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Design of Convergent Switched Systems

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Summary. In this paper we deal with the problem of rendering hybrid/nonlinear systems into convergent closed-loop systems by means of a feedback law or switching rules. We illustrate our approach to this problem by means of two examples: the anti-windup design for a marginally stable system with input saturation, and the design of a switching rule for a piece-wise affine system operating in different modes.

17.1 Introduction

It is well known that any solution of a stable linear time-invariant (LTI) system with a bounded input converges to a unique limit solution that depends only on the input. Nonlinear systems with such a property are referred to as convergent systems. Solutions of the convergent systems "forget" their initial conditions and after some transient time depend on the system input that can be a command or reference signal. One of the main objectives of feedback in controller design is to eliminate the dependency of the system steady-state solutions on initial conditions. This property should be preserved for an admissible class of the inputs that makes the problem of the design of convergent systems an important control problem.

The property of convergency can play an important role in the studies related to the group coordination and cooperative control. Particularly, if each agent from the whole network is described by a convergent model, and if the input signal is identical for all agents, after some transient time all the agents will follow the same trajectory. In other words, the synchronization between the agents from a network will be achieved if each agent is controlled by a local (dependent only on that agent state) feedback aimed at making the agent convergent. An advantage of this synchronization scheme is that it can be achieved via decentralized (local) controllers, i.e. no exchange of information between the agents is required. The main disadvantage, however, stems from the same origin: if the agents operate in different environment, they are perturbed by different disturbances and thus eventually will follow different paths, that is the group of the agents will stay in an asynchronous mode. To overcome this difficulty a communication between the agents can be introduced that

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can be considered as a sort of feedback, usually in a form of the mismatch between the agents states or outputs. In [34, 35] it was demonstrated that for passivity-based design of synchronizing networks via output feedback the convergency of an agent subsystem consistent with some algebraic constraint plays almost the same role as minimumphaseness in the conventional output feedback stabilization problem. This motivates studies related to design of convergent systems via different sorts of feedback.

The property that all solutions of a system "forget" their initial conditions and converge to some steady-state solution has been addressed in a number of publications [8, 39, 32, 20, 41, 22, 10, 11, 13, 2]. In this paper we continue the study originated in [33, 31] on convergency of piece-wise affine (PWA) systems. This class of systems attracted a lot of attention over the last years, see e.g. [3, 17, 18, 38] and references therein. In this paper two extensions are given to the convergency theory as presented in [33, 31]. Each of the extensions is discussed in the context of a suitable application area. First, we present a convergence based approach to the anti-windup controller design for marginally stable systems with input saturation. For such systems one cannot directly use the results on quadratic convergency of PWA continuous systems presented in [33], since there exists no common quadratic Lyapunov function for this kind of system. To tackle the problem we developed a method that allows to establish uniform, but not necessary quadratic convergence. Secondly, we address the problem of switching control for PWA systems operating in different modes, that is to find a switching rule as a function of the state and the input such that the closed loop system demonstrates convergent behavior. This approach is different from the theory discussed in [31], in which the switching rule is assumed to be known.

The paper is organized as follows. In Section 17.2 we recall some basic results on conditions for stability of solutions of nonlinear systems. Section 17.3 gives various definitions of convergent systems. In Section 17.4 we apply the convergency based approach to the anti-windup controller design for a marginally stable system with input saturation. Section 17.5 deals with the design of a switching rule that makes the closed loop system convergent.

17.2 Stability via First Approximation

The study of convergent systems focusses on the stability of solutions of nonlinear systems. Two Lyapunov methods are available for analysis of the stability of solutions, i.e. Lyapunov's indirect and direct method. In this section we present a brief overview of the problem of stability analysis of solutions of nonlinear time-varying systems via its first order approximation (i.e., the indirect Lyapunov method) and at the end we will conclude that the direct Lyapunov method is more promising for analytical purposes.

Consider a classical question of stability analysis of a particular (or all) solution(s) of the following nonlinear time-varying system

$$\dot{x} = F(x,t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{17.1}$$

where F satisfies some regularity assumptions to guarantee the existence of local solutions $x(t, t_0, x_0)$. For the sake of brevity if there are no confusions and if the meaning is apparent we will omit the dependence of the solutions on some parameters, i.e. initial time and data.

Definition 17.1. *A solution $x(t, t_0, \bar{x}_0)$ of system* (17.1)*, defined for all $t \in (t_*, +\infty)$, is said to be*

- stable *if for any $t_0 \in (t_*, +\infty)$ and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $||x_0 - \bar{x}_0|| < \delta$ implies $||x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)|| < \varepsilon$ for all $t \geq t_0$;*
- uniformly stable *if it is stable and the number δ in the definition of stability can be chosen independently of t_0;*
- asymptotically stable *if it is stable and for any $t_0 > t_*$ there exists $\delta = \delta(t_0) > 0$ such that $||x_0 - \bar{x}_0|| < \delta$ for $t_0 > t_*$ implies* $\lim_{t\to\infty} ||x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)|| = 0$*;*
- uniformly asymptotically stable *if it is uniformly stable and there exists $\delta > 0$ (independent of t_0) such that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that $||x_0 - \bar{x}_0|| < \delta$ implies $||x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)|| < \varepsilon$ for all $t \geq t_0 + T$;*
- exponentially stable *if there exist positive δ, C, β such that $||x_0 - \bar{x}_0|| < \delta$ implies*

$$||x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)|| \leq Ce^{-\beta(t-t_0)} ||x_0 - \bar{x}_0||;$$

- uniformly globally asymptotically stable *if it is uniformly asymptotically stable and attracts all solutions starting in $(x_0, t_0) \in \mathbb{R}^n \times (t_*, +\infty)$ uniformly over t_0, i.e. for any $R > 0$ and any $\varepsilon > 0$ there is a $T = T(\varepsilon, R) > 0$ such that if $||x_0|| < R$, then $||x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)|| < \varepsilon$ for all $t \geq t_0 + T$, $t_0 > t_*$.*

Suppose F is continuously differentiable in x and continuous in t. Let

$$A(x, t) = \frac{\partial F(x, t)}{\partial x}$$

be the Jacobi matrix for the function $F(x, t)$ at the point $x \in \mathbb{R}^n$. Let us consider the solutions $x(t, t_0, x_0)$ of the system (17.1) with initial conditions $x(t_0, t_0, x_0) = x_0$ under the assumption that they are well defined for all $t \geq t_*$. Together with system (17.1) consider its first order approximation governed by the following equation

$$\dot{\xi} = A(x(t, t_0, x_0), t)\xi, \quad \xi \in \mathbb{R}^n, \quad t \geq t_*. \tag{17.2}$$

Let $\Phi(t, x_0)$ be a fundamental matrix for system (17.2) with $\Phi(t_0, x_0) = I_n$. The following lemma is crucial for stability analysis of solutions of the nonlinear system (17.1) via first order approximation (17.2).

