The Newton Polygon Method for Differential Equations

José Cano

Dpto. Algebra, Geometría y Topología,
Fac. de Ciencias, Universidad de Valladolid, Spain
jcano@agt.uva.es

Abstract. We prove that a first order ordinary differential equation (ODE) with a dicritical singularity at the origin has a one-parameter family of convergent fractional power series solutions. The notion of a dicritical singularity is extended from the class of first order and first degree ODE's to the class of first order ODE's. An analogous result for series with real exponents is given.

The main tool used in this paper is the Newton polygon method for ODE. We give a description of this method and some elementary applications such as an algorithm for finding polynomial solutions.

1 Introduction

In this paper we give a sufficient condition for a first order ordinary differential equation (ODE) to have a one-parameter family of fractional power series (resp. generalized formal power series) solutions. If the family of solutions have rational exponents it turns out that each one is convergent, hence its sum is the parametrization of an analytical branch curve. The sufficient condition is a generalization of the notion of dicritical singularity of holomorphic foliations in dimension 2. The foliation defined by $a(x,y)\,dx + b(x,y)\,dy = 0$ is dicritical if there exists a dicritical blowup in the reduction of singularities process. It is well known that it is dicritical if and only if there exists a one-parameter family of analytical invariant curves passing through the origin. The above foliation corresponds to the first order and first degree ODE $a(x,y) + b(x,y)\,y' = 0$. We generalize this result for first order and arbitrary degree ODE. The dicritical property is described in terms of the Newton polygon process and in case of first order and first degree agrees with that of foliations.

Briot and Bouquet [1] in 1856 used the Newton polygon method for studying the singularities of first order and first degree ODE's and Fine [2] gives a complete description of the method for general ODE in 1889.

D.Y. Grigoriev and M. Singer [3] use it to give an enumeration of the set of formal power series solution with real exponents of an ODE $F(y) = 0$. They give restrictions (expressed as a sequence of quantifier-free formulas) over parameter C_i and μ_i for $\sum_{i=1}^{\infty} C_i\, x^{\mu_i}$ to be a solution of $F = 0$. The one-parameter families of solutions that we obtain in this paper are simpler than those obtained in [3]

H. Li, P. J. Olver and G. Sommer (Eds.): IWMM-GIAE 2004, LNCS 3519, pp. 18–30, 2005.

for the general case. (See Theorem 2 for a precise description.) In particular the parameter does not appear in the exponents μ_i. When the Newton polygon process to a first order ODE $F(y) = 0$ is applied, it is easy to see that the "first" parameter that we need to introduce is a coefficient and not an exponent (see Proposition 1). The problem arises in subsequent steps, because now we are dealing with a differential equation which has parameters as coefficients. For instance, $xy' - C\,y = 0$ has $y(x) = C_1\,x^C$ as solution and parameter C has "jumped" to the exponent. Here we proof that this phenomenon does not happen.

In a forthcoming paper we will give a complete description of the set of generalized power series solutions of a first order ODE.

In Section 2, we give a detailed description of the Newton polygon process in order to state some basic definitions and notations. This section should be read in connection with our main result, stated and proved in section 3, as a technical guide for the proof. All results in section 2 are not original and can be found in [3, 4, 5, 6, 7, 8]. As an elementary application of the Newton polygon method we have included an algorithm which gives a bound for degree of polynomial solutions of a first order ODE.

2 Description of the Classical Newton Polygon Method

Let K be a field, $\mathbb{C} \subseteq K$. We will denote $K((x))^*$ the field of Puiseux series over K. Hence, the elements of $K((x))^*$ are formal power series $\sum_{i\geq i_0} c_i\,x^{i/q}$, $i \in \mathbb{Z}$, $c_i \in K$ and $q \in \mathbb{Z}_+$. A well-ordered series with real exponents with coefficients in the field K is a series $\phi(x) = \sum_{\alpha\in A} c_\alpha\,x^\alpha$, where $c_\alpha \in K$, and A is a well ordered subset of $\mathbb{R}$. If there exists a finitely generated semi-group Γ of $\mathbb{R}_{\geq 0}$ and $\gamma \in \mathbb{R}$, such that, $A \subseteq \gamma + \Gamma$, then we say that $\phi(x)$ is a *grid-based* series. (This terminology comes from [7].) Let $K((x))^w$ and $K((x))^g$ be the set of well-ordered series and that of grid-based series respectively. Both are differential rings with the usual inner operations and the differential operator

$$\frac{d}{d\,x}(c) = 0, \forall c \in K, \quad \text{and} \quad \frac{d}{d\,x}\left(\sum c_\alpha\,x^\alpha\right) = \sum \alpha\,c_\alpha\,x^{\alpha-1} \ .$$

In fact, both rings are fields by virtue of Theorem 1.

