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Cross-Track Formation Control of Underactuated Autonomous Underwater Vehicles

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Summary. The problem of 3D cross-track control for underactuated 5-degrees-of-freedom (5-DOF) autonomous underwater vehicles (AUV) is considered. The proposed decentralized controllers make the AUVs asymptotically constitute a desired formation that follows a given straight-line path with a given forward speed profile. The proposed controllers consist of two blocks. The first block, which is based on a Line of Sight guidance law, makes every AUV asymptotically follow straight line paths corresponding to the desired formation motion. The second block manipulates the forward speed of every AUV in such a way that they asymptotically converge to the desired formation and move with a desired forward speed profile. The results are illustrated with simulations.

3.1 Introduction

Formation control of marine vessels is an enabling technology for a number of interesting applications. A fleet of multiple autonomous underwater vehicles (AUVs) moving together in a prescribed pattern can form an efficient data acquisition network for surveying at depths where neither divers nor tethered vehicles can be used, and in environments too risky for manned vehicles. This includes for instance oceanographic surveying at deep sea, operations under ice for exploration of Arctic areas and efficient monitoring sub-sea oil installations.

In this paper we study the problem of 3D cross-track formation control for underactuated 5-DOF AUVs that are independently controlled in surge, pitch and yaw. Roughly speaking, this problem can be formulated as follows: given a straight line path, a desired formation pattern, and a desired speed profile, control the AUVs such that asymptotically they constitute the desired formation which then moves along the given path with the desired speed. The relevance of such a formation control problem is justified by the fact that a desired path for autonomous vehicles is usually given by straight lines interconnecting way-points, see, e.g., [9]. At the same time, the speed profile of an autonomous vehicle is often specified independently of the desired path. This makes it possible to decouple the mission planning into two stages: a geometric path planning stage and a dynamic speed assignment stage.

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The cross-track control problem or, more generally, the path following control problem has been investigated in a number of publications. In [10, 3] this problem is considered for a single underactuated 3-DOF surface vessel and the proposed controllers are validated in experiments. In [11], the straight line cross-track control problem is considered for a single 3-DOF underactuated surface vessel. In that paper a nonlinear control law based on so-called Line of Sight (LOS) guidance is proposed and global κ-exponential stability of the cross-track error to a desired straight line path is proven. In [2] the straight line cross-track control problem for 5-DOF underactuated underwater vehicles is solved using an LOS guidance law and a nonlinear feedback controller rendering global κ-exponential stability of the cross-track error. A general path following problem for 6-DOF underactuated underwater vehicles is considered in [7]. The proposed nonlinear control strategy guarantees asymptotic convergence to a desired reference path.

Formation control and cooperative control of various systems have been studied in a large number of recent publications. For a necessarily incomplete list of publications on this subject, see, e.g., [15, 17, 1, 19, 18, 8, 5, 4] and the references therein. Most of the existing works on formation control focus on the high level analysis of achieving a formation. In this case the systems to be controlled are often assumed to have simple dynamics (like fully actuated point masses), which makes controlling every individual system an easy task and allows one to focus on the dynamics of the whole group of controlled systems rather than on the dynamics of individual systems. However, in marine applications the dynamics of individual systems (e.g., ships or AUVs) can be rather complicated to control, especially for underactuated systems. In this case the problem can not be considered only at the level of formation control alone, but must be analyzed both at the levels of individual systems and the whole group. This makes the problem more challenging.

For marine vehicles the formation control problem is considered in [21, 12, 13]. In [21] formation control for a fleet of fully actuated surface ships is considered. The proposed maneuvering-based controllers make each vehicle follow a given parameterized path with an assigned speed. The speed assignment, which depends on the states of all vehicles in the formation, guarantees exponential convergence to the desired formation. Similar ideas are used in [12] for the case of fully actuated AUVs. The control scheme proposed in [12] consists of a feedback controller that stabilizes each vehicle to a given path and a coordination controller that coordinates the motion of the vehicles along the paths. Graph theory is used to allow for different communication topologies in the formation. The problem of formation control for fully actuated marine surface vessels is also considered in [13]. The proposed exponentially stabilizing formation control laws are derived by imposing holonomic inter-vessel constraints and using tools from analytical mechanics.

The main contribution of this paper is the development of a cross-track control scheme for *formation control* of *underactuated* AUVs. This control problem can be decomposed into two sub-problems: **a)** a cross-track control problem and **b)** a coordination control problem. Given a desired formation pattern and a desired straight line path to be followed by the formation, we can define parallel desired straight line paths for each individual AUV, see Fig. 3.1. Then for every AUV we design LOS-based cross-track controllers that make the AUVs converge to the correspond-

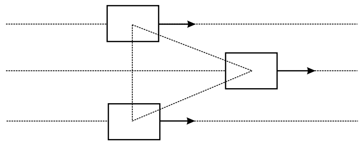


Fig. 3.1. Formation of AUVs.

ing paths. However, without controlling the forward speed of the vehicles in some coordinated manner, the desired formation pattern will not be achieved. To asymptotically achieve the formation pattern, each vehicle must *adapt* its forward speed in such a way that asymptotically all vehicles constitute the desired formation and move with the desired speed. This is the coordination control problem. The low bandwidth of underwater communication links form a serious constraint for cooperative control. To overcome this problem the proposed control scheme requires communication of only one position variable of each AUV among the vehicles. Moreover, it does not require communication links between *all* vehicles, thus significantly reducing the inter-vehicle communication. Similar ideas for 2D formation control of fully actuated marine vehicles are considered in [21] and [12]. In this work, we consider the case of *3D* formation control for *underactuated* underwater vehicles with *full dynamic models*.

The paper is organized as follows. In Section 3.2 we present the AUV model and control problem statement. In Section 3.3 we recall a result on cascaded systems that will be used throughout the paper. Section 3.4 contains a solution of the cross-track control problem for one AUV. In Section 3.5 we solve the coordination control problem for AUV formations. Simulation results are presented in Section 3.6 and conclusions are presented in Section 3.7.

3.2 Vehicle Model and Control Objective

In this section we present the kinematic and dynamic model describing the motion of the class of AUVs studied in this paper. Moreover, we define the notation used throughout the paper and state the control problem to be solved.

3.2.1 AUV Model

We consider an autonomous underwater vehicle (AUV) described by the 5-DOF model [9]

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{\nu} \tag{3.1}$$

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) = \boldsymbol{B}\boldsymbol{\tau}, \tag{3.2}$$

where $\boldsymbol{\eta} \triangleq \mathrm{col}(\boldsymbol{p}^i, \boldsymbol{\Theta}) \in \mathbb{R}^5$ and $\boldsymbol{\nu} \triangleq \mathrm{col}(\boldsymbol{v}^b, \boldsymbol{\omega}^b_{ib}) \in \mathbb{R}^5$. Here $\boldsymbol{p}^i = [x\ y\ z]^T$ is the inertial position of the AUV in Cartesian coordinates and $\boldsymbol{\Theta} = [\theta\ \psi]^T$ is the Euler-angle representation of the orientation of the AUV relative to the inertial frame, where θ and ψ are the pitch and yaw angles, respectively. The roll is assumed to be zero. Vector $\boldsymbol{v}^b = [u\ v\ w]^T$ is the linear velocity of the AUV in the body-fixed coordinate frame, where u, v and w are the surge, sway and heave velocities respectively, and $\boldsymbol{\omega}_{ib}^b = [q\ r]^T$ is the angular velocity of the AUV in the body-fixed coordinate frame, where q and r are the pitch and yaw velocities respectively.