Lemma 17.1 ([21]). *For any two solutions $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ of system* (17.1) *the following estimate*

$$||x(t, t_0, x_0) - x(t, t_0, y_0)|| \leq \sup_{\eta \in B} ||\Phi(t, \eta)|| \; ||x_0 - y_0|| \tag{17.3}$$

is true, where $B = \{\eta \in \mathbb{R}^n \mid ||x_0 - \eta|| \leq ||x_0 - y_0||\}$.

Proof. The system equations can be rewritten in the following form

$$\frac{dx(t,t_0,x_0)}{dt} = F(x(t,t_0,x_0),t).$$

The earlier mentioned assumptions imposed on F imply the solution $x(t,t_0,x_0)$ is differentiable in x_0. From the last equation one gets

$$\frac{d}{dt}\frac{\partial x(t,t_0,x_0)}{\partial x_0} = A(x(t,t_0,x_0),t)\frac{\partial x(t,t_0,x_0)}{\partial x_0}$$

and thus

$$\frac{\partial x(t,t_0,x_0)}{\partial x_0} = \Phi(t,x_0).$$

The multidimensional variant of Lagrange's mean value theorem yields

$$||x(t,t_0,x_0) - x(t,t_0,y_0)|| \leq \sup_{0<\theta<1}\left|\left|\frac{\partial x}{\partial x_0}(t,t_0,x_0+\theta(y_0-x_0))\right|\right|\,||x_0-y_0||.$$

■

The previous lemma is a simple tool that allows to analyze stability of a particular solution $x(t,t_0,\bar{x}_0)$ of nonlinear system (17.1) via its first order approximation (17.2). Indeed, Lyapunov stability of this solution is granted if for all x_0 from a neighborhood of $\bar{x}_0$ the corresponding fundamental matrix $\Phi(t,x_0)$ is bounded as a function of time. If additionally $\lim_{t\to\infty}||\Phi(t,x_0)|| = 0$ then the solution $x(t,t_0,\bar{x}_0)$ is asymptotically stable.

Although this approach can lead to verifiable stability conditions (see [21]) the first attempts to tackle the problem of stability via first order approximation were based on analysis of system (17.2) for *one* given solution $x(t,t_0,\bar{x}_0)$. The latter approach however is only applicable if for system (17.1) some shift transformation $z = x - \bar{x}$ allows to reduce the problem of stability of the solution $x(t,t_0,\bar{x}_0)$ to the problem of stability of the origin of the following nonlinear system

$$\dot{z} = A(t)z + f(t,z), \tag{17.4}$$

where together with mild assumptions on $A(t)$, the nonlinear term $f(t,z)$ must satisfy the following assumption

$$||f(t,z)|| \leq \psi(t)||z||^m, \quad m > 1$$

for a continuous function ψ with zero Lyapunov exponent (in the original statement due to Lyapunov $\psi(t) = \text{const}$). Lyapunov proved that if the first approximation system of (17.4)

$$\dot{\xi} = A(t)\xi \tag{17.5}$$

is *regular* in the sense of Lyapunov (see e.g. [1]) and has negative Lyapunov exponents then the origin of (17.4) is asymptotically stable.

The results of this kind stimulated development of numerical methods to analyze stability via the first Lyapunov method, i.e. the method of Lyapunov exponents.

Researches based on the Lyapunov exponents are particularly popular in physical studies of chaotic systems and numerical calculation of those exponents is now a standard tool, implemented in numerous software packages.

The most difficult assumption to verify in the Lyapunov theorem is the regularity of system (17.5), that implies that all Lyapunov exponents of (17.5) should be strict. Later on this condition was relaxed by Malkin, Chetaev and Massera (see e.g. [24, 6, 25, 26]) who showed that the conclusion of the Lyapunov theorem remains true if the largest Lyapunov exponent of (17.5) is sufficiently negative and strictly bounded from above by

$$-\frac{\alpha}{m-1},$$

where $\alpha \geq 0$ is the so-called irregularity coefficient, that is, in turn, quite difficult to calculate for practical examples.

Another difficulty that should be taken into account for the problem of asymptotic stability via first order approximation is that Lyapunov exponents of system (17.5) are not stable in general (see e.g. [1, 7]). In other words, even an infinitesimal perturbation of (17.5) can change the sign of its Lyapunov exponents. Particularly, this means that the asymptotic stability of the solution $x(t, t_0, \bar{x}_0)$ is not a robust property. Though necessary and sufficient conditions for stability of the Lyapunov exponents are known (see e.g. [5, 27, 1]), they are quite difficult to verify in practice.

Another approach to tackle stability via first order approximation is by means of the general exponent, as originated by Bohl [4]. Particularly, negativity of the general exponent of the norm of the fundamental matrix $||\Phi(t)||$ for (17.5) gives necessary and sufficient conditions for exponential stability of (17.5) [7]. At the same time exponential stability of the first order approximation (17.5) together with some mild assumptions on the nonlinear term $f(t, z)$ gives sufficient conditions for uniform asymptotic stability of the origin of nonlinear system (17.4), see e.g. [37]. Moreover, in contrast to Lyapunov exponents, the general exponents are stable, see [7]. However, as follows from the definition of the general exponents, its numerical computation is a challenging problem.

The brief survey presented above illustrates the main difficulties that arise in the problem of stability via first order approximation if one tries to tackle this problem via method of characteristic exponents (either Lyapunov or Bohl). It seems that from an analytical point of view the most promising method to investigate stability is the direct Lyapunov method. This method also allows to estimate Lyapunov (and Bohl) exponents and gives verifiable conditions for (uniform asymptotic) stability of a solution $x(t, t_0, x_0)$ of nonlinear system (17.1). In the subsequent sections we present recent developments in this field using the concept of convergent systems that is closely related to the concept of stability of all solutions of nonlinear system (17.1).

17.3 Convergent Systems

In this section we give definitions of convergent systems. Those systems are closely related to systems with uniformly globally asymptotically stable solutions and the definitions presented here extend those given by Demidovich [8].

