Probabilistic Transformations of Belief Functions

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Abstract. Alternative approaches to the widely known pignistic transformation of belief functions are presented and analyzed. A series of various probabilistic transformations is examined namely from the point of view of their consistency with rules for belief function combination and their consistency with probabilistic upper and lower bounds. A new definition of general probabilistic transformation is introduced and a discussion of their applicability is included.

Keywords: Belief function, Dempster-Shafer theory, Probabilistic transformation, Pignistic probability, Combination consistency, *ulb*-consistency.

1 Introduction

Belief functions are formalisms widely used for uncertainty representation and processing. For combination of the beliefs the Dempster's rule of combination is used in the Dempster-Shafer theory. Besides, series of modifications of the Dempster's rule were suggested and alternative approaches were created: e. g. Transferable Belief Model (TBM) using the so called non-normalized Dempster's rule [28], combination 'per elements' [5] with its special case — minC combination, see [6], and others. Subsequently, numerous practical applications were suggested and implemented in a wide range of domains.

What is common for their applications? It is an aim to transform the resulting evidence representation by a general belief function to representation by probability for the purpose of easier decision making, resulting beliefs comparison and ordering. Such a probability should be consistent with the original belief function. In fact, we can consider it as a belief function of a special type, so called Bayesian belief function. We call such a transformation as a *probabilistic transformation.*

Frequently only a special case of probabilistic transformation – Pignistic transformation — is used. In the last years several papers on alternative probabilistic transformations have been published [2, 3, 10, 11, 31, 32], and a new justification of pignistic transformation has appeared [29, 30].

This paper summarizes and completes the study of probabilistic transformations presented in [10, 11, 13]. Besides the new original results, Baroni & Vicigs's

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[⋆] Partial support by the COST action 274 TARSKI acknowledged.

L. Godo (Ed.): ECSQARU 2005, LNAI 3571, pp. 539–551, 2005.

results from [2] and Cobb & Shenoy's results [3], the present study includes also Sudano's transformations [31, 32] and Smets' new results [29, 30].

Basic notions, both general, and those from [10] and [11] are introduced in Section 2. Section 3 presents a series of probabilistic transformations from various sources and it shows that some of them are equivalent to other one(s). Section 4 brings a summary of consistencies of the transformations. A new definition of the general probabilistic transformation based on their analysis and a justification of two main alternatives to pignistic transformation is presented in Section 5. A discussion about which transformation should be applied in applications concludes the paper.

2 Preliminaries

2.1 Basic Notions

Let us first recall some basic notions from the theory of belief functions. Let us consider an n-element frame of discernment[1] $\Omega = \{\omega_1, \omega_2, ...\omega_n\}$. A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. If $m(\emptyset) = 0$, we speak about *normalized bba*. A *belief function (BF)* is a mapping $bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. $\mathcal{P}(\Omega)$ is often denoted by 2^Ω. Let us further recall a *plausibility function* $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function*, it is a probability distribution on Ω in fact. If all the focal elements are either singletons or whole Ω (i.e. $|X| = 1$ or $|X| = |\Omega|$), then we speak about a *quasi-Bayesian belief function*, it is something like 'non-normalized probability distribution'.

To underline the cardinality of a frame of discernment, we use the left lower indices, e.g. ${}_{nD}bel(X)$, ${}_{3D}m(X)$, etc., and we speak about nD BF bel, 3D bba m, etc. Let ${}_{2D}0 = (0,0)$ and ${}_{nD}0 = (0, ..., 0)$ denote special BFs bel_0 such that $m_0(\Omega) = 1$, ${}_{2D}0' = (\frac{1}{2}, \frac{1}{2})$ and ${}_{nD}0' = (\frac{1}{n}, ..., \frac{1}{n}, 0, ..., 0)$ denote special BFs $bel_{0'}$ such that $m_{0'}(X) = \frac{1}{n}$ for $|X| = 1$.