Let $F(y_0, \ldots, y_n)$ be a polynomial on the variables $y_0, \ldots, y_n$ with coefficients in $K((x))^g$. The differential equation $F(y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) = 0$ will be denoted by $F(y) = 0$. We are going to describe the classical Newton polygon method for searching solutions of the differential equation $F(y) = 0$ in the field $K((x))^w$. We write F in a unique way as

$$F = \sum a_{\alpha,\rho_0,\ldots,\rho_n}\,x^\alpha\,y_0^{\rho_0}\cdots y_n^{\rho_n}, \quad a_{\alpha,\underline{\rho}} \in K,$$

where $\alpha \in A$ and $\underline{\rho}$ belongs to a finite subset of $\mathbb{N}^{n+1}$. We define the cloud of points of F to be the set $\mathcal{P}(F) = \{P_{\alpha,\underline{\rho}} \mid a_{\alpha,\underline{\rho}} \neq 0\}$, where we denote

$$P_{\alpha,\underline{\rho}} = (\alpha - \rho_1 - 2\rho_2 - \cdots - n\rho_n\, ,\, \rho_0 + \rho_1 + \cdots + \rho_n) \in \mathbb{R} \times \mathbb{N} \ . \qquad (1)$$

The Newton polygon $\mathcal{N}(F)$ of F is the convex hull of the set

$$\bigcup_{P\in\mathcal{P}(F)} (P+\{(a,0)\mid a\geq 0\})\ .$$

We remark that $\mathcal{N}(F)$ has a finite number of vertices and all of them has as ordinate a non-negative integer.

Given a line $L\subseteq \mathbb{R}^2$ with slope $-1/\mu$, we say that μ is the *inclination* of L. Let $\mu\in\mathbb{R}$, we denote $L(F;\mu)$ to be the line with inclination μ such that $\mathcal{N}(F)$ is contained in the right closed half-plane defined by $L(F;\mu)$ and $L(F;\mu)\cap\mathcal{N}(F)\neq\emptyset$. We define the polynomial

$$\Phi_{(F;\mu)}(C)=\sum_{P_{\alpha,\underline{\rho}}\in L(F;\mu)} a_{\alpha,\underline{\rho}}\, C^{\rho_0+\cdots+\rho_n}\,(\mu)_1^{\rho_1}\cdots(\mu)_n^{\rho_n}, \tag{2}$$

where $(\mu)_k=\mu(\mu-1)\cdots(\mu-k+1)$ and $P_{\alpha,\underline{\rho}}$ is as in (1).

2.1 Necessary Initial Conditions

Lemma 1. *Let* $y(x)=c\,x^{\mu}+\cdots$ higher order terms $\cdots\in K((x))^w$ *be a solution of the differential equation* $F(y)=0$*. Then we have that* $\Phi_{(F;\mu)}(c)=0$.

Proof. We have that $F(c\,x^{\mu}+\cdots)=$

$$\sum_{\alpha,\underline{\rho}} a_{\alpha,\underline{\rho}}\, x^{\alpha}\,(c\,x^{\mu}+\cdots)^{\rho_0}(\mu c\,x^{\mu-1}+\cdots)^{\rho_1}\cdots((\mu)_n c\,x^{\mu-n}+\cdots)^{\rho_n}=$$

$$\sum_{\alpha,\underline{\rho}}\left\{A_{\alpha,\underline{\rho}}\; c^{\rho_0+\cdots+\rho_n}\,(\mu)_1^{\rho_1}\cdots(\mu)_n^{\rho_n}\,x^{\alpha-\rho_1-2\rho_2-\cdots-n\rho_n+\mu(\rho_0+\cdots+\rho_n)}+\cdots\right\}=$$

$$\left\{\sum_{\nu(\alpha,\underline{\rho};\mu)=\nu(F;\mu)} a_{\alpha,\underline{\rho}}\; c^{\rho_0+\cdots+\rho_n}\,(\mu)_1^{\rho_1}\cdots(\mu)_n^{\rho_n}\right\}x^{\nu(F;\mu)}+\cdots\text{higher order terms},$$

where $\nu(\alpha,\underline{\rho};\mu)=\alpha-\rho_1-2\rho_2-\cdots-n\rho_n+\mu(\rho_0+\cdots+\rho_n)$, and $\nu(F;\mu)=\min\{\nu(\alpha,\underline{\rho};\mu)\mid a_{\alpha,\underline{\rho}}\neq 0\}$. □