The matrix $\boldsymbol{J}(\boldsymbol{\eta}) \in \mathbb{R}^{5\times 5}$ is the transformation matrix from the body-fixed coordinate frame b to the inertial coordinate frame i. Moreover, $\boldsymbol{M} = \boldsymbol{M}^T > 0$ is the mass and inertia matrix, $\boldsymbol{C}(\boldsymbol{\nu})$ is the Coriolis and centripetal matrix, $\boldsymbol{D}(\boldsymbol{\nu})$ is the damping matrix and $\boldsymbol{g}(\boldsymbol{\eta})$ are the restoring forces and moments due to gravity and buoyancy. The vector $\boldsymbol{\tau} = [\tau_u\ \tau_q\ \tau_r]^T$ is the control input, where τ_u is the surge control, τ_q is the pitch control and τ_r is the yaw control. The matrix $\boldsymbol{B} \in \mathbb{R}^{5\times 3}$ is the actuator matrix. Note that the AUV is underactuated, as only 3 independent controls are available to control 5 degrees of freedom.

By elaborating the differential kinematic equations in (3.1), we obtain ([9]):

$$\dot{x} = u\cos\psi\cos\theta - v\sin\psi + w\cos\psi\sin\theta \tag{3.3a}$$
$$\dot{y} = u\sin\psi\cos\theta + v\cos\psi + w\sin\theta\sin\psi \tag{3.3b}$$
$$\dot{z} = -u\sin\theta + w\cos\theta \tag{3.3c}$$
$$\dot{\theta} = q \tag{3.3d}$$
$$\dot{\psi} = \frac{1}{\cos\theta} r. \tag{3.3e}$$

Due to the Euler angle singularity, Eq. (3.3e) is not defined for $|\theta| = \frac{\pi}{2}$. However, the normal operating conditions for AUVs are $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (for $\frac{\pi}{2} < |\theta| < \pi$ the AUV is upside-down and, moreover, for conventional AUVs the vertical dive corresponding to $\theta = -\frac{\pi}{2}$ is physically impossible). Therefore, our state space for θ will be considered to be $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

In this paper we will use the dynamics model (3.2) in a modified form:

$$\dot{\boldsymbol{\nu}} = -\boldsymbol{M}^{-1}(\boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta})) + \boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{\tau} \triangleq \boldsymbol{f}(\boldsymbol{\nu},\boldsymbol{\eta}) + \bar{\boldsymbol{\tau}}, \tag{3.4}$$

where $\boldsymbol{f} = [f_u, f_v, f_w, f_q, f_r]^T$ and $\bar{\boldsymbol{\tau}} \triangleq \boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{\tau} = [\bar{\tau}_u, \bar{\tau}_v, \bar{\tau}_w, \bar{\tau}_q, \bar{\tau}_r]^T$.

Notice that for a large class of underwater vehicles, the body-fixed coordinate system can be chosen such that $\bar{\tau}_v = \bar{\tau}_w = 0$. This is possible for AUVs having port/starboard symmetry, under the assumption of zero roll. For the corresponding technique applied to surface vessels, see, e.g., [11] and [6]. Moreover, we assume that $\det\left([\boldsymbol{e}_1\ \boldsymbol{e}_4\ \boldsymbol{e}_5]\boldsymbol{M}^{-1}\boldsymbol{B}\right) \neq 0$, such that the mapping $(\tau_u, \tau_q, \tau_r) \mapsto (\bar{\tau}_u, \bar{\tau}_q, \bar{\tau}_r)$ is invertible. Therefore instead of designing controllers for (τ_u, τ_q, τ_r), we will design controllers for $(\bar{\tau}_u, \bar{\tau}_q, \bar{\tau}_r)$.

We assume that for $|u| \leq U_{max}$, where $U_{max} > 0$ is the maximal surge velocity, the sway and heave velocities v and w satisfy the following assumptions

$$|v| \leq C_v U_{max}|r|, \quad |w| \leq C_w U_{max}|q|, \tag{3.5}$$

for some $C_v > 0$, $C_w > 0$, and

$$|v| \leq U_{max}, \quad |w| \leq U_{max}. \tag{3.6}$$

These assumptions can be justified for a slender AUV (with its length much larger than the width/height). In this case, the damping for sway and heave directions (v and w) will be much larger compared to the surge direction (u). Therefore, the forward velocity (satisfying $|u| \leq U_{max}$) becomes dominant, see (3.6). Assumption (3.5) means that for the case of the angular velocities q and r converging to zero (i.e., the heading of the AUV has almost no change), the speeds v and w are damped out because of the hydrodynamical drag in the sway and heave direction and also converge to zero.

3.2.2 Control Objective

In this paper we deal with cross-track control for formations of AUVs. We will design decentralized control laws for n AUVs such that, after transients, the AUVs form a desired formation and move along a desired straight-line path with a given velocity profile, as illustrated in Fig. 3.2. The desired formation is characterized by

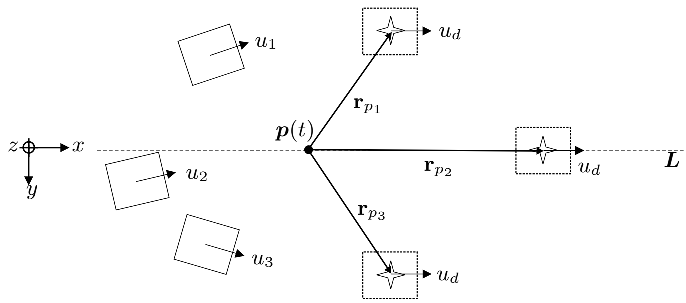


Fig. 3.2. Formation control of AUVs.

a formation reference point $\boldsymbol{p}(t)$ and a set of vectors $\boldsymbol{r}_{p_j}$, $j = 1, \ldots, n$ giving the desired relative positions of the AUVs with respect to the point $\boldsymbol{p}(t)$. The desired path of the formation is given by a straight line $\boldsymbol{L}$. The desired velocity profile is given by a differentiable function $u_d(t)$. The control objective is to guarantee that *asymptotically, i.e., in the limit for* $t \to +\infty$*,*

a) the AUVs constitute the formation, i.e.,

$$\boldsymbol{r}_1(t) - \boldsymbol{r}_{p_1} = \ldots = \boldsymbol{r}_n(t) - \boldsymbol{r}_{p_n} =: \boldsymbol{p}(t),$$

where $\boldsymbol{r}_j$, $j = 1, \ldots, n$, are the position vectors of the AUVs;

b) $\boldsymbol{p}(t)$ follows the desired path $\boldsymbol{L}$ with the desired velocity profile $u_d(t)$, i.e., $\boldsymbol{p}(t) \in \boldsymbol{L}$ and $|\dot{\boldsymbol{p}}(t)| = u_d(t)$, and the orientation of the AUVs are aligned with the desired straight line paths.

By choosing an inertial coordinate system with the x-axis coinciding with the desired straight-line path, i.e., $\boldsymbol{L} = \{(x, y, z) : x \in \mathbb{R}, y = 0, z = 0\}$ (see Fig. 3.2), the control objective can be formalized as follows

$$\lim_{t\to+\infty} y_j(t) - D_{y_j} = 0, \quad j = 1,\ldots,n,$$
$$\lim_{t\to+\infty} z_j(t) - D_{z_j} = 0, \quad j = 1,\ldots,n, \tag{3.7}$$

$$\lim_{t\to+\infty} \theta_j(t) = 0, \quad j = 1,\ldots,n,$$
$$\lim_{t\to+\infty} \psi_j(t) = 0, \quad j = 1,\ldots,n, \tag{3.8}$$

$$\lim_{t\to+\infty} x_1(t) - D_{x_1} = \cdots = \lim_{t\to+\infty} x_n(t) - D_{x_n} = \text{Const} + \int_0^t u_d(s)ds, \tag{3.9}$$

where $[x_j, y_j, z_j]^T$ and $[D_{x_j}, D_{y_j}, D_{z_j}]^T$ are the coordinates of the AUV position vectors $\boldsymbol{r}_j$ and relative position vectors $\boldsymbol{r}_{p_j}$, respectively, in the chosen inertial coordinate system.

