temporal Concept Analysis and other temporal theories, namely theoretical physics, mathematical system theory, automata theory, and temporal logic are discussed.

## 1 Introduction

The purpose of this paper is to present the actual state of Temporal Concept Analysis (TCA), a conceptual granularity theory for the treatment of temporal phenomena. In TCA, not only space and time but also objects and their movements are represented conceptually, including a granularity description based on the notion of formal concepts and conceptual scales. The classical point of view on temporal phenomena is dominated by classical mechanics describing space and time using the continuum of real numbers and by automata theory using an abstract notion of discrete states and transitions without an explicit time description. Therefore, it is desirable to develop a general temporal theory covering continuous as well as discrete temporal systems.

Clearly, such a unification demands a background theory based on general basic concepts. Such a theory emerged from lattice theory, introduced by Garrett Birkhoff [Bir67] as a common generalization of ordered structures in geometry and logic. Rudolf Wille [Wil82] brought a vivid real world relevance into the theory of abstract lattices by his introduction of *formal contexts* and their *concept lattices*. His purpose was to restructure lattice theory in the sense of Hartmut von Hentig's claim to restructure sciences [vHe72]. Concept lattices are used to

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describe the conceptual structures inherent in data tables without loss of information by means of *line diagrams* yielding valuable visualizations of real data.

The conceptual representation of temporal phenomena started with the usual order representation of time as a chain; later on *interordinal scales* proved extremely useful for working with temporal (or spatial) intervals. Based on the idea of an infinite interordinal scale Rudolf Wille introduced *linear continuum structures* "making mathematically explicit the Aristotelian conception of a time continuum" [Wil04].

Clearly, time has to be investigated in connection with a notion of space to represent movements of objects. Based on experiences with real data from psychological and industrial applications the author [Wol00a] combined the idea of a *time granule* like "this morning" with the idea of a *state* by introducing the mathematical notion of a *Conceptual Time System*. That led to a conceptual investigation of the notion of an *object* in the sense of a *spatio-temporal object*. Such an object is given by its *actual objects* which are connected by a *time relation* yielding a *life track* which represents the object [Wol02a, Wol02b]. That led to a purely conceptual understanding of movements of objects in continuous or discrete space and time – without employing the classical algebraic, metric and analytic structures. In the following sections we give a short overview over the main ideas in Temporal Concept Analysis as it is developed now. For that purpose we start with a simple example of a journey.

## 2 Contextual Description of a Journey

In this section we discuss some basic contextual descriptions of temporal and spatial aspects of a journey. In the following we assume that the reader is familiar with the basic definitions in Formal Concept Analysis, in particular with its Conceptual Scaling Theory [GaWi89, GaWi99]. For a short introduction we refer to [Wil97a, Wol94].

### 2.1 John's Journey

In this subsection we start with an example of a typical spatio-temporal description, namely a story about a journey. This example will be used throughout the paper to introduce the main ideas in Temporal Concept Analysis.

> **The Story of John's Journey:** John flies from Frankfurt to Napoli leaving Frankfurt on Thursday, returning on Sunday. John takes a flight on Thursday morning, arriving at Napoli in the afternoon. He visits a conference on Friday morning and the conference dinner on Saturday evening; he leaves Napoli on Sunday afternoon arriving at Frankfurt in the evening.

The following representation of this story does not represent its full linguistic structure. We only try to grasp the spatio-temporal structure and the granularity of the story. First we describe some basic temporal aspects. For that purpose we focus on the days from Thursday to Sunday. To represent the natural ordering of these days we employ a chain with four elements. The contextual representation of such a chain is given in Table 1:

**Table 1.** A formal context for a chain with four elements

| greater or equal | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | × | | | |
| 2 | × | × | | |
| 3 | × | × | × | |
| 4 | × | × | × | × |

Replacing the four numbers by the four days of interest we get from the *abstract ordinal scale* in Table 1 the *concrete scale* for a chain of the four days. The line diagrams in Figure 1 represent the corresponding concept lattices.


**Fig. 1.** Concept lattices of an "abstract" and a corresponding "concrete" scale

Similarly, the day times "morning", "afternoon", "evening" are described by a concept lattice which is a chain with three elements. The direct product of these chains represents the "time schedule of John's journey" (in Figure 2) which is again described as a concept lattice.

We consider the corresponding formal context since it gives us a first hint towards an understanding of the notion of *time granules*. The chosen granularity of the temporal description yields $4 \times 3 = 12$ "possible time granules", for example (Saturday, afternoon), which are the formal objects in Figure 2. Table 2 shows three of the twelve rows of the mentioned context, namely the *time granules* of Saturday:

The complete formal context of this simple and important combination of two scales has as objects all pairs of objects of the two given formal contexts; it has as attribute set the (disjoint) union of the two given attribute sets; and its incidence relation is constructed by copying the given incidence relations, for example: (Saturday, afternoon) gets a cross at all those attributes of the first context where "Saturday" has a cross there, and a cross at all those attributes of the second context where "afternoon" has a cross there. (The formal definition of a semiproduct of two contexts is given in [GaWi99], p.46.)