The Dempster's (conjunctive) rule of combination is given as $(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$,where $K = 1/(1 - \sum_{X\cap Y=\emptyset} m_1(X) m_2(Y)) = \frac{1}{1-\kappa}$, and $m(\emptyset) = 0$, see [26]; putting $K = 1$ and $m(\emptyset) = \sum_{X\cap Y=\emptyset} m_1(X) m_2(Y) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\textcircled{\cap}$, see e.g. [28]. *The disjunctive rule of combination* is given by the formula $(m_1 \textcircled{\cup} m_2)(A) = \sum_{X\cup Y=A} m_1(X) m_2(Y)$, see [19]. *Bayes' rule of probability combination* is defined as a normalized point-wise multiplication of probabilities of singletons. $(P_1 \otimes P_2)(x) = \frac{P_1(x)P_2(x)}{\sum_{y\in\Omega} P_1(y)P_2(y)}$.

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[1] We use the classical Shaferian terminology. Besides, it is also possible to use the new more user-friendly simplification of the terminology suggested by Dempster, see e.g. [15], using a notion *state space* instead of a frame of discernment, and similarly.

2.2 General Definition of Probabilistic Transformations

Let us consider the following very general definition now [2]. A *probabilistic transformation* (or briefly a *probabilization*) is a mapping $T : Bel_{\Omega} \longrightarrow ProbDistr_{\Omega}$, Thus the probabilistic transformation assigns a Bayesian belief function (i.e. probability distribution) to every general one. It is a reason why the transformations of belief functions to probability distributions are sometimes called also Bayesian transformations, see e.g. [33]. As we suppose finite frames of discernments, we can compute $(T(bel))(X) = \sum_{A \in X}(T(bel))(A)$ for any $X \subseteq \Omega$.

The fundamental well know example of a probabilistic transformation is the pignistic transformation $BetT$ and its resulting pignistic probability $BetP$ [3] introduced by Smets. We do not use the name pignistic transformation for the other ones, and we use the general name probabilistic transformation, in accordance with Philippe Smets' wish not to mix new alternatives together with his classical pignistic transformation. Moreover, it allows us to use a more general definition with less assumptions.

2.3 ulb-Consistency and p-Consistency

Probabilistic transformation PT is *ulb-consistent (upper and lower bound consistent)* if its resulting transformed probability TP satisfies the following consistency condition: $Bel(X) \leq TP(X) \leq Pl(X) = 1 - Bel(\overline{X})$. Probabilistic transformation PT is *p-consistent (or probabilistically consistent)* if $PT(m) = m$ for any Bayesian bba m. In other words Bayesian BFs are fix points of p-consistent PTs. p-consistency is in fact ulb-consistency on Bayesian BFs (i.e. weakening of ulb-consistency) because $bel(X) = Pl(X)$ for Bayesian BFs.

2.4 Combination Consistencies

A combination consistency of a PT is based on commutation of a combination rule $\circledast$ with PT, i.e. we obtain the same results if we combine beliefs bel_1 and bel_2 using the combination rule $\circledast$ and perform PT after it as in the case, where we first compute probabilistic transformations of the both input beliefs bel_1 and bel_2 and combine them with the combination rule $\circledast$ after.

Probabilistic transformation PT is $\oplus$-*consistent* if it commutes with the Dempster's rule (with $\oplus$ combination). Analogically [4] PT is $\circledcup$-consistent if it commutes with $\circledcup \circ u$. Where u stands for the nD generalization of the original 2D homomorphism u: ${}_{2D}u(a, b) = (a, b) \circledcup (\frac{1}{a+b}, \frac{1}{a+b}) = (\frac{a}{a+b}, \frac{b}{a+b})$, and its

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[2] For precision of the definition see Section 5.

[3] We denote all transformations with suffix T and related probabilities with P.

[4] It is possible to define analogically other combination consistencies w.r.t. to other combination rules, see e.g. ©-*consistency* [11]. Due to the limitation of applicability of the *consensus operator* © [8, 24] to quasi-Bayesian BFs only [9], we omit a presentation of ©-consistency in this text.

nD generalization $u(x_1, ..., x_n, x_{n+1}, ..., x_{2^n-1}) = (\frac{x_1}{\sum_{i=1}^{n} x_i}, ..., \frac{x_n}{\sum_{i=1}^{n} x_i}, 0, ..., 0)$, see [7, 12].