Definition 17.2. *System* (17.1) *is said to be*

- uniformly convergent *if there is a solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ satisfying the following conditions: (i) $\bar{x}(t)$ is defined and bounded for all $t \in (-\infty, +\infty)$, (ii) $\bar{x}(t)$ is uniformly globally asymptotically stable;*
- exponentially convergent *if it is uniformly convergent and $\bar{x}(t)$ is globally exponentially stable.*

The solution $\bar{x}(t)$ is called a *limit solution.* As follows from the definition of convergence, any solution of a convergent system "forgets" its initial condition and converges to some limit solution which is independent of the initial conditions. In general, if there is a globally asymptotically stable limit solution $\bar{x}(t)$ it may be non-unique, in the sense that there can exist another solution $\tilde{x}(t)$ bounded for all $t \in (-\infty, +\infty)$ that is also globally asymptotically stable. For any two such solutions it obviously follows that $||\bar{x}(t) - \tilde{x}(t)|| \to 0$ as $t \to \infty$. At the same time for *uniformly* convergent systems the limit solution is unique, as formulated below.

Property 17.1 ([30, 29]). If system (17.1) is uniformly convergent, then the limit solution $\bar{x}(t)$ is the only solution defined and bounded for all $t \in (-\infty, +\infty)$.

In systems theory time dependency of the right-hand side of system (17.1) is usually due to some input. This input may represent, for example, a disturbance or a feedforward (reference) control signal. Below we will consider convergence properties for systems with inputs. So, instead of systems of the form (17.1), we consider systems

$$\dot{x} = f(x, w), \tag{17.6}$$

with state $x \in \mathbb{R}^n$ and input $w \in \mathbb{R}^m$. In the sequel we will consider the class $\overline{\mathbb{PC}}_m$ of piecewise continuous inputs $w(t) : \mathbb{R} \to \mathbb{R}^m$ which are bounded for all $t \in \mathbb{R}$. We assume that the function $f(x, w)$ is bounded on any compact set of (x, w) and the set of discontinuity points of the function $f(x, w)$ has measure zero. Under these assumptions on $f(x, w)$, for any input $w(\cdot) \in \overline{\mathbb{PC}}_m$ the differential equation $\dot{x} = f(x, w(t))$ has well-defined solutions in the sense of Filippov. Below we define the convergence property for systems with inputs.

Definition 17.3. *System (17.6) is said to be* (uniformly, exponentially) convergent *if it is (uniformly, exponentially) convergent for every input $w(\cdot) \in \overline{\mathbb{PC}}_m$.*

In this paper we are going to consider the systems of the form (17.1) and (17.6) with non-smooth right hand sides under quite general regularity assumptions that guarantee the existence of solutions in some reasonable sense, e.g. in a sense of Filippov, see e.g. [9, 40]. According to Filippov one can construct a set-valued function $\mathbf{F}(x, t)$ such that an absolutely continuous solution of the differential inclusion $\dot{x} \in \mathbf{F}(x, t)$ is called a solution for system (17.1).

For system (17.1) consider a scalar continuously differentiable function $V(x)$. Define a time derivative of this function along solutions of system (17.1) as follows

$$\dot{V} := \frac{\partial V(x)}{\partial x} \dot{x}(t, t_0, x_0).$$

Since V is continuously differentiable and the solution $x(t, t_0, x_0)$ is an absolutely continuous function of time, the derivative $\dot{V}(x(t, t_0, x_0))$ exists almost everywhere in the maximal interval of existence $[t_0, \bar{T})$ of the solution $x(t, t_0, x_0)$. For the function V we can also define its upper derivative as follows

$$\dot{V}^*(x, t) = \sup_{\xi \in \mathbf{F}(x,t)} \left(\frac{\partial V(x)}{\partial x} \xi \right).$$

Then for almost all $t \in [t_0, \bar{T})$ it follows that

$$\dot{V}(x(t, t_0, x_0)) \leq \dot{V}^*(x(t, t_0, x_0), t).$$

Remark 17.1. Notice that in the domains of continuity of the function $F(x, t)$ the derivative of $V(x)$ along solutions of system (17.1) equals $\dot{V} = \frac{\partial V(x)}{\partial x} F(x, t)$. According to [9] p.155, for a *continuously differentiable* function $V(x)$ it holds that if the inequality

$$\frac{\partial V(x)}{\partial x} F(x, t) \leq 0$$

is satisfied in the domains of continuity of the function $F(x, t)$, then the inequality $\dot{V}^*(x, t) \leq 0$ holds for all $(x, t) \in \mathbb{R}^{n+1}$.

Definition 17.4. *System (17.6) is called quadratically convergent if there exists a matrix $P = P^T > 0$ and a number $\alpha > 0$ such that for any input $w \in \overline{\mathbb{PC}}_m$ for the function $V(x_1, x_2) = (x_1 - x_2)^T P(x_1 - x_2)$ it holds that*

$$\dot{V}^*(x_1, x_2, t) \leq -\alpha V(x_1, x_2), \tag{17.7}$$

where $\dot{V}^*(x_1, x_2, t)$ is the upper derivative of the function $V(x_1, x_2)$ along any two solutions of the corresponding differential inclusion, i.e.

$$\dot{V}^*(x_1, x_2, t) = \sup_{\xi_1 \in \mathbf{F}(x_1, w(t))} \left(\frac{\partial V}{\partial x_1}(x_1, x_2)\xi_1 \right) + \sup_{\xi_2 \in \mathbf{F}(x_2, w(t))} \left(\frac{\partial V}{\partial x_2}(x_1, x_2)\xi_2 \right).$$

Quadratic convergence is a useful tool for establishing the exponential convergence, as follows from the following lemma.

Lemma 17.2 ([30]). *If system (17.6) is quadratically convergent, then it is exponentially convergent.*

The proof of this lemma is based on the following result, which will be also used in the remainder of this paper.

Lemma 17.3 ([39]). *Consider system (17.6) with a given input $w(t)$ defined for all $t \in \mathbb{R}$. Let $\mathcal{D} \subset \mathbb{R}^n$ be a compact set which is positively invariant with respect to dynamics* (17.6). *Then there is at least one solution $\bar{x}(t)$, such that $\bar{x}(t) \in \mathcal{D}$ for all $t \in (-\infty, +\infty)$.*

Note that for convergent nonlinear systems performance can be evaluated in almost the same way as for linear systems. Due to the fact that the limit solution of a convergent system only depends on the input and is independent of the initial conditions, performance evaluation of one solution (i.e. one arbitrary initial state) for a certain input suffices, whereas for general nonlinear systems all initial states need to be evaluated to obtain a reliable analysis. This means that for convergent systems simulation becomes a reliable analysis tool and for example 'Bode-like' plots can be drawn to analyse the system performance. An example of simulation based performance analysis can be found in Sect. 17.5.1.