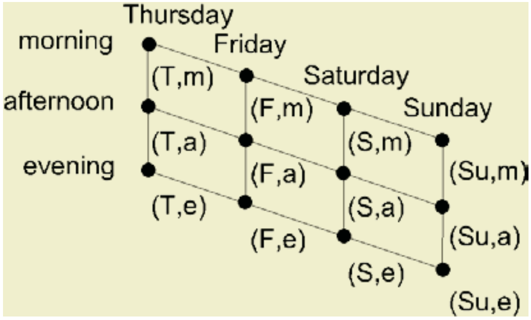


**Fig. 2.** A concept lattice representing the "time schedule of John's journey"

**Table 2.** Three of twelve rows of the so-called semiproduct of the two time scales

| | Thursday | Friday | Saturday | Sunday | morning | afternoon | evening |
|---|---|---|---|---|---|---|---|
| (Saturday, morning) | × | × | × | | × | | |
| (Saturday, afternoon) | × | × | × | | × | × | |
| (Saturday, evening) | × | × | × | | × | × | × |

To represent the spatial descriptions of the journey we construct a scale for the mentioned places "Frankfurt" and "Napoli" . We want to say that the town Napoli is situated south of Frankfurt. That is done in the following formal context:

**Table 3.** An ordinal conceptual scale for the places

| southern of or equal to | Frankfurt | Napoli |
|---|---|---|
| Frankfurt | × | |
| Napoli | × | × |

Clearly, the concept lattice of this context is, as a chain with two elements, a very simple map; a typical plane map with many towns can be represented in the same way by a direct product of two chains with many elements. The metric embedding into the usual plane can be made as fine as necessary; that is not discussed here. In the next section, we continue our example, describing the introduction of *conceptual time systems*, *time granules*, *situations*, and *states*.

## 3 Basic Notions in Temporal Concept Analysis

The author started the conceptual investigation of temporal phenomena with the key-idea that the *states* of a temporal system should be described as the object concepts of a suitable formal context. Since his search for useful descriptions of *states* in Mathematical System Theory, in physics, in Automata Theory, and several other domains did not yield, up to now, a conceptually satisfactory result [Zad64, Arb70, Eil74, Cast98, But99], the notion of a *state* in a *conceptual time system* has been introduced [Wol00a, Wol00b, Wol02b].

### 3.1 Time Granules as Formal Objects, States as Object Concepts

Searching for a general notion of a *state of a system*, we introduce first the definition of a *conceptual time system.* A general system description has to contain the elementary system descriptions that occur when we observe a real system and write down a finite protocol, usually represented as a data table. Therefore, we develop the main ideas in that framework. Indeed, we shall see that even infinite temporal systems can be described in the same way.

Let us imagine that we observe a real system. For a single observation we need some time, may be one minute or only one millisecond. Often we abstract from the duration of an observation and use the notion of a *point of time*, usually represented as a real number.

In the following, we do not assume any internal structure of such a *point of time*, as for example the assumption that it is a real interval or a real number. We just start from a set G; the elements of G are called *time granules.* For describing the observations, we introduce a many-valued context with G as its set of formal objects. In a data table of this many-valued context the row of a time granule g shows in column m the value m(g) of the *measurement* m *at time granule* g.

For clarifying our idea of a *conceptual time system*, we first consider the data table for John's journey in Table 4 where the integers 0,1,. . .,5 represent the six *time granules* which "occurred in the story of John's journey". Their meaning is described by the values in the two columns of the *time part* of the data table. The place of John at each of these time granules is described in the *event part* (or *space part*) of the data table. Together with the previously mentioned scales for the time part and the scale for the event part, we obtain an initial example of a *conceptual time system.*

**Table 4.** A data table of a conceptual time system

| | time part | | event part |
|---|---|---|---|
| time granules | day | day time | place |
| 0 | Thursday | morning | Frankfurt |
| 1 | Thursday | afternoon | Napoli |
| 2 | Friday | morning | Napoli |
| 3 | Saturday | evening | Napoli |
| 4 | Sunday | afternoon | Napoli |
| 5 | Sunday | evening | Frankfurt |

**Definition [Wol00a]: "conceptual time system, situations, states"**
Let $\mathbf{T} := ((G, M, W, I_T), (\mathbf{S}_m \mid m \in M))$ and $\mathbf{C} := ((G, E, V, I), (\mathbf{S}_e \mid e \in E))$ be scaled many-valued contexts on the same object set G. Then the pair $(\mathbf{T}, \mathbf{C})$

is called a *conceptual time system on the set G of time granules.* **T** is called the *time part* and **C** the *event part* or the *space part* of (**T**, **C**). The derived context of **T** ([GaWi89, GaWi99]) is denoted by $\mathbb{K}_T$, the derived context of **C** is denoted by $\mathbb{K}_C$, and the apposition of $\mathbb{K}_T$ and $\mathbb{K}_C$ is denoted by $\mathbb{K}_{TC} := \mathbb{K}_T|\mathbb{K}_C$. It is called the derived context of the conceptual time system (**T**, **C**).
The object concepts of $\mathbb{K}_{TC}$ are called *situations*, the object concepts of $\mathbb{K}_C$ are called *states*, and the object concepts of $\mathbb{K}_T$ are called *time states*. The sets of situations, states, and time states are called *the situation space*, *the state space*, and *the time state space* of (**T**, **C**), respectively. The object concept mappings of $\mathbb{K}_{TC}$, $\mathbb{K}_T$, and $\mathbb{K}_C$ are denoted by $\gamma$, $\gamma_T$, and $\gamma_C$, respectively.

For the conceptual time system of John's journey the derived context $\mathbb{K}_{TC}$ is represented in the next table.

**Table 5.** The derived context of John's journey

| $\mathbb{K}_{TC}$ | time part $\mathbb{K}_T$ | | | | | | | event part $\mathbb{K}_C$ | |
|---|---|---|---|---|---|---|---|---|---|
| time gran. | Thursday | Friday | Saturday | Sunday | morn. | aftern. | evening | Frankfurt | Napoli |
| 0 | × | | | | × | | | × | |
| 1 | × | | | | × | × | | × | × |
| 2 | × | × | | | × | | | × | × |
| 3 | × | × | × | | × | × | × | × | × |
| 4 | × | × | × | × | × | × | | × | × |
| 5 | × | × | × | × | × | × | × | × | |

The subcontext $\mathbb{K}_C$ given by the first column and the two last columns of Table 5 is called the "the event part of $\mathbb{K}_{TC}$". The concept lattice of $\mathbb{K}_C$ is drawn in Figure 3. Its object concepts represent quite well our usual understanding of *states*, namely that each system is at each time granule in exactly one state.