3 Probabilistic Transformations

3.1 Pignistic Transformation

The *pignistic transformation BetT* distributes $m(X)$ equally among all elements of X. It was named and justified by Smets in [27] for Transferable Belief Model (TBM), see [27, 28] in 1990. Nevertheless, the transformation based on the same principle was used by Dubois & Prade [18] as "equidistribution of the values of bba" and by Williams [34] in 1982 already.

The pignistic transformation $BetT$ projects BF bel given bba m to probability $BetP$ defined on the frame of discernment Ω as follows:

$$BetP(A) = \sum_{A \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1 - m(\emptyset)}.$$

It includes normalization and division of bbms assigned to focal elements by their cardinality, non-normalized beliefs used in TBM are admissible.

The justification of the pignistic transformation is based on the assumption of the so called *linearity property*, see e. g. [29, 30], i. e. on commutation of the transformation with a convex combination of beliefs: $T(\alpha m_1 + (1 - \alpha) m_2) = \alpha T(m_1) + (1 - \alpha) T(m_2)$. [5] This property was originally derived from the so called *α-combinability of credibility spaces*, see [27]. In correspondence with the definition of combination consistencies we can call the linearity property assumption as *α-consistency*. No justification of the transformation has been presented by Dubois & Prade or by Williams.

From the definition and justification of the pignistic transformation, we can immediately see that it is *ulb*-consistent and α-consistent. BetT is neither $\oplus$-consistent nor $\textcircled{\cup}$-consistent.

3.2 Plausibility or Cautious Probabilistic Transformation

Let us introduce three different definitions of the main alternative to pignistic probability in this subsection.

Widely known it the following one. The *(normalized) plausibility probabilistic transformation* Pl_T, see e.g. [2] or [3], is defined as a normalized plausibility of singletons [6]. Hence we have

$$Pl\_P(A) = \frac{Pl(A)}{\sum_{B \in \Omega} Pl(B)} = \frac{\sum_{A \in X \subseteq \Omega} m(X)}{\sum_{B \in \Omega} \sum_{B \in X \subseteq \Omega} m(X)}.$$

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[5] The special case of a convex combination of bbas for $\alpha = \frac{1}{2}$ was mentioned as averaging of bbas in [11].

[6] Despite of the fact, that, Cobb and Shenoy introduce it as a new method [3] in 2003, and Sudano also introduces it as *PrNPl* in 2003, it was known already in 1991 [1].

This transformation is called 'the pignistic *probability proportional to normalized plausibility*' ($PrNPl$) by Sudano in [32]. [7]

The *cautious probabilistic transformation* [10, 13] is defined as the Dempster's combination of a belief bel with $0'$: $CautT(bel) = bel \oplus 0'$. It is a generalization of homomorphism h, which corresponds to Hájek & Valdés results on 2D belief functions [21, 22]: ${}_{2D}CautP(A) = \frac{1-m(B)}{2-m(A)-m(B)}$.[8] In the nD case we have:

$$CautP(A) = \frac{\sum_{A \in X} m(X)}{n - \sum_{B \in \Omega, X \subseteq \Omega, B \notin X} m(X)}.$$

Voorbraak's Bayesian transformation (VBT)[9] published in 1989, see [2] and [33], is given by

$$VBP(A) = \frac{\sum_{A \in X} m(X)}{\sum_{Y \subseteq \Omega} (m(Y) \cdot |Y|)}.$$

Theorem 1. *The cautious and plausibility probabilistic transformations and Voorbraak's Bayesian transformation are the same transformations of belief functions to probabilistic distributions, i.e. it holds that* $CautP(A) = Pl\_P(A) = VBP(A)$.

For equality $CautT \equiv Pl\_T$ see [13], and for equality $Pl\_T \equiv VBT$ see [2].

Pl_T is $\oplus$-consistent. It is neither $\circledcirc$-consistent nor α-consistent. Pl_T is neither *ulb*-consistent in general. It is *ulb*-consistent for quasi-Bayesian BFs only; it implies p-consistency in general on nD and *ulb*-consistency on 2D BFs.