17.4 Application of Convergent Systems Analysis to the Anti-windup Problem

The presence of actuator saturation in an otherwise linear closed-loop system can dramatically degrade the performance of that system. This performance degradation is caused by the so-called 'controller windup'.

In the past, several linear and nonlinear anti-windup techniques have been developed to compensate for this windup effect (see e.g. [15, 14, 36, 19, 12]). However, not all approaches, e.g. based on finite (incremental) $\mathcal{L}_2$ gain, are able to guarantee global anti-windup for a marginally stable plant. Here we propose another approach, based on uniform convergency, which is close to that introduced by [12]. The main difference between this and most other approaches is that whereas the other approaches focus on guaranteeing $\mathcal{L}_2$ stability from input to output, this approach focusses on ensuring convergency, i.e. a unique limit solution for each input signal, independent of initial conditions, which is constant (resp. periodic) if the input signal is constant (resp. periodic). In this section we show that under some mild conditions, it is possible to guarantee uniform convergency for an anti-windup scheme with a plant that is an integrator (i.e. marginally stable).

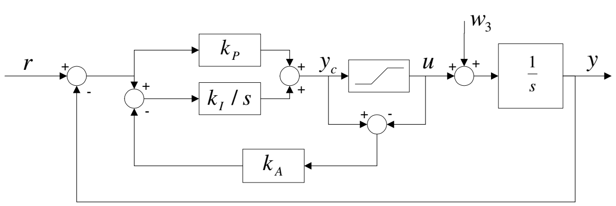


Fig. 17.1. Anti-windup scheme with integrator plant

Consider the system in Fig. 17.1, consisting of an integrator plant with input saturation, a PI-controller and a static anti-windup block. Assuming input r is differentiable, this system can be described as follows

$$\begin{cases} \dot{x} = Ax + Bu + Ew(t) \\ z = Cx \\ u = \mathrm{sat}(y_c) \end{cases} \tag{17.8}$$

where $x = [y\ y_c]^T$, $w(t) = [r(t)\ \dot{r}(t)\ w_3(t)]^T$ and

$$A = \begin{bmatrix} 0 & 0 \\ -k_I & -k_I k_A \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -k_P + k_I k_A \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 1 \\ k_I & k_P & -k_P \end{bmatrix}, \quad C = [0\ \ 1],$$

with $k_I, k_P, k_A > 0$ and $\mathrm{sat}(y_c) = \mathrm{sign}(y_c)\max(1, |y_c|)$.

Definition 17.5. *A continuous function $t \mapsto w(t)$, $w(t) = [w_1(t)\ \ w_2(t)\ \ w_3(t)]^T$ is said to belong to the class $\mathcal{W}$ if there exist nonnegative constants C_1, C_2, C_3, C_4, C_5, with $C_3 < 1$ such that*

1. $\forall t \in \mathbb{R}^1\ |w_1(t)| \le C_1$;
2. $\forall t \in \mathbb{R}^1\ |w_2(t)| \le C_2$;
3. $w_3(t) = w_{3c}(t) + w_{3i}(t)$ *with*
  3a. $\forall t \in \mathbb{R}^1\ |w_{3c}(t)| \le C_3$;
  3b. $\forall t, t_0 \in \mathbb{R}^1\ \left|\int_{t_0}^t w_{3i}(\tau)d\tau\right| \le C_4$;
  3c. $\forall t \in \mathbb{R}^1\ |w_3(t)| \le C_5$.

Theorem 17.1. *If $k_A k_P > 1$ then system* (17.8) *is uniformly convergent for all $w(\cdot) \in \mathcal{W}$.*

First we prove the following lemmata.

Lemma 17.4. *If $k_A k_P > 1$ then system* (17.8) *is uniformly ultimately bounded for all $w \in \mathcal{W}$, that is, given input $w(\cdot)$ from $\mathcal{W}$, there is a number $R > 0$ such that for any solution $x(t, t_0, x_0)$ starting from a compact set Ω there is a number $T(\Omega)$ such that for all $t \ge t_0 + T(\Omega)$ it follows that $||x(t, t_0, x_0)|| < R$.*

Proof. Let $y_i(t) = y(t) - \int_{t_0}^t w_{3i}(\tau)d\tau$. Then the system equations can be rewritten as

$$\begin{cases} \dot{y}_i &= \dot{y} - w_{3i}(t) = \mathrm{sat}(y_c) + w_{3c}(t) \\ \dot{y}_c &= -k_I y_i - k_I k_A y_c + (k_I k_A - k_P)\mathrm{sat}(y_c) + \zeta(t) \end{cases} \tag{17.9}$$

with

$$\zeta(t) = k_I r(t) + k_P \dot{r}(t) - k_P w_3(t) - k_I \int_{t_0}^t w_{3i}(\tau)d\tau.$$

Consider the following Lyapunov function candidate

$$W(y_i, y_c) = \begin{bmatrix} y_i \\ y_c \end{bmatrix}^T P \begin{bmatrix} y_i \\ y_c \end{bmatrix}, \quad \text{with} \quad P = \begin{bmatrix} c & 1 \\ 1 & k_A \end{bmatrix}. \tag{17.10}$$

This function is positive definite and radially unbounded for $k_A c > 1$.

In the linear mode ($\mathrm{sat}(y_c) = y_c$) the time derivative of (17.10) satisfies

$$\dot{W} = \begin{bmatrix} y_i \\ y_c \end{bmatrix}^T Q \begin{bmatrix} y_i \\ y_c \end{bmatrix} + 2 \begin{bmatrix} y_i \\ y_c \end{bmatrix}^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \zeta(t) + 2 \begin{bmatrix} y_i \\ y_c \end{bmatrix}^T P \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_{3c}(t) \qquad (17.11)$$

and the matrix Q is negative definite provided $k_A k_P > 1$ and $c = k_P + k_I k_A$ (or lies in some interval around this value).