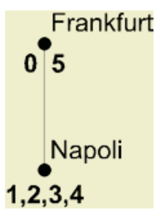


**Fig. 3.** The concept lattice of the event part $\mathbb{K}_C$ for John's journey

To visualize the time states we embed the concept lattice of the time part $\mathbb{K}_T$ into the lattice in Figure 2. The black circles in Figure 4 represent the concepts of the time part; the black ones which are numbered represent the six time states. In the right part of Figure 4, we have drawn some arrows indicating the temporal sequence in which John's journey happens. That will be discussed more extensively in the next subsection.

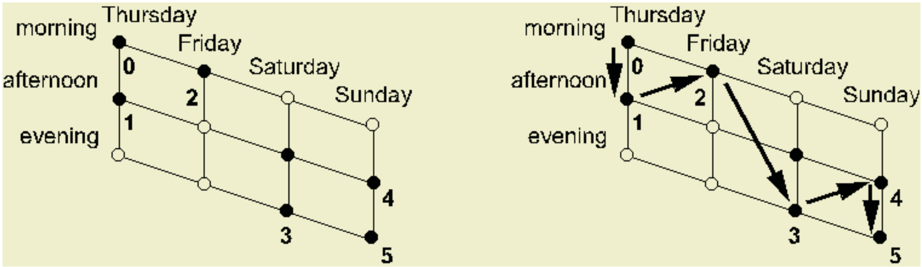


**Fig. 4.** Embedding the concept lattice of the time part into the "time schedule"

### 3.2 The Time Relation, Transitions, and Life Tracks

For our approach it is important that the notion of a *state* is introduced in a meaningful way without using an ordering of *the time*. What is *the time* in a conceptual time system? We have introduced several temporal notions, namely *time granules* and a *time part*, whose attributes are interpreted as *time measurements* and their scales as *time scales*. Thence we have mathematically defined *situations*, *states*, and *time states*. But for all that we did not need any notion of an ordering. In the example, we have used the integers 0,1,. . .,5 to represent time granules. We have written them down in the first column of Table 5 in their natural ordering. But since the sequence of the names of the (formal) objects in a data table of a (many-valued) context is not represented in the mathematical definition of a (many-valued) context we have to make it explicit formally.

### 3.3 The Time Relation

In many temporal systems we wish to express the "natural temporal ordering". To investigate carefully the conceptual role of temporal orderings we have to decide where we should introduce some ordinal structure; there are three main possibilities: in the time scales, on the time values, or on the time granules. In the following we describe a simple way to represent "the temporal ordering" by introducing a relation R, called the *time relation*, on the set G of time granules of a conceptual time system. Then we speak of a *conceptual time system with a time relation* (**CTST**).

**Definition [Wol02a]: "conceptual time system with a time relation"**
Let (**T**, **C**) be a conceptual time system on G and R $\subseteq$ G $\times$ G. Then the triple (**T**, **C**, R) is called a *conceptual time system (on G) with a time relation.*

To distinguish clearly between some order theoretic and graph theoretic notions we again look at the conceptual time system of John's journey. On the set G := {0,1,2,3,4,5} of its time granules we introduce the relation R := {(0,1),(1,2),(2,3),(3,4),(4,5)}, shortly described as $0 \to 1 \to 2 \to 3 \to 4 \to 5$. Clearly, in that example the directed graph (G,R) is a directed path. It is not yet an ordered set since it is neither reflexive nor transitive (but antisymmetric). The reflexive and transitive closure of it is just the usual natural order on the set G. The chosen time relation R is just the neighborhood relation of that ordered set.

As in the example of John's journey, in standard applications the set G of time granules will be finite, say G := {0,. . .,n-1}; then the time relation usually will be chosen as the neighborhood relation on these integers. If G is an interval of the usual real order, we emphasize taking the real order relation as the time relation since the neighborhood relation of the ordered set of the real numbers is empty. Now we are ready to introduce *transitions* and *life tracks* in conceptual time systems with a time relation.

### 3.4 Transitions

The basic idea of a *transition* is a "step from one point to another". We shall use *transitions* in several *spaces*, mainly in the *situation space*, the *state space*, and the *time state space*. The idea is to generate these transitions by the *R-transitions* (g,h) which are by definition the elements of the time relation R.

That is demonstrated for John's journey in the right part of Figure 4. Each arrow in Figure 4 represents a "transition of John" and is described by the R-transition (g,h) and by the pair of points say (f(g), f(h)) to which g and h are mapped. In this example the mapping f is the object concept mapping of the time part, which maps a time granule onto its object concept in the time state space.

In general, for any **CTST** and any mapping f from the set G of time granules into some other set X we define an *f-transition of the* ***CTST*** *in the set X* as a pair ((g,h), (f(g), f(h))) of two pairs, namely an R-transition and its image under f. That allows for describing "multiple transitions" between two given states (or situations or time states) at different time granules.

### 3.5 Life Tracks

The transitions in Figure 4 form a *life track* of John. To introduce life tracks mathematically we shall define a life track as a set which is structured by the induced time relation. In the three diagrams of Figure 5 John's life track is represented by the bold arrows. The thin arrows show the not yet told journey of John's wife, Mary. The formal representation of persons like John and Mary as subsystems will be discussed in the next section.