The set $\mathrm{NIC}(F)=\{(c,\mu)\mid c\in\bar{K}, c\neq 0,\ \mu\in\mathbb{R},\ \Phi_{(F;\mu)}(c)=0\}$, where $\bar{K}$ is the algebraic closure of K, is called the set of necessary initial conditions for F. We give a precise description of this set using the Newton polygon of F.

For each $\mu\in\mathbb{R}$, $L(F;\mu)\cap\mathcal{N}(F)$ is either a side or a vertex. Assume that $L(F;\mu)\cap\mathcal{N}(F)=S$ is a side. We call $\Phi_{(F;\mu)}(C)$ the *characteristic* polynomial of F associated to the side S. Let $A_S=\{c\in\bar{K}\mid c\neq 0,\ \Phi_{(F;\mu)}(c)=0\}$. We say that the side S is of type (0) if $A_S=\emptyset$, of type (I) is A_S is a finite set and of type (II) if $A_S=\bar{K}$. We have that $A_S\times\{\mu\}=\mathrm{NIC}(F)\cap(\bar{K}\times\{\mu\})$.

Let $p=(a,h)$ be a vertex of $\mathcal{N}(F)$, and let $\mu_1<\mu_2$ be the inclinations of the adjacent sides at p. For any μ such that $\mu_1<\mu<\mu_2$, we have that $L(F;\mu)\cap\mathcal{N}(F)=\{p\}$. Then $\Phi_{(F;\mu)}(C)=C^h\,\Psi_{(F;p)}(\mu)$, where

$$\Psi_{(F;p)}(\mu)=\sum_{P_{\alpha,\underline{\rho}}=p} a_{\alpha,\underline{\rho}}\,(\mu)_1^{\rho_1}\cdots(\mu)_n^{\rho_n}\ , \tag{3}$$

Table 1. Necessary initial conditions for the equation $F(y) = 0$ of example 1

vertices

μ	$\Psi_{(F;p)}(\mu)$	type
$-\infty < \mu < -1/2$	$\mu^3\,(\mu-1)^2$	0
$-1/2 < \mu < 0$	0	IV
$0 < \mu < 2$	$\mu^3(\mu-1)$	III
$2 < \mu < 4$	$\mu - 4$	0
$4 < \mu < \infty$	-1	0

sides

μ	$\Phi_{(F;\mu)}(C)$	type
$\mu = -1/2$	$-9/32\,C^5$	0
$\mu = 0$	0	II
$\mu = 2$	$2\,C(4C+1)$	I
$\mu = 4$	-1	0

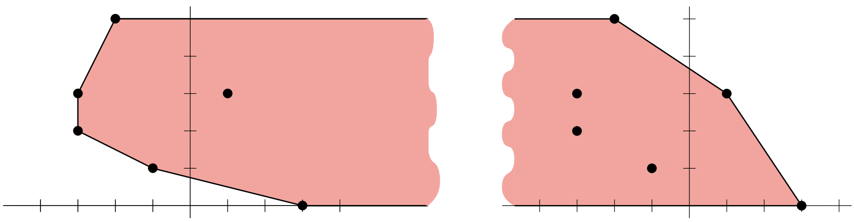

Fig. 1. The Newtonpolygon of F at the origin (left) and at infinity (right)

where $P_{\alpha,\underline{\rho}}$ is as in (1). The polynomial $\Psi_{(F;p)}(\mu)$ is called the *indicial* polynomial associated to p. Let $A_p = \{\mu \in (\mu_1, \mu_2) \mid \Psi_{(F;p)}(\mu) = 0\}$. Then $\mathrm{NIC}(F) \cap \bar{K} \times (\mu_1, \mu_2) = \bar{K} \times A_p$. There are again three possibilities. Either $A_p = \emptyset$, in this case we say that the vertex p is of type (0); or A_p is a finite set, then p is of type (III); or $A_p = (\mu_1, \mu_2)$, and we call p of type (IV).