The above stated control problem will be solved in two steps. First, for each AUV we design independent cross-track controllers guaranteeing that the cross-track control goal (3.7) is achieved, the orientation of the AUVs satisfy (3.8), and that the remaining dynamics in the x direction track certain speed reference commands u_{cj} (j corresponds to the jth AUV). At the second stage, we specify control laws for u_{cj} that coordinate the AUVs in the x-direction to asymptotically constitute the desired formation and make the speed of the formation track the desired speed profile $u_d(t)$, as specified in the coordination control goal (3.9).

3.3 Preliminaries

In this section we recall some results on cascaded systems of the form

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, t) + \boldsymbol{g}(\boldsymbol{x}_1, \boldsymbol{x}_2, t)\boldsymbol{x}_2 \tag{3.10}$$

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_2, t). \tag{3.11}$$

Prior to formulating the results we give the following definition.

Definition 3.1 ([16]). *System $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is called exponentially stable in any ball if for any $r > 0$ there exist $k = k(r) > 0$ and $\alpha = \alpha(r) > 0$ such that $|\boldsymbol{x}(t)| \leq k|\boldsymbol{x}(t_0)|\mathrm{e}^{-\alpha(t-t_0)}$.*

Theorem 3.1 ([16]). *System $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is exponentially stable in any ball if and only if it is globally uniformly asymptotically stable and locally exponentially stable.*

The next result directly follows from [20] (Theorem 7 and Lemma 8). It will be used throughout this paper.

Theorem 3.2. *System (3.10), (3.11) is exponentially stable in any ball if the following conditions are satisfied:* **a)** *systems $\dot{\boldsymbol{x}}_1 = f_1(\boldsymbol{x}_1, t)$ and (3.11) are both exponentially stable in any ball;* **b)** *there exists a quadratic positively definite function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\frac{\partial V}{\partial \boldsymbol{x}_1} \boldsymbol{f}_1(\boldsymbol{x}_1, t) \leq 0, \quad \forall\ \boldsymbol{x}_1,\ t \geq t_0; \tag{3.12}$$

c) *the interconnection term $\boldsymbol{g}(\boldsymbol{x}_1, \boldsymbol{x}_2, t)$ satisfies $\forall t \geq t_0$*

$$|\boldsymbol{g}(\boldsymbol{x}_1, \boldsymbol{x}_2, t)| \leq \rho_1(|\boldsymbol{x}_2|) + \rho_2(|\boldsymbol{x}_2|)|\boldsymbol{x}_1|, \tag{3.13}$$

where $\rho_1, \rho_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are some continuous functions.

3.4 Cross-Track Control of One AUV

In this section, we develop a cross-track controller for one AUV based on a line of sight (LOS) guidance algorithm and nonlinear controller design. Since we are dealing with only one AUV, the index j referring to the AUV's number is omitted.

3.4.1 Line of Sight Guidance

Line of sight (LOS) guidance is often used in practice for path control of marine vehicles. In this section we propose to use an LOS guidance law to meet the cross-track control goal (3.7) and the orientation control goal (3.8). First, following (3.7), we define the cross-track error as

$$\begin{bmatrix} e_y \\ e_z \end{bmatrix} \triangleq \begin{bmatrix} y - D_y \\ z - D_z \end{bmatrix}. \tag{3.14}$$

To study the cross-track error dynamics, we differentiate (3.14) with respect to time and use (3.1) to obtain:

$$\begin{bmatrix} \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} u \sin\psi \cos\theta + v \cos\psi + w \sin\theta \sin\psi \\ -u \sin\theta + w \cos\theta \end{bmatrix}. \tag{3.15}$$

The right-hand side of system (3.15) contains no control inputs. To regulate the cross-track error to zero, we will control the surge speed u, the pitch angle θ and the yaw angle ψ in such a way that the cross-track error converges to zero. This will be done with the help of LOS guidance. For LOS guidance, we pick a point that lies a distance $\Delta > 0$ ahead of the vehicle, along the desired path. The angles describing the orientation of the xz- and xy-projection of the line of sight are referred to as the LOS angles. With reference to Fig. 3.3, the LOS angles are given by the following two expressions:

$$\theta_{LOS}(t) = \tan^{-1}\left(\frac{e_z(t)}{\Delta}\right), \qquad \psi_{LOS}(t) = \tan^{-1}\left(\frac{-e_y(t)}{\Delta}\right). \tag{3.16}$$

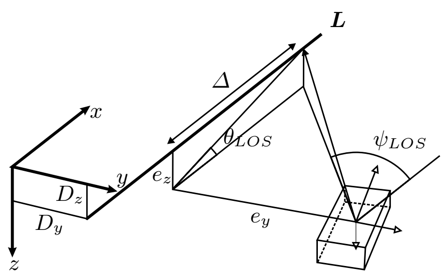


Fig. 3.3. Illustration of the LOS angles.

In the next subsections we will propose three controllers. The first controller regulates the surge speed u to asymptotically track some commanded speed signal $u_c(t)$. The second controller makes the pitch angle θ track θ_{LOS}. We will show that this will result in the cross-track error e_z and pitch angle θ exponentially converging to zero. The third controller makes the yaw angle ψ asymptotically track ψ_{LOS}. This will make the cross-track error e_y and the yaw angle ψ exponentially converge to zero.

3.4.2 Surge Control

The AUV considered in this work is underactuated. It can be actuated only in surge, pitch and yaw. In order to make the system controllable in other degrees of freedom (sway and heave), we need to ensure that the surge speed $u(t)$ is separated from zero. This will be achieved by a controller that makes $u(t)$ asymptotically track a speed reference command $u_c(t)$ that lies strictly within the bounds $u_c(t) \in (U_{min}, U_{max})$,

$t \geq t_0$, for some $U_{max} > U_{min} > 0$. The speed reference command u_c satisfying the above mentioned condition will be specified and used at a later stage to achieve the coordination control goal (3.9).

The controller for tracking the commanded signal u_c is given by

$$\bar{\tau}_u := -f_u(\boldsymbol{\nu}, \boldsymbol{\eta}) + \dot{u}_c - k_u(u - u_c), \tag{3.17}$$

where $k_u > 0$ is the controller gain. As follows from (3.4), this yields the linear GES tracking error dynamics $\dot{\tilde{u}} = -k_u \tilde{u}$, where $\tilde{u} := u - u_c$.

Notice that since $u_c(t)$ lies strictly within (U_{min}, U_{max}), $\forall t \geq 0$, and $u(t) \rightarrow u_c(t)$ exponentially and without overshoot, there exists a $t_0 \geq 0$ such that $u(t) \in [U_{min}, U_{max}]$, $\forall t \geq t_0$. Therefore, in the following sections we assume that $u(t) \in [U_{min}, U_{max}]$, $\forall t \geq t_0$.