Figure 5 shows three related diagrams labelled by the names "states", "time states", and "situations". The time granules of John are represented in bold; those of Mary are thin; they are drawn only in the situation space; they can be reconstructed in the state space and in the time state space by projection from the situation space – which will be discussed later. The "state space" in the upper left of Figure 5 tells us that John and Mary make a journey from Frankfurt to Napoli and back. The time state space in the form of the schedule in Figure 2 tells us when they make their transitions. In the direct product of these two

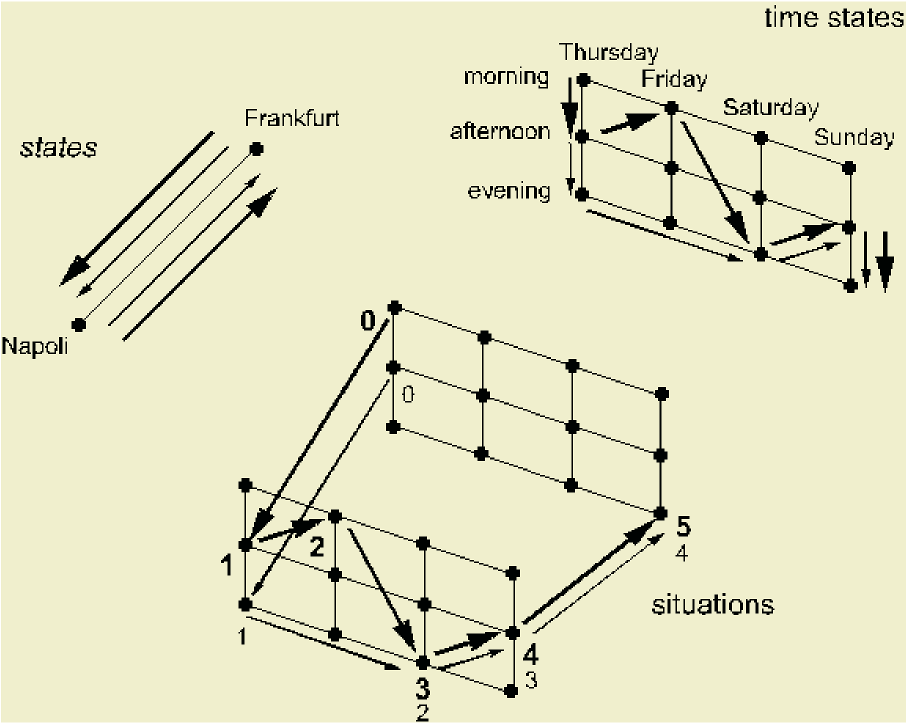


**Fig. 5.** John's and Mary's journey

spaces we see all "situations" as object concepts in the "situation space" of the journey. To be clear, we just tell the story of Mary's journey:

> **The Story of Mary's Journey:** Mary takes a flight from Frankfurt to Napoli on Thursday afternoon arriving at Napoli in the evening. She visits the conference dinner on Saturday evening and leaves Napoli on Sunday afternoon arriving at Frankfurt in the evening.

To prepare the definition of a "life track of a **CTST**" we assume that we are interested in some mapping f (for example the object concept mapping) from the set G of time granules into some other set X (for example the set of states or the set of situations).

**Definition: "transitions and life tracks"**
Let (**T**,**C**,R) be a conceptual time system on G with a time relation. Then any pair (g,h) $\in$ R is called an *R-transition on G.* Let X be a set and f: G $\rightarrow$ X, then f induces the mapping $f_R$ : R $\rightarrow$ { (f(g), f(h)) | (g,h) $\in$ R } where $f_R$((g,h)) := (f(g), f(h)). The element ( (g,h), (f(g),f(h)) ) $\in$ $f_R$ is called the *f-induced R-transition on X leading from the start point* (g, f(g)) *to the endpoint* (h, f(h)). The set f = {(g,f(g))| g $\in$ G} is called the *life track of f in X.*

Now we are interested in some special choices of f. Let $\mathbb{K}_{TC} := \mathbb{K}_T|\mathbb{K}_C$ be the derived context of the CTS (**T**, **C**). For the object concept mapping $\gamma$: G $\rightarrow$ $\gamma$G of $\mathbb{K}_{TC}$ the $\gamma$-induced R-transitions on the situation space $\gamma$G are called the *R-transitions on* $\gamma G$. In the same way the R-transitions on the *state space* $\gamma_C G$ and on the *time state space* $\gamma_T G$ are defined as induced by the corresponding object concept mappings $\gamma_C$ and $\gamma_T$.

In the following definition we introduce on the life track an isomorphic copy of the time relation R:

**Definition: "the life track digraph (f, $R_f$)"**
Let (T, C, R) be a conceptual time system on G with a time relation. Let X be a set and f: G $\rightarrow$ X, then the relation $R_f$ is defined on the life track f = {(g,f(g))| g $\in$ G} by

$$(g,f(g))\ R_f\ (h,f(h)) :\Leftrightarrow g\ R\ h.$$

The directed graph (f, $R_f$) is called the *life track digraph of R.*

The life track digraph (f, $R_f$) is isomorphic to (G,R). Hence, if R is an order relation on G, the relation $R_f$ is an isomorphic order relation on the life track f. If (G,R) is a chain, then (f, $R_f$) is an isomorphic chain yielding the usual trajectories in dynamical systems as defined for example in [Kr98], p.8. If (G,R) represents a directed graph-theoretic path, then (f, $R_f$) is an isomorphic path; representing that path on the set X (for example the state space) using labels (as in Figure 3) we get a directed graph with point labels and usually with loops (x,x). In Figure 5 we have omitted the loops (in the state diagram).

In the next section, we introduce "objects" or "persons", like John and Mary, as subsystems.