Let $(c, \mu) \in \mathrm{NIC}(F)$. Then μ is associated with either a side or a vertex of $\mathcal{N}(F)$. We say that (c, μ) is of type (I)–(IV) depending on the type of the vertex or of the side associated with μ.

Example 1. Let $F(y) = x^3\, y_0^2\, y_1\, y_2^2 - y_1^3 + y_0\, y_1\, y_2 + x^{-1}\, y_0\, y_1^2 + x\, y_2^2 - y_1\, y_2 - y_1 + 4x^{-1}\, y_0 - x^3\, y_0\, y_1\, y_2 - x^3$.

The necessary initial conditions for $F(y)$ are described in Table 1 and its Newton polygon is drawn on the left hand of Fig. 1.

Proposition 1. *Let $F(x, y, y') \in K((x))^w[y, y']$ be a polynomial first order ordinary differential equation. Let p be a vertex of $\mathcal{N}(F)$. Then p is not of type (IV).*

Proof. Let $p = (\alpha, h)$. Let F_p be the sum of all monomials of F whose corresponding point is p. We have that $F_p = \sum_{i=0}^{h} a_i\, x^\alpha\, y^i\, (xy')^{h-i}$. Then the indicial polynomial associated with the vertex p is $\Psi_{(F;p)}(\mu) = \sum_{i=0}^{h} a_i\, \mu^{h-i}$. If this polynomial has an infinite number of solutions, then $a_i = 0$ for all $0 \le i \le h$. Then F_p is identically null, which is an absurd because p belongs to the cloud of points of F. □

2.2 The Newton Polygon at Infinity and Polynomial Solutions

Let us assume that the coefficients of $F(y)$ are rational functions. For any point x_0, in order to find formal solutions in the variable $x - x_0$ we only need to perform the change of variable $\bar{x} = x - x_0$ in the differential equation and work as before.

At infinity, we have two possibilities. Either we perform the change of variable $z = 1/x$ in the differential equation and we work as above, or we just look for series of the form $y(x) = \sum_{i=0}^{\infty} c_i\, x^{\mu_i}$, $\mu_i > \mu_{i+1}$, for all i. In this case we do not need to do any change to the original equation, but the monomials involved in the necessary initial condition are those which give the greatest order. Hence, the Newton polygon of $F(y)$ at infinity $\mathcal{N}_\infty(F)$ is defined as the convex hull of $\bigcup_{P \in \mathcal{P}(F)} (P + \{(a,0) \mid a \leq 0\})$, $L_\infty(F;\mu)$ being the line with slope $-1/\mu$ which supports $\mathcal{N}_\infty(F)$, and $\Phi^\infty_{(F;\mu)}(C)$ as in (2) substituting $L_\infty(F;\mu)$ for $L(F;\mu)$. Then Lemma 1 holds substituting "lower order terms" for "higher ..." and $\Phi^\infty_{(F;\mu)}$ for $\Phi_{(F;\mu)}$. Using this version of Lemma 1 and Prop. 1, we have the following

Algorithm
Input: A first order ODE $F(y) = 0$ with rational coefficients.
Output: A bound for the degree of the polynomial solutions of $F(y) = 0$.
Let p the top vertex of $\mathcal{N}_\infty(F)$. Return N, where N is greater than any real root of $\Psi_{(F;p)}(\mu)$, and such that $N \geq \mu$, where μ is the inclination of the non-horizontal side of $\mathcal{N}_\infty(F)$ adjacent to p.

We may apply the above algorithm for any ODE provided the top vertex of $\mathcal{N}_\infty(F)$ is not of type (IV). For instance, the equation in Example 1 satisfies this hypothesis, so that 3/2 as a bound for the degree of its polynomial solutions.

2.3 The Newton Polygon Process

Given a differential polynomial $F(y) \in K((x))^g[y_0, \ldots, y_n]$, the Newton polygon process constructs a tree $\mathcal{T}$. The root of $\mathcal{T}$ is τ_0. For each node τ of the tree there are three associated elements: a differential polynomial $F_\tau(y)$ with coefficients in $\bar{K}((x))^g$, an element $c_\tau \in \bar{K}$, and $\mu_\tau \in \mathbb{R} \cup \{-\infty, \infty\}$. For the root τ_0, we have $F_{\tau_0}(y) = F(y)$, $c_{\tau_0} = 0$ and $\mu_{\tau_0} = -\infty$.