3.3 Belief or Disjunctive Probabilistic Transformation

In [10], the *disjunctive probabilistic transformation DisjT* has been presented which has been defined on 2D frames so that it commutes with $\circledcirc \circ u$, $DisjP(\{A\}) = \frac{m(\{A\})}{m(\{A\})+m(\Omega-\{A\})}$. Its nD generalization [13] is given by the following formula:

$$DisjP(\{A\}) = \frac{m(A)}{\sum_{X \in \Omega} m(X)}.$$

A *(normalized) belief probabilistic transformation* Bel_T [11] is defined as a normalization of beliefs of singletons (bbms of singletons), i.e. by the same

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[7] This name does not correspond to Smets' wish of using the name of the pignistic transformation, besides it does not satisfy all assumptions required from Smets' pignistic transformation, either the original [27, 28] or the recent ones [29, 30]. For this reason we eliminate the word 'pignistic' from the name of the transformation and add a letter T (or P) to abbreviation of the transformation (or resulting probability) to obtain $PrNPlT$ (or $PrNPlP$) to be consistent with the other names. The same holds also for the other Sudano's transformations, see [31, 32].

[8] This 2D transformation was used already in the Expert System Shell EQUANT-PC in late 80's, see [20].

[9] Voorbraak proposed VBT not for decision making, but for approximation of BFs.

formula. Thus it is evident that $Bel\_T \equiv DisjT$. We have to note that Bel_T is not defined if $\sum_{X \in \Omega} m(X) = 0$; we can complete its definition analogically to the proportional transformation, see later, but such a definition breaks the $\circledcirc$-consistency which was a motivation for definition of $DisjT$. Further, we have to note that Bel_T is significantly sensitive to the bbms of singletons because it ignores completely the bbms of non-singleton focal elements.

Bel_T is $\circledcirc$-consistent, it is not $\oplus$-consistent. It is neither α-consistent nor *ulb*-consistent in general. It is *ulb*-consistent only for quasi-Bayesian BFs; it implies p-consistency in general on nD and *ulb*-consistency on 2D BFs.

3.4 Proportional Probabilistic Transformations

Proportional transformations take bbm $m(A)$ of a singleton A and add to it proportional parts of $m(X)$ for all its supersets $A \subset X$. From this assumption it is obvious that these *proportional probabilistic transformations* are *ulb*-consistent.

If the proportionalization is computed with respect to the beliefs of singletons, we speak about the *proportional belief probabilistic transformation* $Prop_{Bel}T$, see [11, 13]:

$$Prop_{Bel}P(A) = \sum_{A \in X \subseteq \Omega} \frac{m(A)}{\sum_{B \in X} m(B)} \cdot m(X).$$

If $\sum_{B \in X} m(B) = 0$, then $|X|$ is used instead of it and thus $m(X)$ is relocated per the same portions among all elements of X in such a case.

The equivalent *proportional belief transformation* $PrBlT$, see [31, 32], is based on the same idea as $Prop_{Bel}T$, also the formula for computing of $PrBlP$ corresponds to that for computing $Prop_{Bel}P$. Hence $PrBlT \equiv Prop_{Bel}T$.

In order to correct a statement from [11], we have to note that the equivalence $Bel\_T \equiv Prop_{Bel}T$ holds on 2D and nD quasi-Bayesian BFs only, see [14].

$Prop_{Bel}P(A)$ is defined for all BFs, but similarly to Bel_T it is also significantly sensitive to the bbms of singletons. To improve it, the *stepwise proportional belief probabilistic transformation* $StProp_{Bel}T$ or simply *stepwise belief transformation* $StBel\_T$ has been defined in [11]. Bbms $m^{(i-1)}(X)$ for $|X| = (n+1-i)$ are proportionally relocated in the i-th step among $m^{(i)}(Y)$ for $Y \subset X$, $|Y| = (n-i)$. $m^{(i)}(Z) = m^{(i-1)}(Z) = m(Z)$ for $|Z| < n-i$, and $m^{(i)}(Z) = 0$ for $|Z| > n-i$. If $\sum_{Y \subset X, |Y|=|X|} m(Y) = 0$ then $|X|$ is used instead of it, thus $m(X)$ is relocated per the same portions among all Y in such a case.