3.4.3 Pitch Control

In this section, we propose a control law for the pitch control $\bar{\tau}_q$ that guarantees $\theta(t) \rightarrow \theta_{LOS}(t)$. We derive the pitch tracking error dynamics by differentiating $\tilde{\theta} := \theta - \theta_{LOS}$ with respect to time and using (3.3d), (3.16):

$$\dot{\tilde{\theta}} = \dot{\theta} - \dot{\theta}_{LOS} = q - \frac{\Delta}{e_z^2 + \Delta^2}\dot{e}_z = q - \frac{\Delta}{e_z^2 + \Delta^2}(-u \sin\theta + w \cos\theta). \tag{3.18}$$

We choose q as a *virtual* control input with the desired trajectory for q given by

$$q_d = \frac{\Delta}{e_z^2 + \Delta^2}(-u \sin\theta + w \cos\theta) - k_\theta \tilde{\theta}, \tag{3.19}$$

where $k_\theta > 0$. Inserting $q = q_d + \tilde{q}$ into (3.18), where $\tilde{q} := q - q_d$, then gives:

$$\dot{\tilde{\theta}} = -k_\theta \tilde{\theta} + \tilde{q}. \tag{3.20}$$

The pitch rate error dynamics is obtained by differentiating $\tilde{q}$ with respect to time and using (3.4):

$$\dot{\tilde{q}} = \dot{q} - \dot{q}_d = f_q(\boldsymbol{\nu}, \boldsymbol{\eta}) + \bar{\tau}_q - \dot{q}_d \tag{3.21}$$

We choose the feedback linearizing control law

$$\bar{\tau}_q = \dot{q}_d - f_q(\boldsymbol{\nu}, \boldsymbol{\eta}) - k_q \tilde{q}, \tag{3.22}$$

where $k_q > 0$. This results in

$$\dot{\tilde{q}} = -k_q \tilde{q}. \tag{3.23}$$

Notice that the closed-loop dynamics (3.20), (3.23) is a linear system with eigenvalues $-k_\theta < 0$ and $-k_q < 0$. Therefore, (3.20) and (3.23) is GES. The controller (3.22), with q_d given by (3.19), requires the measurement of $\dot{u}$ and $\dot{w}$. For AUVs equipped with inertial navigation systems, these accelerations are measured.

Next, we derive an estimate of $|w|$, which will be used in the next section. Notice that $q_d = \dot{\theta}_{LOS} - k_\theta \tilde{\theta}$. Thus, $|q| \leq |q_d| + |q - q_d| \leq |\dot{\theta}_{LOS}| + k_\theta |\tilde{\theta}| + |\tilde{q}|$. One can compute $\dot{\theta}_{LOS}$ from (3.16) and obtain the estimate $|\dot{\theta}_{LOS}| \leq \Delta |\dot{e}_z| / (e_z^2 + \Delta^2) \leq |\dot{e}_z| / \Delta$. Substitution of this estimate into the obtained estimate of $|q|$ and then into (3.5) gives

$$|w| \leq C_w U_{max}(|\dot{e}_z|/\Delta + k_\theta |\tilde{\theta}| + |\tilde{q}|). \tag{3.24}$$

3.4.4 Analysis of the e_z Dynamics

In this section we analyze the e_z-dynamics of the AUV and show that the controller (3.22) by making $\theta(t) \to \theta_{LOS}(t)$ also makes $e_z(t)$ converge exponentially to zero. Subsequently this implies that $\theta_{LOS}(t)$ and therefore $\theta(t)$ converge to zero (see (3.16)). The differential equation for e_z given in (3.15) can be written as

$$\dot{e}_z = -u \sin \theta_{LOS} + w \cos \theta_{LOS} + \delta(\theta, \theta_{LOS}, u, w)\tilde{\theta}, \tag{3.25}$$

where $\delta(\theta, \theta_{LOS}, u, w) := (u(\sin \theta_{LOS} - \sin \theta) + w(\cos \theta - \cos \theta_{LOS}))/\tilde{\theta}$. Notice that by the mean value theorem and by assumption (3.6) we have

$$|\delta(\theta, \theta_{LOS}, u, w)| \leq |u| + |w| \leq 2U_{max}, \tag{3.26}$$

provided $|u| \leq U_{max}$. Notice that this condition holds because $u(t) \in [U_{min}, U_{max}]$, $\forall t \geq t_0$, as discussed in Section 3.4.2. In the sequel we will write δ without its arguments. Substituting the expressions of θ_{LOS} from (3.16) into (3.25), we obtain

$$\dot{e}_z = -u \frac{e_z}{\sqrt{e_z^2 + \Delta^2}} + w \frac{\Delta}{\sqrt{e_z^2 + \Delta^2}} + \delta\tilde{\theta}. \tag{3.27}$$

System (3.27) can be considered as a nominal system perturbed through the term $\delta\tilde{\theta}$ by the GES dynamics (3.20), (3.23). The next theorem provides a result on stability of these systems.

Theorem 3.3. *Consider system (3.27) in cascade with (3.20), (3.23). Let w satisfy assumptions (3.5) and (3.6) and suppose $u(t) \in [U_{min}, U_{max}]$ for all $t \geq t_0$ with $U_{min} > 0$ and $U_{max} < \Delta/C_w$. Then system (3.27), (3.20), (3.23) is exponentially stable in any ball.*

Proof: Consider the Lyapunov function candidate $V := 1/2|e_z|^2$. Its derivative along solutions of (3.27) satisfies

$$\begin{aligned} \dot{V} &= \frac{-u|e_z|^2 + \Delta e_z w}{\sqrt{e_z^2 + \Delta^2}} + e_z\delta\tilde{\theta} \leq -\frac{u|e_z|^2 + \Delta|e_z||w|}{\sqrt{e_z^2 + \Delta^2}} + |e_z||\delta||\tilde{\theta}| \\ &\leq -\frac{u|e_z|^2}{\sqrt{e_z^2 + \Delta^2}} + |e_z||w| + |e_z|2U_{max}|\tilde{\theta}|. \end{aligned} \tag{3.28}$$

In the last inequality we have used inequality (3.26) and the upper bound on u. Substituting inequality (3.24), which holds since w satisfies assumption (3.5), into (3.28) and using the upper bound on u, we obtain

$$\dot{V} \leq -\frac{u|e_z|^2}{\sqrt{e_z^2 + \Delta^2}} + |e_z|C_w U_{max}(|\dot{e}_z|/\Delta + k_\theta|\tilde{\theta}| + |\tilde{q}|) + |e_z|2U_{max}|\tilde{\theta}|. \tag{3.29}$$

Notice that $|\dot{V}| = |e_z||\dot{e}_z|$. Therefore,

$$\dot{V} \leq -\frac{U_{min}|e_z|^2}{\sqrt{e_z^2 + \Delta^2}} + \frac{C_w U_{max}}{\Delta}|\dot{V}| + |e_z|(\alpha_q|\tilde{q}| + \alpha_\theta|\tilde{\theta}|), \tag{3.30}$$

where $\alpha_q := C_w U_{max}$ and $\alpha_\theta := (C_w k_\theta + 2)U_{max}$. Hence,

$$\dot{V}\left(1 - \text{sign}\dot{V}\frac{C_w U_{max}}{\Delta}\right) \le -\frac{U_{min}|e_z|^2}{\sqrt{e_z^2 + \Delta^2}} + |e_z|(\alpha_q|\tilde{q}| + \alpha_\theta|\tilde{\theta}|). \tag{3.31}$$

Denote $\beta^+ := (1 + C_w U_{max}/\Delta)^{-1}$ and $\beta^- := (1 - C_w U_{max}/\Delta)^{-1}$. Since $U_{max} < \Delta/C_w$, we have $\beta^+ > 0$ and $\beta^- > 0$. Hence

$$\dot{V} \le -\beta^+\frac{U_{min}|e_z|^2}{\sqrt{e_z^2 + \Delta^2}} + \beta^-|e_z|(\alpha_q|\tilde{q}| + \alpha_\theta|\tilde{\theta}|). \tag{3.32}$$

Next we consider the comparison system

$$\dot{V} = -\beta^+\frac{U_{min}2V}{\sqrt{2V + \Delta^2}} + \beta^-\sqrt{2V}(\alpha_q|\tilde{q}| + \alpha_\theta|\tilde{\theta}|). \tag{3.33}$$