In the saturated mode ($\mathrm{sat}(y_c) = \mathrm{sign}(y_c)$) the time derivative of W equals

$$\begin{aligned} \frac{\dot{W}}{2} =& - k_I y_i^2 - 2k_I k_A y_i y_c - k_I k_A^2 y_c^2 + (c - k_P + k_I k_A) y_i \mathrm{sign}(y_c) \\ &+ (1 - k_A k_P + k_A^2 k_I)|y_c| + (y_i + k_A y_c)\zeta(t) + (c y_i + y_c) w_{3c}(t). \end{aligned}$$

Denote $\mu = y_i + k_A y_c$. Then

$$\begin{aligned} \frac{\dot{W}}{2} =& - k_I \mu^2 + (c - k_P + k_I k_A)\mu \mathrm{sign}(y_c) - (c - k_P + k_I k_A) k_A |y_c| \\ &+ (1 - k_A k_P + k_A^2 k_I)|y_c| + \mu\zeta(t) + (c\mu + (1 - k_A c) y_c)\, w_{3c}(t), \end{aligned}$$

so that

$$\frac{\dot{W}}{2} \leq -k_I \mu^2 + |\mu| C_6 + (1 - k_A c)(1 - C_3)|y_c|, \qquad (17.12)$$

with

$$C_6 = |c - k_P + k_I k_A| + k_I C_1 + k_P (C_2 + C_5) + k_I C_4 + c C_3.$$

Combining (17.11) and (17.12) one can apply the Yoshizawa theorem on ultimate boundedness (see e.g. [16]) since $k_A c > 1$, $C_3 < 1$ and $k_A k_P > 1$. This completes the proof. ■

Consider a system consisting of two copies of (17.8) with identical inputs:

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \mathrm{sat}(y_{c1}) \\ \mathrm{sat}(y_{c2}) \end{bmatrix} + \begin{bmatrix} E \\ E \end{bmatrix} w(t), \qquad (17.13)$$

with $x = [y_1\ y_{c1}\ y_2\ y_{c2}]^T$. Define the function $\xi(t)$ as follows

$$\xi(t) = \begin{cases} \frac{\mathrm{sat} y_{c1}(t) - \mathrm{sat} y_{c2}(t)}{y_{c1}(t) - y_{c2}(t)} & \text{if } \ y_{c1}(t) \neq y_{c2}(t); \\ 1 & \text{if } \ y_{c1}(t) = y_{c2}(t) \ \ \& \ \ |y_{c1}(t)| < 1 \ \ \& \ \ |y_{c2}(t)| < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Since the function $\mathrm{sat}(\cdot)$ satisfies the incremental sector condition it follows that $\forall t$ $0 \leq \xi(t) \leq 1$. We need the following result.

Lemma 17.5. *Given $w(\cdot)$ from $\mathcal{W}$, for any solution $x(t, t_0, x_0)$ of (17.13) starting from some compact set Ω there exist $\delta = \delta(\Omega) > 0$, $\bar{T} = \bar{T}(\Omega) > 0$, such that for all $t \geq t_0$ it follows that*

$$\int_t^{t+\bar{T}} \xi(s)ds \geq \delta.$$

Proof. First consider $y_{c1_{max}} := \limsup_{t\to\infty} |y_{c1}(t)| < 1$. From the previous lemma this means that after some finite $T(\Omega)$ both subsystems of (17.13) approach the linear mode and will stay in this mode for all $t \geq T(\Omega)$. If both $|y_{c2}| < 1$ and $|y_{c1}| < 1$ then $\xi = 1$ and the result follows.

Secondly, consider the opposite: $y_{c1_{max}} \geq 1$. From the system equations it follows that for any $t \geq t_0$, $T > 0$

$$y_1(t+T) - y_1(t) = \int_t^{t+T} \mathrm{sat} y_{c1}(s)ds + \int_t^{t+T} w_3(s)ds$$

and therefore

$$\frac{1}{T}\int_t^{t+T} \mathrm{sat} y_{c1}(s)ds = \frac{y_1(t+T) - y_1(t)}{T} - \frac{1}{T}\int_t^{t+T} w_3(s)ds. \tag{17.14}$$

From Lemma 17.4 and the assumption imposed on the signal $w_3(t)$ it follows that by making T sufficiently large one can make the first term of the right hand side arbitrarily small and the second term strictly smaller than 1 by absolute value. In other words, for sufficiently large $\bar{T} = \bar{T}(\Omega)$ there is an α that can be chosen independently of t_0, $0 < \alpha < 1$ such that

$$\left|\frac{1}{\bar{T}}\int_t^{t+\bar{T}} \mathrm{sat} y_{c1}(s)ds\right| \leq \alpha.$$

Due to the mean value theorem there is a $\eta \in (t, t+\bar{T}]$ such that $|\mathrm{sat} y_{c1}(\eta)| \leq \alpha$. From Lemma 17.4 we know that the time derivative of $y_{c1}(t)$ is bounded and therefore the function $\mathrm{sat} y_{c1}(t)$ is uniformly continuous on $[t_0, \infty)$. Now choose some $\varepsilon > 0$ such that $\alpha + \varepsilon < 1$. This is always possible since $\alpha < 1$. Since $\mathrm{sat} y_{1c}(t)$ is uniformly continuous there is a number $\Delta t > 0$ such that

$$|\mathrm{sat} y_{1c}(\tau)| \leq \alpha + \varepsilon < 1, \quad \forall \tau \in [\eta - \Delta t,\ \eta + \Delta t].$$

This number Δt can be chosen independently of t_0 since the right hand side of (17.13) and hence $\dot{y}_{1c}(\tau)$ is uniformly bounded. Among all possible Δt choose the largest possible satisfying $\Delta t \leq \bar{T}$. Now, integrating nonnegative ξ from t till $t + \bar{T}$ yields

$$\int_t^{t+\bar{T}} \xi(s)ds \geq \int_{\max\{t, \eta-\Delta t\}}^{\min\{t+\bar{T}, \eta+\Delta t\}} \xi(s)ds.$$

Since $t \leq \eta \leq t + \bar{T}$ it follows that

$$\min\{t+\bar{T}, \eta+\Delta t\} - \max\{t, \eta - \Delta t\} \leq \min\{\bar{T}, \Delta t\} = \Delta t$$

and

$$\int_{\max\{t,\eta-\Delta t\}}^{\min\{t+\bar{T},\eta+\Delta t\}} \xi(s)ds \geq \Delta t \xi_{min},$$

where ξ_{min} is the lowest bound for ξ under restriction that $|\text{sat}y_{1c}| \leq \alpha+\varepsilon < 1$. ξ approaches this bound if $|y_{1c}| = \alpha+\varepsilon$ and $|y_{2c}| = y_{c1_{max}} \geq 1$. Therefore,

$$\xi_{min} = \frac{1-\alpha-\varepsilon}{y_{c1_{max}}-\alpha-\varepsilon} > 0$$

and hence

$$\int_t^{t+\bar{T}} \xi(s)ds \geq \Delta t \frac{1-\alpha-\varepsilon}{y_{c1_{max}}-\alpha-\varepsilon} > 0.$$

The last inequality implies the statement of Lemma (17.5) with $\delta = \Delta t \frac{1-\alpha-\varepsilon}{y_{c1_{max}}-\alpha-\varepsilon}$.
■