If the proportionalization is computed with respect to the plausibilities of singletons, we speak about the *proportional plausibility probabilistic transformation* $Prop_{Pl}T$, see [11], which is defined by

$$Prop_{Pl}P(A) = \sum_{A \in X \subseteq \Omega} \frac{Pl(A)}{\sum_{B \in X} Pl(B)} \cdot m(X).$$

The equivalent *proportional plausibility transformation* $PrPlT$ [31, 32] is based on the same idea as $Prop_{Pl}T$, also the formula for computing of $PrPl$ corresponds to that for computing $Prop_{Pl}P$. Hence $PrPlT \equiv Prop_{Pl}T$.

Two other probabilistic proportional transformations are defined by Sudano in [31], see also [32]. *Probability deficiency transformation PraPlT* and iterative *proportional self-consistent probabilistic transformation PrScT*.

$$PraPlP(A) = m(A) + \frac{1 - \sum_{B\in\Omega} m(B)}{\sum_{B\in\Omega} Pl(B)} \cdot Pl(A).$$

$PraPlT$ is equal to $PrPlT$ and $Prop_{Pl}T$ on 2D and on nD qBBFs, but it does not satisfy our introductive assumption of proportional probabilistic transformations. Moreover, it is not *ulb*-consistent in general, even if its *ulb*-consistency is assumed and claimed in [31] [10]. Nevertheless, $PraPlT$ satisfies the weaker p-consistency.

$$PrScP(A) = \sum_{A\in X} \frac{PrScP(A)}{\sum_{B\in X} PrScP(B)} \cdot m(X).$$

$PrScT$ transformation satisfies our assumption, thus it is really *ulb*-consistent.

Sudano's *hybrid pignistic probability transformation PrHybT* [32] is also *ulb*-consistent.

$$PrHybP(A) = \sum_{A\in X} \frac{PraPlP(A)}{\sum_{B\in X} PraPlP(B)} \cdot m(X).$$

Analogically to starting a proportional transformation from the bbms or the beliefs of singletons $m(a) = bel(A)$ and adding some proportions of $m(X)$ to it for $A \in X$, we can start from $Pl(A)$ and remove some proportions of $m(X)$ from it, see [11, 14].

4 Summary of Consistencies of Probabilistic Transformations

The reason of defining the new transformations in [11] was an endeavour to find a probabilistic transformation which is both $\oplus$-consistent and *ulb*-consistent or $\circledcup$-consistent and *ulb*-consistent. This endeavour was unsuccessful, on contrary it is possible to prove the following theorem.

Theorem 2. *(i) Pl_T is the only $\oplus$-consistent probabilistic transformation.*
(ii) Bel_T is the only $\circledcup$-consistent PT which is also p-consistent.
(iii) BetT is the only α-consistent PT which is also p-consistent and satisfies Smets' assumptions of Anonymity and of Impossible event, see Section 5 and [30].

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[10] A counter-example: $m(\{a\}) = m(\{b\}) = m(\{c\}) = 0.1,\ m(\{a,b\}) = 0.7$, we obtain $PrPl(\{a\}) = PrPl(\{b\}) = 0.4294$ and $PrPl(\{c\}) = 0.1412 > 0.1 = Pl(\{c\})$.

For proofs of (i) and (ii) see [14], (iii) follows Smets' necessity of pignistic transformation [30]. From Theorem 2 the following corollary immediately follows.

Corollary 1. *(i) There does not exist any probabilistic transformation which is both $\oplus$-consistent and ulb-consistent in full generality. The only exception is normalized plausibility transformation Pl_T on the domain of quasi-Bayesian belief functions.*
(ii) There does not exist any probabilistic transformation which is both $\textcircled{\cup}$-consistent and ulb-consistent in full generality. The only exception is normalized belief transformation Bel_T on the domain of quasi-Bayesian belief functions.
(iii) There does not exist any $\oplus$- or $\textcircled{\cup}$-consistent probabilistic transformation which satisfies Smets' assumptions of pignistic transformation.
(iv) The pignistic transformation is neither compatible with the Dempster's rule $\oplus$ nor with the disjunctive rule of combination $\textcircled{\cup}$. (We mean compatibility in the sense of combination of pignistic probabilities).

Hence there is no need to look for another new probabilistic transformation. We can summarize consistencies of probabilistic transformations in Table 1.