If we show that system (3.33), (3.20), (3.23) is exponentially stable in any ball provided $V(t_0) \ge 0$, then by the comparison lemma [14] we conclude that system (3.27), (3.20), (3.23) is exponentially stable in any ball. Notice that system (3.33), (3.20), (3.23) can be considered as a cascaded connection of the nominal system

$$\dot{V} = -\frac{2\beta^+ U_{min}}{\sqrt{2V + \Delta^2}}V, \tag{3.34}$$

with GES system (3.20), (3.23) through the interconnection term $g := \beta^-\sqrt{2V}(\alpha_q|\tilde{q}| + \alpha_\theta|\tilde{\theta}|)$. One can easily see that system (3.34) is exponentially stable in any set $V(t_0) \in [0, R]$, $R \ge 0$ with the quadratic Lyapunov function V^2. The interconnection term g can be estimated by $|g| \le (1 + 2V)\beta^-(\alpha_q|\tilde{q}| + \alpha_\theta|\tilde{\theta}|)$. Therefore by Theorem 3.2 the cascade (3.33) and (3.20), (3.23) is exponentially stable in any ball. By the comparison lemma [14] the system (3.27), (3.20), (3.23) is exponentially stable in any ball.This concludes the proof of the theorem. □

3.4.5 Yaw Control

In this section, we propose a control law for the yaw control $\bar{\tau}_r$ that guarantees that $\psi \to \psi_{LOS}$ exponentially. We derive the yaw tracking error dynamics by differentiating $\tilde{\psi} := \psi - \psi_{LOS}$ with respect to time and using (3.3e):

$$\dot{\tilde{\psi}} = \dot{\psi} - \dot{\psi}_{LOS} = \frac{1}{\cos\theta}r + \frac{\Delta}{e_y^2 + \Delta^2}\dot{e}_y \tag{3.35}$$

$$= \frac{1}{\cos\theta}r + \frac{\Delta}{e_y^2 + \Delta^2}(u\sin\psi\cos\theta + v\cos\psi + w\sin\theta\sin\psi). \tag{3.36}$$

We choose r as a *virtual* control input and choose the desired trajectory for r as

$$r_d = -\cos\theta\frac{\Delta}{e_y^2 + \Delta^2}(u\sin\psi\cos\theta + v\cos\psi + w\sin\theta\sin\psi) - k_\psi\tilde{\psi}\cos\theta, \tag{3.37}$$

where $k_\psi > 0$. Inserting $r = r_d + \tilde{r}$ into (3.36), where $\tilde{r} := r - r_d$, then gives

$$\dot{\tilde{\psi}} = -k_\psi \tilde{\psi} + \frac{1}{\cos\theta}\tilde{r}. \tag{3.38}$$

We derive the yaw rate error dynamics by differentiating $\tilde{r}$ with respect to time and using (3.4):

$$\dot{\tilde{r}} = \dot{r} - \dot{r}_d = f_r(\boldsymbol{\nu}, \boldsymbol{\eta}) + \bar{\tau}_r - \dot{r}_d. \tag{3.39}$$

We choose the feedback linearizing control law

$$\bar{\tau}_r = \dot{r}_d - f_r(\boldsymbol{\nu}, \boldsymbol{\eta}) - k_r \tilde{r}, \tag{3.40}$$

where $k_r > 0$. This results in the GES linear dynamics

$$\dot{\tilde{r}} = -k_r \tilde{r}. \tag{3.41}$$

System (3.38), (3.41) can be viewed as a cascade of two GES linear systems interconnected through the term $\tilde{r}/\cos\theta(t)$. If $\theta(t)$ lies in a compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \geq t_0$, then $1/\cos\theta(t)$ is bounded and the interconnection term satisfies $|\tilde{r}/\cos\theta(t)| \leq C\tilde{r}$ for some constant $C > 0$. Therefore system (3.38), (3.41) is GES.

Just like in the case of pitch control, here we give an estimate of $|v|$, which will be used in the next section. Notice that $r = \cos\theta\dot{\psi} = \cos\theta\dot{\psi}_{LOS} + \cos\theta(\dot{\psi} - \dot{\psi}_{LOS})$. As follows from (3.36) and (3.37), $\cos\theta(\dot{\psi} - \dot{\psi}_{LOS}) = \tilde{r} - k_\psi \cos\theta\tilde{\psi}$. Therefore, $|r| \leq |\dot{\psi}_{LOS}| + |\tilde{r}| + k_\psi|\tilde{\psi}|$. One can compute $\dot{\psi}_{LOS}$ from (3.16) and obtain the estimate $|\dot{\psi}_{LOS}| \leq \Delta|\dot{e}_y|/(e_y^2 + \Delta^2) \leq |\dot{e}_y|/\Delta$. Substitution of this estimate into the obtained estimate of $|r|$ and then into (3.5) gives

$$|v| \leq C_v U_{max}(|\dot{e}_y|/\Delta + k_\psi|\tilde{\psi}| + |\tilde{r}|). \tag{3.42}$$

3.4.6 Analysis of the e_y Dynamics

In this section we consider the e_y-dynamics of the AUV and show that the controller (3.40) by making $\psi(t) \to \psi_{LOS}(t)$ also makes $e_y(t)$ converge exponentially to zero. This, in turn, implies that $\psi_{LOS}(t)$ and $\psi(t)$ converge to zero. The differential equation for e_y given in (3.15) can be written as

$$\dot{e}_y = u\sin\psi_{LOS} + v\cos\psi_{LOS} + \delta_\psi(u, v, \psi, \psi_{LOS})\tilde{\psi} + \delta_y(u, w, \psi, \theta)\theta \tag{3.43}$$

where, $\delta_\psi(u, v, \psi, \psi_{LOS}) := u(\sin\psi - \sin\psi_{LOS})/\tilde{\psi} + v(\cos\psi - \cos\psi_{LOS})/\tilde{\psi}$ and $\delta_y(u, w, \psi, \theta) := (u\sin\psi(\cos\theta - 1) + w\sin\theta\sin\psi)/\theta$. Notice that by the mean value theorem and by assumption (3.6) we have

$$|\delta_y(u, w, \psi, \theta)| \leq |u| + |w| \leq 2U_{max} \tag{3.44}$$

$$|\delta_\psi(u, v, \psi, \psi_{LOS})| \leq |u| + |v| \leq 2U_{max} \tag{3.45}$$

provided $|u| \leq U_{max}$. The pitch angle θ can be written as

$$\theta = \tilde{\theta} + \theta_{LOS} = \tilde{\theta} + \delta_z(e_z)e_z,$$

where $\delta_z(e_z) := \tan^{-1}(e_z/\Delta)/e_z$ is a globally bounded function. In the sequel we will write δ_z, δ_y and δ_ψ without their arguments. Substituting the expression for θ into (3.43) together with the expression of ψ_{LOS} from (3.16) gives

$$\dot{e}_y = -u\frac{e_y}{\sqrt{e_y^2 + \Delta^2}} + v\frac{\Delta}{\sqrt{e_y^2 + \Delta^2}} + \delta_\psi\tilde{\psi} + \delta_y\tilde{\theta} + \delta_y\delta_z e_z. \tag{3.46}$$

System (3.46) can be viewed as a nominal system perturbed through the terms $\delta_\psi\tilde{\psi}$, $\delta_y\tilde{\theta}$ and $\delta_y\delta_z e_z$ by the $(\tilde{\psi}, \tilde{r})$-dynamics (3.38), (3.41), which is GES provided $\theta(t)$ lies in a compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \geq t_0$, and by the e_z and $(\tilde{\theta}, \tilde{q})$-dynamics (3.27), (3.20), (3.23), which is exponentially stable in any ball by virtue of Theorem 3.3. Notice that the nominal dynamics of system (3.46) is identical to the nominal dynamics of system (3.27) and the estimate (3.42) is identical to the estimate (3.24). Therefore, using the same arguments as in the proof of Theorem 3.3 we can formulate the following result for the e_y-dynamics.