Let τ be a node of $\mathcal{T}$ which is not a leaf. We are going to describe all its descendant nodes. First, if $y = 0$ is a solution of $F_\tau(y) = 0$, then there is a descendant σ of τ, which is a leaf and for which $F_\sigma = F_\tau$, $\mu_\sigma = \infty$ and $c_\sigma = 0$. The other descendant nodes of τ are in a bijective correspondence with the set $D_\tau = \{(c,\mu) \in \mathrm{NIC}(F_\tau); \mu > \mu_\tau\}$. For each $(c,\mu) \in D_\tau$, there is a descendant node σ for which $F_\sigma(y) = F_\tau(cx^\mu + y)$, $c_\sigma = c$ and $\mu_\sigma = \mu$.

The above tree depends on F and on the field K. If necessary, we shall write $\mathcal{T}(F)$ or even $\mathcal{T}(F;K)$ to clearly state which tree we are referring to. For instance, let $F = y'$, we may consider $K = \mathbb{C}$ or $L = \mathbb{C}(C)$. The tree $\mathcal{T}(F;L)$, has for each rational function $R(C)$ a node σ with $c_\sigma = R(C)$ and $\mu_\sigma = 0$.

As usual, the level of a node σ is k if the path of $\mathcal{T}$ from τ_0 to σ is $\tau_0, \tau_1, \ldots, \tau_k = \sigma$. We say that the tree $\mathcal{T}$ is *discrete* if for each $k \geq 0$, the number of nodes of level k is finite.

Example 2. Let $F(y)$ be as in example 1. We will describe here some nodes of the tree $\mathcal{T} = \mathcal{T}(F;\mathbb{C})$. The tree $\mathcal{T}$ has only one node of level zero: the root node τ_0. We have $F_{\tau_0}(y) = F(y)$, $c_{\tau_0} = 0$ and $\mu_{\tau_0} = -\infty$. From Table 1, we have that $\mathrm{NIC}(F_{\tau_0})$ is the set

$$D_{\tau_0} = \{(c,\mu)\,;\, -1/2 < \mu < 0,\, c \in \mathbb{C}\} \cup \{(c,\mu)\,;\, c \in \mathbb{C},\, \mu \in \{0,1\}\} \cup \{(-1/4,2)\}$$

Since $y = 0$ is not a solutions of $F(y) = 0$, then the descendant nodes of τ_0 are in bijective correspondence with set D_{τ_0}. In particular, the tree $\mathcal{T}$ is not discrete. Let us consider σ_1 and σ_2 the descendant nodes of τ_0 corresponding to the elements $(2,0)$ and $(-1/3,0)$ of D_{τ_0} respectively. Thus, $F_{\sigma_1}(y) = F(2+y)$, $F_{\sigma_2}(y) = F(-1/3+y)$ and $\mu_{\sigma_1} = \mu_{\sigma_2} = 0$. Using the Newton polygon of F_{σ_1} and that of F_{σ_2}, we see that $D_{\sigma_1} = D_{\sigma_2} = \{(2\sqrt{-1},1),(-2\sqrt{-1},1)\}$, and $y = 0$ is not a solution of neither $F_{\sigma_1}(y) = 0$ nor $F_{\sigma_2}(y) = 0$. Hence, each σ_i, for $i = 1$ or $i = 2$, has exactly two descendant nodes $\sigma_{i,j}$, with $j = 1$ or $j = 2$. Now we have that $D_{\sigma_{2,j}} = \emptyset$ and $y = 0$ is not a solution of $F_{\sigma_{2,j}}(y) = 0$. Thus, $\sigma_{2,j}$ is a leaf of $\mathcal{T}$ because it has no descendant nodes. Lemma 1 implies that there are not solutions of $F(y) = 0$ of type $-1/3 + \sqrt{-1}\,x + \cdots$, where dots stands for higher order terms. We remark that $\mu_{\sigma_{2,j}} = 1 \neq \infty$. In following subsection we will see that branches of $\mathcal{T}$ with a leaf σ satisfying $\mu_\sigma \neq \infty$ does not correspond to solutions of $F(y) = 0$. On the other hand, we have that $D_{\sigma_{1,j}} = \{(3/14,2)\}$ and $y = 0$ is not a solution $F_{\sigma_{1,j}}(y) = 0$, hence $\sigma_{1,j}$ has exactly one descendant node $\sigma_{2,j}$, where $c_{\sigma_{2,j}} = 3/14$ and $\mu_{\sigma_{2,j}} = 2$. Moreover, after the next subsection, we will be able to prove that there are two branches $(\sigma_{k,j})_{k\geq 0}$ of $\mathcal{T}$, where $j = 1,2$ and we put $\sigma_{0,j} = \tau_0$, each one corresponding to a solution of $F(y) = 0$ of the form $2 + (-1)^j\sqrt{-1}\,x + 3/14\,x^2 + \cdots$.