## 4 Objects as Subsystems

In Figure 5 we have visualized the life tracks of two persons. Since we represented John's journey as a CTST we would like to do the same for Mary. Hence the question arises of how to combine two CTST's in a meaningful way; for example, in such a way that the life tracks of these two systems appear in the same space; then the systems should have the same many-valued attributes and the same scales. In this case the tables are arranged in subposition, for example, the table of Mary is just written under the table of John. The formal definition of subposition of formal contexts can be found in [GaWi99], p.40. The subposition of many-valued contexts is defined analogously.

The following Table 6 shows the many-valued context of "John's and Mary's journey" where we introduced "actual objects"; for example, (John,5) describes "John at time granule 5". To obtain the life tracks of John and Mary as drawn in Figure 5 we introduce the time relation on the set of actual objects by:

$$(J,0) \rightarrow (J,1) \rightarrow (J,2) \rightarrow (J,3) \rightarrow (J,4) \rightarrow (J,5)$$
$$(M,0) \rightarrow (M,1) \rightarrow (M,2) \rightarrow (M,3) \rightarrow (M,4).$$

**Table 6.** The data table of John's and Mary's journey

| | time part | | event part |
|---|---|---|---|
| time granules | day | day time | place |
| (J,0) | Thursday | morning | Frankfurt |
| (J,1) | Thursday | afternoon | Napoli |
| (J,2) | Friday | morning | Napoli |
| (J,3) | Saturday | evening | Napoli |
| (J,4) | Sunday | afternoon | Napoli |
| (J,5) | Sunday | evening | Frankfurt |
| (M,0) | Thursday | afternoon | Frankfurt |
| (M,1) | Thursday | evening | Napoli |
| (M,2) | Saturday | evening | Napoli |
| (M,3) | Sunday | afternoon | Napoli |
| (M,4) | Sunday | evening | Frankfurt |

Together with the above mentioned scales and the previously mentioned time relation Table 6 shows a first example of a "conceptual time system with actual objects and a time relation" (CTSOT). Its derived context yields the concept lattice indicated in Figure 5 with the two life tracks of John and Mary.

The following definition of a CTSOT was introduced by the author in [Wol02a].

**Definition: "CTSOT"**
**"conceptual time systems with actual objects and a time relation"**
Let P be a set (of "persons", or "objects") and G a set (of "time granules") and $\Pi \subseteq$ P $\times$ G. Let (**T**, **C**) be a conceptual time system on $\Pi$ and R $\subseteq \Pi \times \Pi$. Then the tuple (P, G, $\Pi$, **T**, **C**, R) is called a *conceptual time system (on $\Pi \subseteq P \times G$) with actual objects and a time relation*, in short a CTSOT. For each object p $\in$ P the set $p^{\Pi}$ := {g $\in$ G | (p,g) $\in \Pi$} is called the *time of p in $\Pi$*. Then the set $\mathbf{R}_p$ := {(g,h) | ((p,g), (p,h)) $\in$ R } is called the set of *R-transitions of p* and the relational structure ($p^{\Pi}$, $\mathbf{R}_p$) is called the *time structure of p*.

The subsystem of the "rows of a single person p" can be described as a CTST. The previously described definitions of situations, states, time states, transitions, and life tracks can be used to describe the corresponding notions for a CTSOT. The formal definitions are given in [Wol02a]. Here we mention the definition of the life track of an object.

**Definition: "life track of an object"**
Let (P, G, $\Pi$, **T**, **C**, R) be a CTSOT, and p $\in$ P. Then for any mapping f: $\{p\} \times p^{\Pi} \to X$ (into some set X) the set f = {((p,g),f(p,g))|$g \in p^{\Pi}$} is called the *f-life track of p*.

The two most useful examples for such mappings are the object concept mappings $\gamma$ and $\gamma_C$ of the derived contexts $\mathbb{K}_T|\mathbb{K}_C$ and $\mathbb{K}_C$ of the CTST (**T**, **C**,

R) on $\Pi$, each of them restricted to the set $\{p\} \times p^{\Pi}$ of actual objects. They are called the *life track of p in the situation space* and the *life track of p in the state space* respectively.

Clearly, there are other possibilities for describing the subsystems of the "persons", for example by introducing a new many-valued attribute "PERSON" with the names of the persons as values, more precisely PERSON(p,g):= p. Then the scale for the many-valued attribute PERSON can be chosen to represent hierarchies for persons, for example the membership hierarchy of a family, where the family itself can be understood as a "general person" or "general object". That led the author recently to a conceptual understanding of particles, waves and wave packets in "Conceptual Semantic Systems" [Wol04a]. The connection between CTSOTs, Conceptual Semantic Systems, conceptual graphs and power context families as introduced by Wille [Wil97b] will be discussed elsewhere.

## 5 Conceptual Granularity Reasoning

Now we are ready to discuss some basic aspects of "conceptual granularity reasoning". First, we study an example. In colloquial speech we conclude from "John took a flight on Sunday to Frankfurt" that "John took a flight at the weekend to Germany". For that kind of reasoning we use our "background knowledge" that "Sunday belongs to the weekend" and "Frankfurt belongs to Germany". Clearly, we cannot conclude from any judgement by replacing some concepts by superconcepts that the new statement is also valid, for example the judgement that "the regions of two towns are disjoint" does not imply that "the regions of the counties of these towns are disjoint". Therefore, we take some first cautious steps to investigate granularity reasoning.

With respect to CTSOTs, we are interested in statements about life tracks and granularity. In the example of the life track of "John" in the situation space in Figure 5 we see that we get the life track of "John" in the time scale "by projection" from the life track of "John" in the situation space. This leads to the conjecture that "the life track of a person can be mapped by a suitable projection onto the life track of that person in some factor space". Indeed, there is such a general projection which is called the "closure function" in [Ern82].