Proof of Theorem 17.1: For system (17.8) consider the Lyapunov function:

$$V(x) = x^T \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} x \geq 0, \tag{17.15}$$

with $x = [y_1 \; y_{c1} \; y_2 \; y_{c2}]^T$, P as defined in (17.10), $c = k_P + k_I k_A$ and therefore $k_A c > 1$. Denote $e_1 = y_1 - y_2$, $e_2 = y_{c1} - y_{c2}$ and $\varphi = \text{sat}(y_{c1}) - \text{sat}(y_{c2})$. Then the derivative of V satisfies

$$\begin{aligned}
\frac{\dot{V}}{2} &= -k_I e_1^2 - 2k_I k_A e_1 e_2 - k_I k_A^2 e_2^2 + 2k_I k_A e_1 \varphi + (1 + k_I k_A^2 - k_P k_A) e_2 \varphi \\
&= -k_I (e_1 + k_A e_2)^2 + 2k_I k_A \varphi (e_1 + k_A e_2) - k_I k_A^2 \varphi^2 \\
&\quad - k_I k_A^2 e_2 \varphi - (k_P k_A - 1) e_2 \varphi + k_I k_A^2 \varphi^2 \\
&= -k_I \left(e_1 + k_A (e_2 - \varphi)\right)^2 - (k_P k_A - 1) e_2 \varphi - k_I k_A^2 (e_2 \varphi - \varphi^2).
\end{aligned}$$

Since $\text{sat}(\cdot)$ satisfies the $[0,1]$-incremental sector condition

$$e_2 \varphi - \varphi^2 \geq 0,$$

it follows that

$$\dot{V} \leq 0$$

and uniform stability of all solutions $y_1(t), y_{1c}(t)$ of (17.8) is proven. The last inequality is not sufficient to prove quadratic stability of all solutions. However, the exponential convergence from a given compact set can be deduced from the previous lemma. Indeed,

$$\begin{aligned}
\frac{\dot{V}}{2} &= -k_I e_1^2 - 2k_I k_A e_1 e_2 - k_I k_A^2 e_2^2 + 2k_I k_A e_1 \varphi + (1 + k_I k_A^2 - k_P k_A) e_2 \varphi \\
&= k_I (1 - \xi(t)) \left(-e_1^2 - 2k_A e_1 e_2 - k_A^2 e_2^2\right) - k_I \xi(t) e_1^2 - (k_P k_A - 1) \xi(t) e_2^2 \\
&\leq -k_I \xi(t) e_1^2 - (k_P k_A - 1) \xi(t) e_2^2.
\end{aligned}$$

It follows then that V satisfies the following inequality

$$\dot{V} \leq \lambda_{\max} \xi(t) V,$$

in which $\lambda_{\max} < 0$ is the largest solution of the following generalized eigenvalue problem

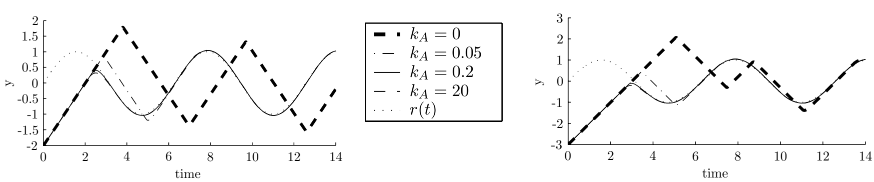
$$\det \left(2 \begin{bmatrix} -k_I & 0 \\ 0 & -(k_P k_A - 1) \end{bmatrix} - \lambda P \right) = 0.$$

Hence $\int_{t_0}^{t_0+\bar{T}} \lambda_{\max} \xi(t) \leq \lambda_{\max} \delta < 0$ with δ from the statement of Lemma 17.5. Using the Gronwall-Bellman lemma (see e.g. [1]) one can see that $V \to 0$ as $t \to \infty$ uniformly in time and uniformly in the initial conditions from Ω. Since Ω is an arbitrary compact set and due to Lemma 17.4, all solutions are globally uniformly asymptotically stable. Due to Lemma 17.3 there is a bounded solution $\bar{x}(t)$ defined on the whole time interval $(-\infty, +\infty)$ and thus system (17.8) is uniformly convergent for all $w(\cdot) \in \mathcal{W}$. Note that due to Property 17.1 this solution $\bar{x}(t)$ is the unique solution bounded on $(-\infty, +\infty)$. ■

As one can see, our analysis is based on a PE-like (persistency of excitation) property that follows from Lemma 17.5. More advanced results in this direction can be found in [28, 23].

17.4.1 Example: Influence of Parameter k_A on System Dynamics

Theorem 17.1 states that system (17.8) is uniformly convergent for $k_A > 1/k_P$. In this example, we consider system (17.8) with $k_P = 10$, $k_I = 20$ and $w_3(t) = 0$, and evaluate the system behavior for several values of k_A. Note that the values of k_P and k_I are chosen in such a way that the system without the saturation has a satisfactory performance.



(a) $x_0 = (-2, 0)$

(b) $x_0 = (-3, 0)$

Fig. 17.2. System output for input $r(t) = \sin(t)$ and different values of k_A

In Fig. 17.2 the system output y is plotted for four different values of k_A and the input signal $r(t) = \sin(t)$. Figures 2(a) and 2(b) display the results for two different initial conditions of the system. For $k_A = 0$ (i.e. no anti-windup) the two initial conditions result in two different limit solutions. An other observation is that for the other values of k_A (even for $k_A = 0.05 < 1/k_P$) the system seems to have a unique limit solution for different initial conditions. However, to verify the existence of a unique limit solution for $k_A = 0.05$ we should be able to evaluate all initial conditions and input signals, since it is possible that for another initial condition or input signal the limit solution for $k_A = 0.05$ is not unique, or we should be able to expand Theorem 17.1 such that it holds for $k_A < 1/k_P$ as well.

17.5 Quadratic Convergence of Switched Systems

Consider the switched dynamical system

$$\dot{x} = A_i x + B_i w(t), \quad i = 1, \ldots, k, \tag{17.16}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(\cdot) \in \overline{\mathbb{PC}}_m$ is the input. Suppose the collection of matrices $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ is given, and A_i is Hurwitz for all $i = 1, \ldots, k$. The general problem is to find a switching rule such that the closed loop system is uniformly (exponentially) convergent. In this section, we focus on a switching rule that is based on static state feedback, i.e. $i = \sigma(x)$. Note that in a similar way dynamic state feedback and static/ dynamic output feedback can be considered. These approaches are subject for future research.