Theorem 3.4. *Consider the e_y-dynamics (3.46) in cascade with the $(\tilde{\psi}, \tilde{r})$-dynamics (3.38), (3.41) and with the e_z and $(\tilde{\theta}, \tilde{q})$-dynamics (3.27), (3.20), (3.23). Let w and v satisfy assumptions (3.5) and (3.6) and suppose $u(t) \in [U_{min}, U_{max}]$ for all $t \geq t_0$ with $U_{min} > 0$ and $U_{max} < \min\{\Delta/C_w, \Delta/C_v\}$. Then the overall closed-loop system system (3.46), (3.38), (3.41), (3.27), (3.20), (3.23) is exponentially stable in any ball provided $\theta(t)$ lies in a compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \geq t_0$.*

Under the conditions of this theorem, we obtain that the cross-track errors $e_y(t)$ and $e_z(t)$ and the orientation angles $\theta(t)$ and $\psi(t)$ converge to zero exponentially, i.e., the cross-track control goal (3.7) and the orientation control goal (3.8) are achieved. The remaining dynamics is in the x-direction. These dynamics are analyzed in the next section.

3.4.7 Analysis of the x Dynamics

The x-dynamics can be written as

$$\begin{aligned}\dot{x} &= u + u(1 - \cos\psi\cos\theta) - v\sin\psi + w\cos\psi\sin\theta \\ &= u_c + \tilde{u} + u\frac{(1 - cos\psi)}{\psi}\psi + u\cos\psi\frac{(1 - \cos\theta)}{\theta}\theta - v\frac{\sin\psi}{\psi}\psi + w\cos\psi\frac{\sin\theta}{\theta}\theta.\end{aligned} \tag{3.47}$$

Recall that $\psi = \psi_{LOS} + \tilde{\psi} = \frac{\tan^{-1}(-e_y/\Delta)}{e_y}e_y + \tilde{\psi}$ and $\theta = \theta_{LOS} + \tilde{\theta} = \frac{\tan^{-1}(e_z/\Delta)}{e_z}e_z + \tilde{\theta}$. Substituting these expressions into (3.47), we obtain

$$\dot{x} = u_c + \boldsymbol{h}(e_z, e_y, \theta, \psi, u, w, v)\boldsymbol{\chi}, \tag{3.48}$$

where $\boldsymbol{\chi} := (\tilde{u}, e_z, e_y, \tilde{\theta}, \tilde{\psi})^T$. The interconnection matrix $\boldsymbol{h}$ can easily be obtained from the expressions given above. Notice that since the functions $\sin\alpha/\alpha$, $(1 - \cos\alpha)/\alpha$ and $\tan^{-1}(\alpha)/\alpha$ are globally bounded, and because of assumption (3.6) $|v|$ and $|w|$ are bounded by U_{max}, we conclude that $\boldsymbol{h}$ is globally bounded. As follows from the previous sections, the variable $\boldsymbol{\chi}$ exponentially converges to zero provided that $u(t)$ and $\theta(t)$ satisfy the conditions in Theorem 3.4. Therefore, the speed in the x-direction asymptotically tracks the speed reference command $u_c(t)$.

3.5 Coordination Control of Multiple AUVs

In the previous sections we have designed a cross-track controller that guarantees that every AUV in closed loop with this controller achieves the cross-track control goal (3.7) with the remaining dynamics in the x-direction given by (3.48). In order to achieve the coordination control goal (3.9), for each AUV we will use the freedom we have in choosing the commanded speed signal u_{cj}, $j = 1, \dots, n$. Since in this section we are dealing with multiple AUVs, we will use the subscript j to denote the AUV's number.

Recall that the cross-track control goal (3.7) is achieved provided that the commanded speed for each AUV lies inside the set (U_{min}, U_{max}), i.e.,

$$u_{cj}(t) \in (U_{min}, U_{max}), \quad \forall \ t \geq 0, \ j = 1, \dots, n. \tag{3.49}$$

In this section we must therefore design control laws for u_{cj}, $j = 1, \dots, n$, that satisfy these constraints at the same time as they guarantee that all AUVs achieve the coordination control goal (3.9). To satisfy (3.9) the AUVs have to adjust their forward speed to asymptotically converge to the desired formation pattern and move with the desired speed profile $u_d(t)$. This means that they may either have to speed up or wait for other AUVs to obtain the desired formation before they collectively reach the desired speed $u_d(t)$.

Here, we make a natural assumption that the desired speed profile lies within (U_{min}, U_{max}), i.e., there exists $a > 0$ such that

$$u_d(t) \in [U_{min} + a, U_{max} - a], \quad \forall t \geq 0. \tag{3.50}$$

To solve the coordination problem (3.9), we propose the following control law for u_{cj}:

$$u_{cj} = u_d(t) - g\left(\sum_{i=1}^{n} \gamma_{ji}(x_j - x_i - d_{ji})\right), \quad j = 1, \dots, n. \tag{3.51}$$

Here $d_{ji} \triangleq D_{x_j} - D_{x_i}$ correspond to the distances along the x-axis between the jth and ith AUVs in the formation. The linkage parameters γ_{ji} are nonnegative and satisfy $\gamma_{ij} = \gamma_{ji}$, $\gamma_{ii} = 0$. The function $g(x)$ is a continuously differentiable non-decreasing function with a bounded derivative satisfying $g'(0) > 0$, $g(0) = 0$ and $g(x) \in (-a, a)$, where a is the parameter defined in (3.50). Notice that under these assumptions on g and under the assumption on the desired speed profile $u_d(t)$ (3.50), the proposed u_{cj} satisfy the condition (3.49) for all values of its arguments. The function g can be chosen, for example, equal to $g(x) \triangleq 2a/\pi \tan^{-1}(x)$.

The dynamics of (3.48) in closed loop with (3.51) are given by the equations

$$\dot{x}_j = u_d(t) - g\left(\sum_{i=1}^{n} \gamma_{ji}(x_j - x_i - d_{ji})\right) + \boldsymbol{h}_j \boldsymbol{\chi}_j, \quad j = 1, \dots, n. \tag{3.52}$$

It can be easily verified that after the change of coordinates $\bar{x}_j \triangleq x_j - D_{x_j} - \int_0^t u_d(s)ds$, $j = 1, \dots, n$, system (3.52) is equivalent to

$$\dot{\bar{x}}_j = -g\left(\sum_{i=1}^{n} \gamma_{ji}(\bar{x}_j - \bar{x}_i)\right) + \boldsymbol{h}_j \boldsymbol{\chi}_j, \quad j = 1, \ldots, n. \tag{3.53}$$

To simplify this system, we will rewrite it in the vector form. To this end, let us introduce the following notations $\bar{\mathbf{x}} := (\bar{x}_1, \ldots, \bar{x}_n)^T$, the function $\mathbf{g}(\bar{\mathbf{x}})$ and the matrix $\boldsymbol{\Gamma}$ given by

$$\mathbf{g}(\bar{\mathbf{x}}) := \begin{pmatrix} g(\bar{x}_1) \\ \vdots \\ g(\bar{x}_n) \end{pmatrix}, \quad \boldsymbol{\Gamma} := \begin{pmatrix} \sum_{j=1}^{n} \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1n} \\ -\gamma_{21} & \sum_{j=1}^{n} \gamma_{2j} & \cdots & -\gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{n1} & -\gamma_{n2} & \cdots & \sum_{j=1}^{n} \gamma_{nj} \end{pmatrix}.$$

Then, system (3.53) can be written in the vector form

$$\dot{\bar{\mathbf{x}}} = -\mathbf{g}(\boldsymbol{\Gamma}\bar{\mathbf{x}}) + \boldsymbol{H}\boldsymbol{\varepsilon}, \tag{3.54}$$

Where $\boldsymbol{\varepsilon} := [\boldsymbol{\chi}_1^T, \ldots, \boldsymbol{\chi}_n^T]^T$ and the matrix $\boldsymbol{H}$ is a block-diagonal matrix with $\boldsymbol{h}_j$, $j = 1, \ldots, n$, on its diagonal. Since $\boldsymbol{h}_j$, is a globally bounded function of the states of the jth AUV, $j = 1, \ldots, n$, the matrix $\boldsymbol{H}$ is a globally bounded function of the states of all n AUVs. Notice that matrix $\boldsymbol{\Gamma}$ has the property $\boldsymbol{\Gamma}\mathbf{v}_1 = 0$, where $\mathbf{v}_1 := (1, 1 \ldots, 1)^T$. Therefore, $\boldsymbol{\Gamma}$ has a zero eigenvalue with $\mathbf{v}_1$ being the corresponding eigenvector.