Table 1. Consistencies of probabilistic transformations

	$\oplus$-consistency	$\textcircled{\cup}$-consistency	α-consistency	ulb-consistency	p-consistency
Pl_T	$\oplus$-consistent	no	no	2D BFs nD qBBFs	yes
Bel_T^*	no	$\textcircled{\cup}_I$-consistent	no	2D BFs nD qBBFs	yes
$BetT$	no	no	α-consistent	ulb-consistent	yes
$Prop_{Bel}T$	no	2D BFs - $(0,0)$ nD qBBFs - $_{nD}0$	no	ulb-consistent	yes
$StBel\_T$	no	2D BFs - $(0,0)$ nD qBBFs - $_{nD}0$	no	ulb-consistent	yes
$Prop_{Pl}T$	no	no	no	ulb-consistent	yes
$PraPlT$	no	no	no	2D BFs nD qBBFs	yes

* Bel_T is not defined for BFs such that $\sum_{A \in \Omega} m(A) = 0$.
qBBFs stands for quasi Bayesian belief functions.
All these transformations are $\oplus$-, $\textcircled{\cup}$-, and α-consistent on nD Bayesian BFs.

We have to recall the following equivalencies: $Pl\_T \equiv CautT \equiv VBT \equiv PrNPlT$, $Bel\_T \equiv DisjT$, $Prop_{Bel}T \equiv PrBlT$, and $Prop_{Pl}T \equiv PrPlT$. On 2D BFs and on nD quasi-Bayesian BFs (qBBFs) it holds further $Bel\_T \equiv Prop_{Bel}T \equiv StBel\_T$, and $Prop_{Pl}T \equiv PrPlT \equiv PraPlT$. The equivalency $\oplus \equiv \textcircled{\cup} \circ u \equiv \otimes$ holds on general nD Bayesian BFs, see [12].

5 Justification of Probabilistic Transformations

The recent justification of pignistic transformation is presented in [29, 30]. Let us make a general justification of the probabilistic transformations, which have been studied in this text.

Let us assume that a general probabilistic transformation PT is a function from the set of all belief functions to the Bayesian ones, i. e. to the set of probabilistic distributions on Ω. $PT(m) = P$, where $P(X) = PT(m)(X) = m'(X)$. It includes Smets' assumption of *Credal-Pignistic Link*, see Proposition 3.1 in [30]. Smets' assumption of *Efficiency*, see Proposition 4.1 in [30], also holds because $P(\Omega) = \sum_{A\in\Omega} P(A) = \sum_{A\in\Omega} m'(A) = bel'(\Omega) = 1$. All the studied transformations are p-consistent, thus we can, without lost of generality, assume this very natural assumption which requires that Bayesian BFs are transformed back to themselves. It corresponds to the Smets' *Projectivity* assumption, see Proposition 3.2 from [30].

All our probabilistic transformations satisfy also the Smets' assumption of *Anonymity*, i.e. independence of the result of transformation on permutation of elements of Ω, see Proposition 4.2 in [30], and the assumption of *Impossible event* requiring probability of an impossible event equal to zero, see Proposition 4.3 in [30].

The *Linearity assumption*, see Proposition 1.1 in [30], i.e. α-consistency in our terminology, is the only Smets' assumption that we do not include in our general assumptions. We can summarize our assumptions to the following definition.

Definition 1. *A function PT from the set of all belief functions to the set of the Bayesian ones is called* probabilistic transformation *of belief functions if it satisfies:*

(i) *p-consistency, i. e.* $PT(bel) = bel$ *for any Bayesian BF* bel,
(ii) $PT(bel)(X) = 0$ *for any impossible event* X, *i.e. for* X *such that* $Pl(X) = 0$,
(iii) *anonymity, i.e.* $TP(bel^*)(R(X)) = P^*(R(X)) = P(X) = TP(bel)(X)$, *for any permutation* R *of elements of* Ω *and BF* bel^* *given by* $m^*(R(X)) = m(X)$.