**Definition: "closure function"**
Let (V, $\leq$) be an ordered set and T $\subseteq$ V such that each subset S $\subseteq$ T has an infimum in T, i.e. $\forall_{S \subseteq T} \exists_{t \in T} t = \inf S$. Then the mapping

$$\pi\text{: V} \rightarrow \text{T defined by } \pi(x) := \inf\{y \in T | x \leq y\}$$

is called the *closure function from* (V, $\leq$) *onto* T.

Clearly, $\pi$ is a projection from V onto T, i.e. $\pi^2 = \pi$, since $\pi$(t) = t for all t $\in$ T. Furthermore, $\pi$(x) $\geq$ x. In the special case that V is the power set P(X) of a set X, and T is a closure system on X, then the corresponding closure function is just the closure operator of the closure system T.

Using the closure function we now prove the following Life Track Lemma:

**Life Track Lemma:**
Let (P, G, $\Pi$, **T**, **C**, R) be a CTSOT and p $\in$ P.
Let $\mathbb{K}_{TC} := \mathbb{K}_T|\mathbb{K}_C$ be the derived context of the given CTSOT. The object set of $\mathbb{K}_{TC}$ is $\Pi$, let M be its set of attributes, and I its incidence relation. Let **B** denote the set of all formal concepts of $\mathbb{K}_{TC}$ and for N $\subseteq$ M let $\mathbf{B}_N$ denote the set of all formal concepts of the subcontext $\mathbf{K}_N$:= ($\Pi$, N, I$\cap$($\Pi$ $\times$ N)) and $\gamma$, $\gamma_N$ the object concept mappings of $\mathbb{K}_{TC}$ and $\mathbf{K}_N$ respectively. Let $\varphi$: $\mathbf{B}_N \rightarrow \mathbf{B}$ be the meet-preserving order embedding satisfying $\varphi$(A,B):= (A,$\text{A}^I$). Then the closure function $\pi$: $\mathbf{B} \rightarrow \varphi\mathbf{B}_N$ satisfies

$$\pi\gamma = \varphi\gamma_N$$

and the *extended closure function* $\tau := \varphi^{-1}\pi$ satisfies

$$\tau\gamma = \gamma_N$$

and maps each object concept $\gamma$(p,g) of the actual person (p,g) onto the object concept $\gamma_N$(p,g) and therefore the $\gamma$-life track of p in the situation space onto the $\gamma_N$-life track of p in the factor space $\mathbf{B}_N$ obtained by restricting the attribute set M to the subset N.

Clearly, if we restrict the situation space to the state space by omitting all attributes of the time part, the corresponding extended closure function maps the life track of a person in the situation space onto the life track of the same person in the state space.

**Proof of the Life Track Lemma:**
First, we mention that $\varphi$: $\mathbf{B}_N \rightarrow \mathbf{B}$ is a meet-preserving order embedding ([GaWi99], p.98), hence the set $\varphi\mathbf{B}_N$ has the property that each of its subsets has an infimum in $\varphi\mathbf{B}_N$. Therefore, the closure function $\pi$: $\mathbf{B} \rightarrow \varphi\mathbf{B}_N$ exists and satisfies for any actual object (p,g) that the extent of $\pi(\gamma$(p,g)) can be described by the following formula (where we use J:= I$\cap$($\Pi$ $\times$ N))
$\bigcap\{\text{C} \mid (\text{p,g})^{II} \subseteq \text{C}, (\text{C,C}^J) \in \mathbf{B}_N\} = \bigcap\{\text{C} \mid (\text{p,g}) \in \text{C}, (\text{C,C}^J) \in \mathbf{B}_N\} = (\text{p,g})^{JJ}$
since (p,g) $\in$ $(\text{p,g})^{II} \subseteq (\text{p,g})^{JJ}$. Using that $\varphi(\gamma_N$(p,g)) has the same extent as $\gamma_N$(p,g), namely $(\text{p,g})^{JJ}$ we get $\pi(\gamma(\text{p,g}))=\varphi(\gamma_N(\text{p,g}))$ and that proves the Life Track Lemma.

## 6 Applications and Computer Programs

Temporal Concept Analysis was developed by the author motivated by many applications of Formal Concept Analysis in practice [SpWo91, WoSt93, Wol95a]. To improve process control the formal representation of the temporal structure of processes was necessary. After having introduced the notions of conceptual time systems, states, and situations many previously studied examples could be represented much clearer. The introduction of transitions and life tracks led to valuable computer animations of processes. We demonstrate two examples, one from my long cooperation with the psychoanalyst Dr. Norbert Spangen-