For system (3.54) the control goal (3.9) can be stated in the equivalent form

$$\bar{\mathbf{x}}(t) \to \eta \mathbf{v}_1, \text{ as } t \to +\infty \tag{3.55}$$

for some $\eta \in \mathbb{R}$. Now we can formulate the main result of this section.

Theorem 3.5. *Consider system (3.54) coupled with the cross-track dynamics of every AUV through $\boldsymbol{H}\boldsymbol{\varepsilon}$. Suppose the conditions of Theorem (3.4) hold for every AUV and the zero eigenvalue of matrix $\boldsymbol{\Gamma}$ has multiplicity one. Then control goal (3.55) is achieved.*

Proof: Consider the matrix $\boldsymbol{\Gamma}$. From the structure of $\boldsymbol{\Gamma}$, by Gershgorin's theorem [22] we obtain that all eigenvalues of $\boldsymbol{\Gamma}$ lie in the closed right half of the complex plane. Since $\boldsymbol{\Gamma}$ is symmetric, all its eigenvalues are real and, by the condition of the theorem, only one of them is zero. Therefore, all the other eigenvalues are positive. Since $\boldsymbol{\Gamma}$ is symmetric, one can choose an orthogonal matrix $\boldsymbol{S}$, i.e., such that $\boldsymbol{S}^{-1} = \boldsymbol{S}^T$, satisfying $\boldsymbol{\Gamma} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^T$, where

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{I}_{(n-1)} \end{pmatrix},$$

with $\boldsymbol{I}_{(n-1)}$ being the $(n-1)$-dimensional identity matrix. The corresponding matrix $\boldsymbol{S}$ equals $\boldsymbol{S} := [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, where $\mathbf{v}_1$ is the eigenvector corresponding to the zero eigenvalue and $\mathbf{v}_2, \ldots, \mathbf{v}_n$ are the orthogonal eigenvectors corresponding to the remaining positive eigenvalues λ_j, $j = 2, \ldots, n$, and normalized according to $|\mathbf{v}_j|^2 = 1/\lambda_j$. Substituting this factorization of $\boldsymbol{\Gamma}$ into (3.54), we obtain, after the change of coordinates $\hat{\mathbf{x}} := \boldsymbol{S}^T \bar{\mathbf{x}}$,

$$\dot{\hat{\mathbf{x}}} = -\boldsymbol{S}^T \mathbf{g}\left(\boldsymbol{S}\boldsymbol{\Lambda}\hat{\mathbf{x}}\right) + \boldsymbol{S}^T \boldsymbol{H}\boldsymbol{\varepsilon}. \tag{3.56}$$

Denote $\boldsymbol{\Xi} := [\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n]$. Then $\boldsymbol{S}^T = [\mathbf{v}_1, \boldsymbol{\Xi}]^T$. Denote the first component of $\hat{\mathbf{x}}$ by ζ and the $(n-1)$-dimensional vector of the remaining components by $\boldsymbol{\xi}$, i.e., $\hat{\mathbf{x}} = [\zeta, \boldsymbol{\xi}^T]^T$. By the structure of $\boldsymbol{\Lambda}$ we have $\boldsymbol{S}\boldsymbol{\Lambda}\hat{\mathbf{x}} = \boldsymbol{\Xi}\boldsymbol{\xi}$. With this new notation we can write system (3.56) in the following form:

$$\dot{\zeta} = -\mathbf{v}_1^T \mathbf{g}\left(\boldsymbol{\Xi}\boldsymbol{\xi}\right) + \mathbf{v}_1^T \boldsymbol{H}\boldsymbol{\varepsilon} \tag{3.57}$$

$$\dot{\boldsymbol{\xi}} = -\boldsymbol{\Xi}^T \mathbf{g}\left(\boldsymbol{\Xi}\boldsymbol{\xi}\right) + \boldsymbol{\Xi}^T \boldsymbol{H}\boldsymbol{\varepsilon}. \tag{3.58}$$

From these equations we see that the $\boldsymbol{\xi}$-dynamics are decoupled from ζ. The $\boldsymbol{\xi}$-dynamics can be considered as the nominal dynamics

$$\dot{\boldsymbol{\xi}} = -\boldsymbol{\Xi}^T \mathbf{g}\left(\boldsymbol{\Xi}\boldsymbol{\xi}\right) \tag{3.59}$$

coupled through $\boldsymbol{\Xi}^T \boldsymbol{H}\boldsymbol{\varepsilon}$ with the cross-track dynamics of the variables e_{yj}, v_j, e_{zj}, w_j, θ_j, q_j, ψ_j, r_j, $\tilde{u}_j$ of every AUV. These cross-track dynamics are exponentially stable in any ball provided that $u_j(t) \in [U_{min}, U_{max}]$ and $\theta_j(t)$ lies in a compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$, for $j = 1, \ldots, n$. We will show that system (3.58) in cascade with the cross-track dynamics of all AUVs is exponentially stable in any ball provided that the above mentioned conditions on $u_j(t)$ and $\theta_j(t)$ are satisfied. This will be shown using Theorem 3.2. Since the coupling matrix $\boldsymbol{\Xi}^T \boldsymbol{H}$ is globally bounded and the cross-track dynamics of all AUVs is exponentially stable in any ball (under the conditions on $u_j(t)$ and $\theta_j(t)$ stated above), we only need to show that the nominal system is exponentially stable in any ball with a quadratic Lyapunov function satisfying (3.12).

Consider the Lyapunov function $V(\boldsymbol{\xi}) = 1/2|\boldsymbol{\xi}|^2$. It's derivative along solutions of (3.59) equals

$$\dot{V} = -\boldsymbol{\xi}^T \boldsymbol{\Xi}^T \mathbf{g}\left(\boldsymbol{\Xi}\boldsymbol{\xi}\right). \tag{3.60}$$

Denote $\boldsymbol{\vartheta} := \boldsymbol{\Xi}\boldsymbol{\xi}$. Then $\dot{V} = -\boldsymbol{\vartheta}^T \mathbf{g}(\boldsymbol{\vartheta})$. By elaborating this expression we obtain $\dot{V} = -\sum_{i=1}^{n} \vartheta_i g(\vartheta_i)$. Notice that by the conditions imposed on the function g we have $xg(x) > 0$ for all $x \in \mathbb{R}$ satisfying $x \neq 0$. Therefore $\dot{V} = 0$ if and only if $\boldsymbol{\vartheta} = 0$. At the same time, since rank$\boldsymbol{\Xi} = (n-1)$, it holds that $\boldsymbol{\vartheta} = \boldsymbol{\Xi}\boldsymbol{\xi} = 0$ if and only if $\boldsymbol{\xi} = 0$. Hence, $\dot{V}$ as a function of $\boldsymbol{\xi}$ is negative definite. This implies that system (3.59) is GAS. In fact, since system (3.59) is autonomous, it is GUAS. Let us show that system (3.59) is locally exponentially stable (LES). The system matrix $\boldsymbol{A}$ of system (3.59) being linearized at the origin equals $\boldsymbol{A} = -\boldsymbol{\Xi}^T \frac{\partial \boldsymbol{g}}{\partial \bar{\boldsymbol{x}}}(0)\boldsymbol{\Xi}$. By the construction of $\boldsymbol{g}(\bar{\boldsymbol{x}})$, we obtain $\frac{\partial \boldsymbol{g}}{\partial \bar{\boldsymbol{x}}}(0) = g'(0)\boldsymbol{I}_n$. Since $\boldsymbol{\Xi}$ consists of orthogonal eigenvectors $\mathbf{v}_i$ normalized by $|\mathbf{v}_i|^2 = 1/\lambda_i$, where $\lambda_i > 0$, $i = 2, \ldots, n$, we obtain $\boldsymbol{A} = -g'(0)\mathrm{diag}(1/\lambda_2, \ldots, 1/\lambda_n)$, which is Hurwitz, because $g'(0) > 0$ by the definition of $g(x)$. Therefore, the linearized system (3.59) is GES, which implies that system (3.59) itself is LES. By Theorem 3.1 system (3.59) is exponentially stable in any ball.