Suppose a common Lyapunov matrix $P = P^T > 0$ exists that satisfies the following inequalities

$$A_i^T P + P A_i < 0, \quad i = 1, \ldots, k. \tag{17.17}$$

Consider the following switching rule

$$\sigma(x, w) = \arg\ \min_i \{x^T Z_{ix} x + x^T Z_{iw} w\}, \tag{17.18}$$

in which $Z_{iw} = 4PB_i$ and Z_{ix} are matrices to be defined.

Theorem 17.2. *If there exist a $P = P^T > 0$, $\alpha > 0$, and $Z_{1x}, \ldots, Z_{kx}$ and if*

$$Z_{ix} \neq Z_{jx} \text{ and/or } Z_{iw} \neq Z_{jw} \quad \forall i, j \leq k,\ i \neq j, \tag{17.19}$$

such that

$$\begin{bmatrix} PA_i + A_i^T P - (Z_{ix} - Z_{jx}) & -(A_i^T P + PA_j) \\ -(A_j^T P + PA_i) & PA_j + A_j^T P + (Z_{ix} - Z_{jx}) \end{bmatrix} \leq -\alpha \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} \tag{17.20}$$

for all $i, j \leq k,\ i \neq j$, then the switching rule (17.18) *with matrices $Z_{1x}, \ldots, Z_{kx}$ makes system* (17.16) *quadratically convergent.*

Proof. First, note that condition (17.19) implies that the set of discontinuities of the right-hand side of the closed loop system has zero measure, which means that a

Filippov solution exists for the closed loop system. Let P be a common Lyapunov matrix for the collection $\{A_1, \ldots, A_k\}$ and consider the Lyapunov function candidate

$$V(x_1, x_2) = (x_1 - x_2)^T P (x_1 - x_2). \tag{17.21}$$

If $\sigma(x_1, w) = \sigma(x_2, w)$ the inequality

$$\dot{V} \leq -\varepsilon V, \quad \varepsilon > 0 \tag{17.22}$$

is obviously satisfied. Let $\sigma(x_1, w) = p$ and $\sigma(x_2, w) = q$, such that the derivative of (17.21) can be written as

$$\begin{aligned} \dot{V} = {} & x_1^T(A_p^T P + PA_p)x_1 + x_2^T(A_q^T P + PA_q)x_2 - x_1^T(A_p^T P + PA_q)x_2 \\ & -x_2^T(PA_p + A_q^T P)x_1 + 2x_1^T PB_p w + 2x_2^T PB_q w - 2x_1^T PB_q w - 2x_2^T PB_p w. \end{aligned} \tag{17.23}$$

Using the switching rule (17.18) the following constraint functions can be defined

$$\begin{aligned} S_1(x, w) &= \frac{1}{2}\left(x_1^T(Z_{px} - Z_{qx})x_1 + x_1^T(Z_{pw} - Z_{qw})w\right) \leq 0, \\ S_2(x, w) &= \frac{1}{2}\left(x_2^T(Z_{qx} - Z_{px})x_2 + x_2^T(Z_{qw} - Z_{pw})w\right) \leq 0. \end{aligned}$$

The system is quadratically stable (see Definition 17.4) if

$$\dot{V} \leq -\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for all (x, w) satisfying $S_1(x, w) \leq 0$ and $S_2(x, w) \leq 0$. Using the $\mathcal{S}$-procedure, the previous condition is satisfied if the following inequality holds

$$\dot{V} - S_1 - S_2 \leq -\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{17.24}$$

This inequality is equivalent to (17.20). ■

Remark 17.2. Note that (17.20) is an LMI with design variables P, $Z_{1x}, \ldots, Z_{kx}$ and α, which can be solved efficiently using available LMI toolboxes.

Remark 17.3. In case $B_i = B$ for all modes, then the switching rule (17.18) is independent of the input. This implies that under the conditions stated in Theorem 17.2 the system can be made convergent without regarding the input, even if the input for example represents a disturbance signal.

Although Theorem 17.2 gives sufficient conditions for quadratic convergence, it does not give insight for what collection of matrices $\{A_1, \ldots, A_k\}$ a switching law can be found. In the case that we define in advance both Z_{ix} and Z_{iw}

$$Z_{ix} = A_i^T P + PA_i, \quad Z_{iw} = 4PB_i, \quad \forall i = 1, \ldots, k, \tag{17.25}$$

then Theorem 17.2 can be simplified as follows.

Theorem 17.3. *If there exist a $P = P^T > 0$ satisfying* (17.17) *and if*

$$Z_{ix} \neq Z_{jx} \text{ and/or } Z_{iw} \neq Z_{jw} \quad \forall i, j \leq k, \; i \neq j \tag{17.26}$$

and

$$P(A_i - A_j) - (A_i - A_j)^T P = 0 \quad \forall i, j \leq k, \tag{17.27}$$

then the switching rule (17.18) *makes system* (17.16) *quadratically convergent.*

Proof. Consider the same notations as in the proof of Theorem 17.2. Note that due to (17.27),

$$\begin{aligned} &x_1^T(A_p^T P + PA_q)x_2 + x_2^T(PA_p + A_q^T P)x_1 = \\ &\frac{1}{2}x_1^T(A_p^T P + PA_p + A_q^T P + PA_q)x_2 + \frac{1}{2}x_2^T(A_p^T P + PA_p + A_q^T P + PA_q)x_1. \end{aligned} \tag{17.28}$$

Combining (17.23) with (17.28) and using notations (17.25) gives

$$\begin{aligned} \dot{V} = &\frac{1}{2}x_1^T Z_{px}x_1 + \frac{1}{2}x_1^T Z_{px}x_1 + \frac{1}{2}x_1^T Z_{pw}w \\ &+\frac{1}{2}x_2^T Z_{qx}x_2 + \frac{1}{2}x_2^T Z_{qx}x_2 + \frac{1}{2}x_2^T Z_{qw}w \\ &-\frac{1}{2}x_1^T(Z_{px} + Z_{qx})x_2 - \frac{1}{2}x_2^T(Z_{px} + Z_{qx})x_1 \\ &-\frac{1}{2}x_2^T Z_{pw}w - \frac{1}{2}x_1^T Z_{qw}w. \end{aligned}$$

The switching rule (17.18) implies that

$$\begin{aligned} \frac{1}{2}\left(x_1^T Z_{px}x_1 + x_1^T Z_{pw}w\right) &\leq \frac{1}{2}\left(x_1^T Z_{qx}x_1 + x_1^T Z_{qw}w\right) \\ \frac{1}{2}\left(x_2^T Z_{qx}x_2 + x_2^T Z_{qw}w\right) &\leq \frac{1}{2}\left(x_2^T Z_{px}x_2 + x_2^T Z_{pw}w\right) \end{aligned}$$

and therefore

$$\dot{V} \leq \frac{1}{2}(x_1 - x_2)^T(Z_{px} + Z_{qx})(x_1 - x_2) \leq -\alpha V$$

for some $\alpha > 0$. ■

Remark 17.4. Note that condition (17.27) is always satisfied for symmetric matrices A_i, $i = 1, \ldots, k$.