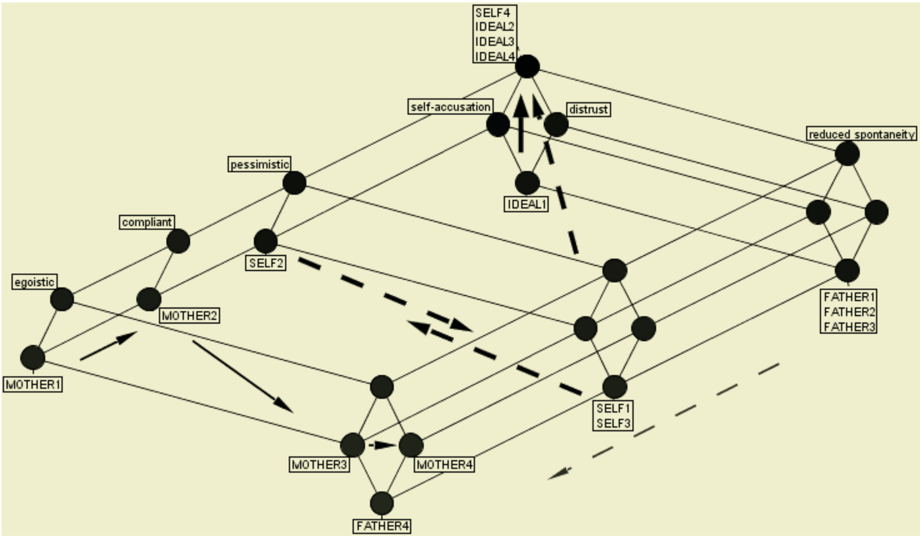


**Fig. 6.** The development of an anorectic young woman and her family over about two years

berg (then at the Sigmund Freud Institute Frankfurt, Germany) who is working in psychosomatic process research [Spa90]; the other example demonstrates an application in the multi-dimensional visualization of processes in a chemical distillation column. Finally, we briefly mention the main computer programs for TCA.

### 6.1 The Development of an Anorectic Young Woman

The following example in Figure 6 describes the development of an anorectic young woman (SELF), her father, mother, and her self ideal (IDEAL) during a period of about two years. The underlying formal context was constructed by the psychoanalyst Spangenberg on the basis of four repertory grids taken about each half year from the beginning (time granule 1) until the end (time granule 4) of the psychoanalytic treatment of his patient. SELF1, the self at the beginning of the treatment, has the attributes “distrust” and “reduced spontaneity”, SELF2 “pessimistic” and “self-accusation”, SELF3 is in the same state as SELF1, and SELF4 reaches the state of IDEAL2,3,4. Indeed, the patient was healthy again at this point in time. It is remarkable that the life tracks of FATHER and MOTHER start from quite different states and end in similar states, the FATHER having all negative attributes of that context. For further information the reader is referred to [Spa90, SpWo91, SpWo93].

### 6.2 A Chemical Process in a Distillation Column

The diagram in Figure 7 demonstrates a visualization of a chemical process in a distillation column over a period of 20 days.

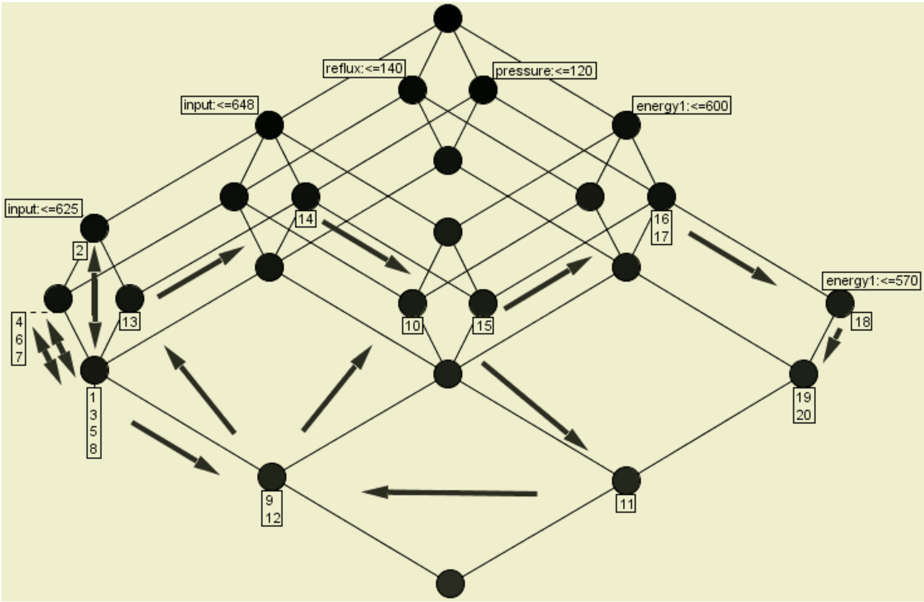


**Fig. 7.** A chemical process represented in a 4-dimensional state space

From such diagrams the process can be understood quite well taking the attributes used by the experts. A short and coarse description of that process might be:

> Starting on the first day from a state of low input, low reflux and low pressure, but high energy1, the system switched at low input between low and high pressure; from day 9 to 12 it visited in a circular movement states of middle and low energies; finally it came at middle resp. high input to states of middle resp. low energy1, low pressure and low reflux.

Typically, in such applications the experts suggest first a coarse granularity by a few "cuts" like "energy1≤600". After having studied the concept lattice with a coarse granularity it is usually refined, depending on the data and on the interest of the experts. That leads in a few steps to valuable visualizations of multidimensional processes. For further information the reader is referred to [Wol95a, Wol00b].

### 6.3 Computer Programs

The state of the art in the graphical representation of concept lattices by computer programs is mainly represented by two tools. The first one is the NaviCon Decision Suite with the main programs Anaconda, Toscana, and Cernato from NaviCon AG (Frankfurt). Its extended Java-version ToscanaJ contains the program ELBA for the construction of conceptual scales, which are used in the main program ToscanaJ for the generation of nested line diagrams. For drawing transition diagrams as in Figure 6 and 7 the temporal component of the program Siena can be employed; Siena can also be used for the presentation of animations of conceptual time systems. For further information the reader is referred to [Bec95].

# 7 Connections to Other Temporal Theories

In this paper, it is impossible to mention all relevant connections to other temporal theories. Therefore, I describe here only the main relations between Temporal Concept Analysis and some of the most important temporal theories.

In contrast to the following theories, TCA has a general granularity tool allowing for a common conceptual notation for finite as well as for infinite temporal systems. The introduction of *actual objects* and their *time relation* in CTSOTs is a new approach to understand the relation between objects, space and time.

## 7.1 Classical Physics and Quantum Theory

In this subsection some basic aspects of classical physics and quantum theory are related to Temporal Concept Analysis. First, we discuss the roles of scales and objects.