Applying Theorem 3.2, we conclude that system (3.58) in cascade with the cross-track dynamics of all AUVs is exponentially stable in any ball provided that $u_j(t) \in [U_{min}, U_{max}]$ and $\theta_j(t)$ lies in a compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$, for $j = 1, \ldots, n$. This, in turn, implies that the right-hand side of (3.57) exponentially tends to zero.

By integrating (3.57), we obtain that $\zeta(t) \to \eta$, where $\eta \in \mathbb{R}$ is some constant. Recall that $\bar{\mathbf{x}} = \boldsymbol{S}\hat{\mathbf{x}} = \mathbf{v}_1\zeta + \boldsymbol{\Xi}\xi$. Since $\boldsymbol{\xi}(t) \to 0$ and $\zeta(t) \to \eta$, we obtain $\bar{\mathbf{x}}(t) \to \eta\mathbf{v}_1$. □

Remark 1. Note that for the jth AUV, the overall controller, which consists of the cross-track controller and the coordination controller (3.51), requires only the communication of the x positions from those AUVs that correspond to non-zero entries in the jth row of the interconnection matrix $\boldsymbol{\Gamma}$. At the same time, the condition on $\boldsymbol{\Gamma}$ imposed in Theorem 3.5 allows for many zero entries in $\boldsymbol{\Gamma}$, meaning that all-to-all communication is not required. These properties of the proposed controllers are very important in low bandwidth and unreliable underwater communication.

Remark 2. The proposed controllers (3.17), (3.22) and (3.40) are based on feedback linearization. In practice, however, exact cancelation is not possible due to inevitable model uncertainties. This can lead to steady-state errors. Partly this problem can be solved by omitting the cancelation of the dissipative damping terms.

3.6 Simulations

The proposed formation control scheme has been implemented in Simulink™ and simulated using a 6-DOF model of the HUGIN AUV from FFI and Kongsberg Maritime. The control scheme is simulated for the case of three AUVs.

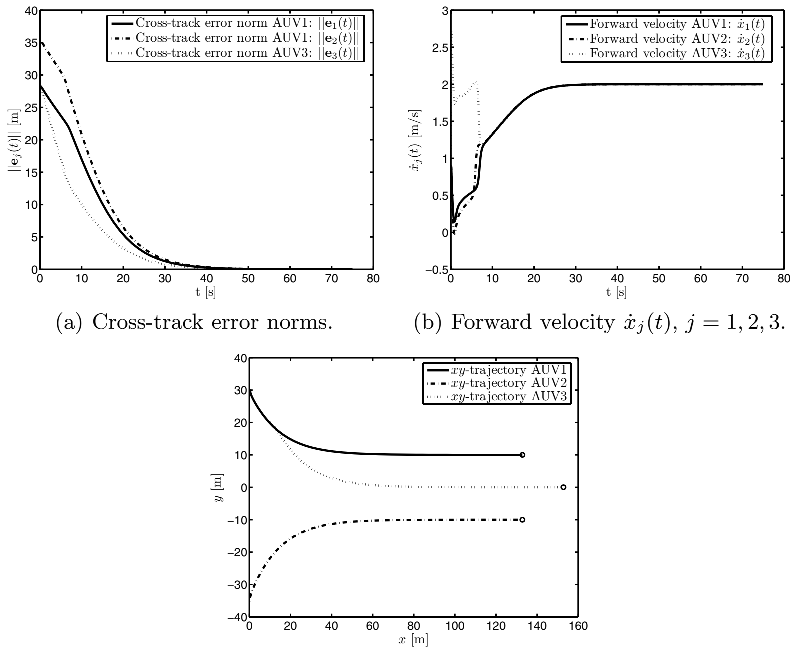
The desired formation is chosen to be given by $(D_{x1}, D_{y1}, D_{z1}) = (0, 10, 0)$, $(D_{x2}, D_{y2}, D_{z2}) = (20, -10, 0)$, $(D_{x3}, D_{y3}, D_{z3}) = (20, 0, 0)$, see Section 3.2.2. The desired straight-line path coincides with the x-axis. The desired formation speed is chosen as $u_d = 2.0$ m/s. The initial cross-track errors are chosen as $(e_{y1}(0), e_{z1}(0)) = (20, 20)$, $(e_{y2}(0), e_{z2}(0)) = (-25, -25)$ and $(e_{y3}(0), e_{z3}(0)) = (20, 20)$. The initial surge speed is chosen as $u_j(0) = 0.5$ m/s, $j = 1, 2, 3$, while the initial sway and heave velocities are chosen as $v_j(0) = w_j(0) = 0$, $j = 1, 2, 3$. All vehicles are given zero initial pitch and yaw angle, i.e., $\theta_j(0) = \psi_j(0) = 0$, $j = 1, 2, 3$.

The controller gain k_u in (3.17) is chosen as $k_u = 10$, k_θ and k_q in (3.22) are chosen as $k_\theta = 3$ and $k_q = 50$ and the controller gains k_ψ and k_r in (3.40) are chosen as $k_\psi = 3.5$ and $k_r = 50$. The linkage parameters γ_{ij} are set to $\gamma_{12} = \gamma_{13} = \gamma_{23} = 4$ and the function $g(x)$ is chosen as $g(x) = \frac{2}{\pi}\tan^{-1}(x) \in [-1, 1]$.

The simulation results are shown in Fig. 3.4(a)-3.4(c). Figure 3.4(a) shows the cross-track error norm of each vehicle, Fig. 3.4(b) shows the inertial velocity in the x-direction, i.e., $\dot{x}_j$, of each vehicle and Fig. 3.4(c) shows the xy-trajectory of each vehicle. The presented simulation results clearly show that the control goals (3.7)-(3.9) are achieved. The AUVs asymptotically constitute the desired formation that moves along the desired path with the prescribed velocity profile.

3.7 Conclusions

In this paper we have considered the problem of 3D cross-track formation control for underactuated AUVs. The proposed decentralized control laws guarantee convergence of the underwater vehicles to a desired formation moving with a desired



(a) Cross-track error norms.

(b) Forward velocity $\dot{x}_j(t)$, $j = 1, 2, 3$.

(c) xy-trajectories.

Fig. 3.4. Simulation results

formation speed along a desired straight-line path. This control problem has been solved in two steps. The first step is the design of a cross-track controller. For each AUV such a controller, which is based on the Line of Sight guidance law, guarantees convergence to the desired path for this AUV in the formation. It has been proved that for any initial condition of an AUV, the convergence to the desired path is exponential (yet depending on the initial conditions). Moreover, this controller guarantees that the forward velocity of the AUV tracks some speed reference command, which is designed at the second step. Control laws for the speed reference command are designed for each AUV. These controllers asymptotically align the AUVs in the direction of the desired path in such a way that they constitute the desired formation and move synchronously with the desired speed profile.

The performance of the proposed formation control scheme has been investigated for the case of three AUVs through numerical simulations in Simulink™with a model of the HUGIN AUV. The simulations have demonstrated the validity of the obtained theoretical results.

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