Theorem 3. *Let us assume all the assumptions from Definition 1. The following holds:*

(i) *If we add an assumption (iv-a) of* α*-consistency, we obtain a justification of the pignistic transformation* $BetT$.
(ii) *If we add an assumption (iv-c) of* $\oplus$*-consistency, we obtain a justification of the normalized plausibility transformation* Pl_T.
(iii) *If we add an assumption (iv-d) of* $\textcircled{\cup}$*-consistency, we obtain a justification of the normalized belief transformation* Bel_T.

The proofs of the statements immediately follow Definition 1, Theorem 2, and properties of the transformations. Note that both Cobb & Shenoy's *Invariance with respect to combination* and *Idempotency* [3] follow the assumption (iv-c) of $\oplus$-consistency.

The addition of an assumption of the *ulb*-consistency does not justify any unique probabilistic transformation. On the other hand, it excludes Pl_T and Bel_T, hence we do not assume any *ulb*-consistency in our new definition of probabilistic transformations.

6 Applicability of Probabilistic Transformations

Several probabilistic transformations have been presented and compared in this text. None of them is the best of all in general. Thus a natural question arises: Which probabilistic transformation should be used in our applications? As the answer is not unique, we will discuss it in this section.

The answer depends on the reason why we want to compute the probabilistic transformation and how we want to use it: Whether our goal is only to find the most prospective element of the frame of discernment or whether we have some specific assumptions to the result, and what operations we want to perform with the resulting probability.

Let us assume that we have all our evidence represented with BFs, i.e. that there is no other explicit nor implicit information about bbms assigned to multi-element focal elements. If we want to use a transformed probability for betting, we have to follow the Smets' necessity of pignistic transformation and compute pignistic probabilities. Nevertheless, we have to use them strictly on the pignistic level and to keep in mind that we cannot handle pignistic probabilities like the Bayesian BFs and combine them with the conjunctive or disjunctive rule of combination and similarly.

If we assume that the belief corresponds to lower probability and the plausibility to upper probability, we have to use some of the *ulb*-consistent probabilistic transformations. Similarly as before, we have to keep in mind that we have left the credal level and that we cannot handle probabilities as Bayesian BFs. If we, moreover, assume the α-consistency, then it is the only possibility of the pignistic probability again.

If we assume or want to be prepared for a combination of the resulting probabilities with the conjunctive combination, we have to use $\oplus$-consistent transformation, i.e. Pl_T. It is just the case of Cobb & Shenoy's assumptions. Similarly, if we assume disjunctive or α-combination of the resulting probabilities we have to use $\textcircled{\cup}$- or α-consistent transformation, i.e Bel_T or $BetT$ respectively.

If we are interested in selection of the most plausible element we have to use normalized plausibility transformation Pl_T. For determining the most believable element we have to use normalized belief Bel_T or preferably its stepwise version $StBel\_T$. In the case where $\textcircled{\cup}$ rule and Bel_T are used, we can handle probability as a Bayesian belief and combine it with $\textcircled{\cup}$. While in the case $StBel\_T$ we have to keep in mind that the credal level was left.

In the case of general looking for the most prospective element of the frame of discernment (without any other assumption) we can select a transformation with regard to its interpretation, see [10, 13].

If we have some other information on the domain, on the belief functions which are transformed or some special requirements to the resulting probabilities, we can use some special probabilistic transformation.

We assume that the evidence about application domain is represented with belief functions. It is called the *credal level* by Smets. By applying the pignistic transformation we leave this level and move us to the *pignistic level.* In the case that we do not assume α-consistency and do not use the pignistic transformations, we cannot speak longer about the pignistic level than about the *probabilistic level* or, more generally, about the *decisional level* of a representation and a solution of the decisional task.

7 Conclusion

A series of probabilistic transformations of belief functions have been analyzed and compared in this text, namely from the point of view of combination consistencies. They have different pros and cons. It has been shown that there does not exist a probabilistic transformation which is the best in general.

A new definition of probabilistic transformations which covers all the investigated transformations has been presented.

A particular discussion about which transformation should be applied in applications concludes the paper. It has been shown that both the Smets' approach of the necessity of the pignistic transformation and the Cobb & Shenoy's necessity of the normalized plausibility transformation are right within their assumptions which are mutually different. Besides, the other assumptions tend to other alternative solutions.

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