The great success of classical physics is based on the Euclidean space together with its differentiable structure. The points in that space are used as "places for objects" showing that the Euclidean space is employed as a scale for the embedding of objects – but that is not made explicit by general theoretical notions for objects and scales. Clearly, the classical scale types on the real numbers [LKST90, Wol95b] are well-known also to many physicists; but a general investigation of not only infinite but also finite scales with the purpose of developing a physical granularity theory for combining the discrete measurements with the continuous theory in a theoretical way is not known to the author.

The "space occupied by an object" is described as a subset of the Euclidean space $\mathbb{R}^3$ and the "time of an event" as a subset of the time axis $\mathbb{R}$ – but such a granularity structure causes problems. Indeed, Einstein mentioned some of these problems in his "granularity remark" in the 1905 – paper introducing the theory of special relativity [Ein05], Footnote on page 893 (translated by the author):

> The inaccuracy which lies in the concept of simultaneity of two events at (about) the same place and which has to be bridged also by an abstraction, shall not be discussed here.

I believe that a theory (and not only a well-developed practice) of granularity in physics could lead to a better understanding of many problems related to the meaning of limits (like velocities and energies), and to the understanding of inaccuracy and Heisenberg's uncertainty relation. The *problem of time* as discussed in [But99] and [ButIsh99], page 147, could be embedded into a general granularity theory for objects in space and time. Recent investigations in

TCA might be a starting point for such a development: the introduction of a granularity not only for space and time but also for the objects, as for example, persons as members of a family, led the author to a mathematical definition of wave packets, yielding definitions of particles and waves as special examples [Wol04a]. These definitions cover the continuous as well as the discrete waves, as for example, electro-magnetic waves as well as waves of influenza represented in discrete data.

### 7.2 Mathematical System Theory, Turing Machines, and Automata Theory

The formalization of the ideas of Bertalanffy [Ber69] led to Mathematical System Theory (cf. Kalman [KFA69], Mesarovic [MeTa75], Lin [Lin99]). As pointed out by Zadeh [Zad64] Mathematical System Theory did not find a satisfactory notion of state, and Lin [Lin99] writes that there is no generally accepted notion of a system. The recent developments in TCA might be a first step towards a better understanding of *states* and *systems*.

The introduction of computers was accompanied by the development of a theory of computation initiated by Post [Pos36] and Turing [Tur36]. Their *computing machines* are now known as *Turing machines*. It was shown recently by Wolff and Yameogo [WoYa05] that any Turing machine can be represented by a suitable "Turing CTSOT" such that for each possible input of the Turing machine the uniquely determined sequence of computation steps is represented as the life track of the input word in the state space of that Turing CTSOT. The conceptual role of the instructions of the Turing machine is understood as a set of background implications of the derived context of the Turing CTSOT.

The investigation of *computing machines* led to the development of automata theory which is mainly concerned with finite automata as described for example by Arbib [Arb70] and Eilenberg [Eil74]. The *continuous* time of physics was replaced by a *discrete* time, the set of *states* was introduced axiomatically as a set of things without an explicit definition in terms of time, but these *states* can be connected by *labelled transitions*. Finite paths from an *initial state* to some *final* (or *terminal*) *states* are used to describe *runs* of the *machine*. Automata can be described by CTSOTs such that the states, the transitions, and the successful paths of an automaton are represented by the states, the transitions, and the life tracks of a suitable CTSOT. For further information the reader is referred to [Wol02b].

### 7.3 Temporal Logic and Conceptual Temporal Logic

Temporal Logic in the sense of Gabbay [GHR94] and van Benthem [vBe95] is developed as a general logic for temporal phenomena. After having introduced Temporal Concept Analysis as a theory for handling temporal phenomena on the basis of mathematically defined conceptual time systems, time granules, states, situations, and life tracks I could discuss at the 9th International Symposium on Temporal Representation and Reasoning (TIME'02) in July 2002 in Manchester with Dov Gabbay the relations between Temporal Logic and Temporal Concept Analysis. His central idea of a "branching time" is described in [GHR94], page 86:

> We should, therefore, pay special attention to discrete future branching past-linear flows of time.

This "tree structure" of time might be extended to a more general framework as for example to the temporal scales in TCA (which may be chosen as trees in their usual lattice representation). But the basic idea of a "branching time" is independent of the time scale since it is based on the idea of branching possible future life tracks; those life tracks can be easily represented in a CTSOT with an arbitrarily given time scale.

The main difference between Temporal Logic and TCA seems to be that Temporal Logic is designed as a "logic" for arbitrary temporal models while TCA yields a general description of temporal models. It seems to be desirable to combine the classical Temporal Logic with TCA towards a "Conceptual Temporal Logic" which, for instance, could include a tool for the representation of relational logic using for example power context families or relational conceptual time systems. Then the CTSOTs (or more general temporal structures) could be models in that Conceptual Temporal Logic having general logical tools for spatio-temporal granularity reasoning in those conceptual structures.

## 8 Conclusion and Future Research

Temporal Concept Analysis is based on mathematically defined notions of conceptual time systems, states, situations, transitions, and life tracks of objects such that continuous and discrete temporal phenomena can be described in the same conceptual framework.

Future research in TCA should develop not only the just mentioned Conceptual Temporal Logic but also the temporal aspects of relational logic. Furthermore, the connections to other temporal theories should be clarified. Especially, applications in physics might yield progress in understanding temporal phenomena as for example further discussions about particles and waves including interference of waves. The formal representation of granularity might be a powerful tool for understanding Heisenberg's uncertainty relation in a more general framework. The *problem of time* in quantum theory might become better understandable with the tools of TCA too.

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