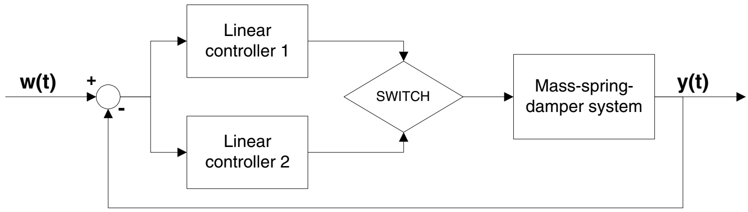


Fig. 17.3. Graphical representation of system (17.29)

17.5.1 Example: Performance of a Convergent Switched System

In this example we show how a simulation-based performance analysis can be realized for a switched systems that is made convergent by the design of a switching rule.

Consider the switched system

$$\begin{aligned} \dot{x} &= A_i x + B_i w(t), \quad i = 1, 2 \\ y &= Cx \end{aligned} \tag{17.29}$$

which represents for example a mass-spring-damper system with two linear controllers (see Fig. 17.3). Here, $x(t) \in \mathbb{R}^3$ is the state, $w(\cdot) \in \overline{\mathbb{PC}}_1$ is the input, $C = [\,1\ 0\ 0\,]$, and

$$A_1 = \begin{bmatrix} -7 & 2 & -6 \\ -10 & -3 & -5 \\ 7 & 4 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 13 \\ 15 \\ -6 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -6 & 3 & -8 \\ -9 & 0 & -8 \\ 5 & 1 & -9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}.$$

For the common Lyapunov matrix

$$P = \begin{bmatrix} 0.5163 & -0.1655 & -0.0038 \\ -0.1655 & 0.2609 & 0.0321 \\ -0.0038 & 0.0321 & 0.2669 \end{bmatrix} > 0$$

and

$$Z_{1x} - Z_{2x} = \begin{bmatrix} -0.3584 & -0.0312 & 0.5055 \\ -0.0312 & -0.5208 & 0.7083 \\ 0.5055 & 0.7083 & 2.2239 \end{bmatrix}$$

conditions (17.19) and (17.20) are satisfied, with $\alpha = 1$. Switching rule (17.18) thus makes the system (17.29) quadratically convergent,

$$\dot{V} \leq -(x_1 - x_2)^2 \leq -1.6704(x_1 - x_2)^T P(x_1 - x_2) = -1.6704V.$$

Subsequently, the fact that

$$\lambda_{\min}(P)(x_1 - x_2)^2 \leq (x_1 - x_2)^T P(x_1 - x_2) \leq \lambda_{\max}(P)(x_1 - x_2)^2,$$

with $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ the minimum and maximum eigenvalue of P respectively, leads to the following upper bound

$$\begin{aligned} |x_1(t) - x_2(t)| &\leq \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\; |x_1(0) - x_2(0)|\; e^{\frac{-1.6704}{2}t} \\ &\leq 1.8671\; |x_1(0) - x_2(0)|\; e^{-0.8352t}. \end{aligned} \tag{17.30}$$

In order to analyse the performance of this switched system, only one solution of the system needs to be evaluated, since the limit solution of this (convergent) system is independent of its initial conditions. In Fig. 17.4 the performance of the switched system is compared with the performance of the two corresponding linear systems, i.e., $\dot{x} = A_1 x + B_1 w(t)$ and $\dot{x} = A_2 x + B_2 w(t)$. The performance measure applied here is the integrated tracking error:

$$\sqrt{\frac{\int_{t_l}^{t_l+T} (w(t) - y(t))^2\, dt}{\int_{t_l}^{t_l+T} w(t)^2 dt}}, \tag{17.31}$$

where T is a time period that is long enough to obtain a good average of the tracking error and t_l is a moment in time for which all considered solutions are close enough to the limit solution. The time t_l is in this example determined visually, but a bound can be calculated as well using (17.30). The performance is evaluated for the following input signals:

$$w(t) = \sin(bt), \quad b \in [10^{-2}, 10^3].$$

From Figure 17.4 it can be concluded that for the considered performance mea-

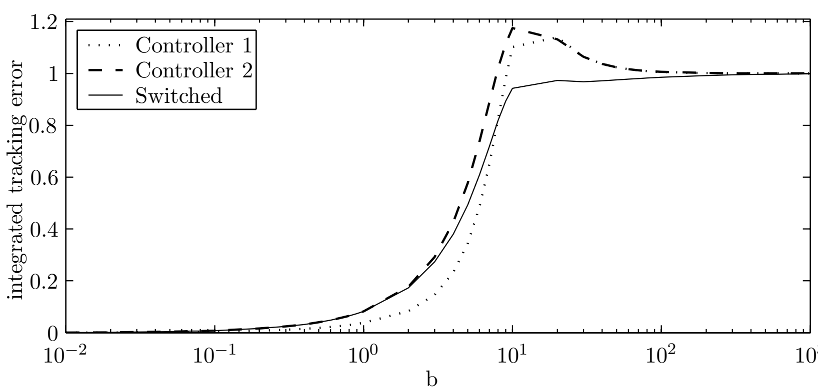


Fig. 17.4. Performance of switched system

sure (17.31) the switched system performs better then the linear systems for the input range $b \in [10^1, 10^3]$, i.e. the switched system is less sensitive to high frequencies in the input signal. More important, however, is the fact that the performance of the switched system *can* be determined by means of simulation, which is practically impossible for most nonlinear/switched systems that are not convergent.

17.6 Conclusions

In this paper we considered the following problem definition for piece-wise affine systems: is it possible to design a feedback law and/or switching rule such that the resulting closed-loop system is convergent? We have investigated this problem for two areas of interest, i.e. the anti-windup design for a marginally stable plant with input saturation and the class of switched linear systems.

For an integrator plant with input saturation, we proved that by a simple static anti-windup rule a uniformly convergent closed-loop system can be obtained. It is also noted that within the range of the anti-windup rule for which the system is convergent, performance of the closed-loop system can be optimized using simulation.

Furthermore, we demonstrated the use of convergency in the class of switched linear systems. It is proved that by definition of the switching rule the switched system can made convergent if the linear subsystems satisfy certain conditions. For the convergent switched system, a performance evaluation has been shown feasible using a Bode-like plot.

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