2.3.1 Relation Between Branches of $\mathcal{T}$ and Solutions of $F(y) = 0$

The tree $\mathcal{T}$ has three types of branches: (a) infinite ones; (b) finite branches whose leaf σ has $\mu_\sigma = \infty$; and (c) finite branches whose leaf σ has $\mu_\sigma < \infty$. In this paragraph we will see that there exists a one to one correspondence between formal power series solution of $F(y) = 0$ and the set of branches of $\mathcal{T}$ of types (a) or (b).

Let $\tau_0, \tau_1, \ldots, \tau_k, \tau_{k+1}$ be a finite branch of $\mathcal{T}$ such that $\mu_{\tau_{k+1}} = \infty$. Let $\phi(x) = \sum_{i=1}^{k} c_{\tau_i}\, x^{\mu_{\tau_i}}$. We have that $F_{\tau_k}(y) = F(\phi(x)+y)$. Since $\mu_{\tau_{k+1}} = \infty$ we have that $y = 0$ is a solution of $F_{\tau_k}(y) = 0$. Hence, $\phi(x)$ is a solution of $F(y) = 0$.

Reciprocally, let $\phi(x) = \sum_{i=1}^{k} c_i\, x^{\mu_i}$ be a solution of $F(y) = 0$. By Lemma 1, there exists a finite branch $\tau_0, \tau_1, \ldots, \tau_k, \tau_{k+1}$ of $\mathcal{T}$ with $c_{\tau_i} = c_i$ and $\mu_i = \mu_{\tau_i}$ for $1 \leq i \leq k$.

Let $\tau_0, \tau_1, \ldots, \tau_k$ be a finite branch of $\mathcal{T}$ such that $\mu_{\tau_{k+1}} < \infty$. Let $\phi(x) = \sum_{i=1}^{k} c_{\tau_i}\, x^{\mu_{\tau_i}}$. We have that $F_{\tau_k}(y) = F(\phi(x)+y)$. Since τ_k is a leaf, one has $\mathrm{NIC}(F_{\tau_k}) \cap \bar{K} \times (\mu_{\tau_k}, \infty) = \emptyset$, and $y = 0$ is not a solution. By Lemma 1, this means that there is no solution $y(x)$ of $F(y) = 0$ such that $y(x) = \phi(x) + \psi(x)$ and with the terms of $\psi(x)$ of order greater than μ_{τ_k}. Hence, the above branch does not correspond to any solution of $F(y) = 0$.

In view of Lemma 1, it is obvious that if $y(x) = \sum_{i=1}^{\infty} c_i\, x^{\mu_i}$, where $c_i \neq 0$ for all i, is a solution of $F(y) = 0$, then there is an infinite branch $(\tau_i)_{i\geq 0}$ of $\mathcal{T}$. The reciprocal result is the following

Theorem 1. *Let $F(y) \in K((x))^g[y_0, \ldots, y_n]$. Let $B = (\tau_i)_{i\geq 0}$ be an infinite branch of the tree $\mathcal{T}$. Let $\phi(x) = \sum c_{\tau_i}\, x^{\mu_{\tau_i}}$. Then $\phi(x) \in \bar{K}((x))^g$ and is a solution of $F(y) = 0$. In particular, if all the exponents of $\phi(x)$ are rational, then they have a greatest common denominator and $\phi(x)$ is a Puiseux series.*

See [3, 4, 7, 8] for proofs of this theorem in different settings. They are based on the stabilization of the Newton polygon process, which also gives a recurrent formula for the coefficients of $\phi(x)$ which we will need later.

2.3.2 Stabilization of the Newton Polygon Process

In this paragraph we will see that given an infinite branch $(\tau_i)_{i\geq 0}$ of $\mathcal{T}$, there exists i_1, such that, for $i \geq i_1$, the node τ_i has only one descendant, in other words, the coefficient $c_{\tau_{i+1}}$ and the exponent $\mu_{\tau_{i+1}}$ are completely determined. We will give a recurrent formula for the coefficients $c_{\tau_{i+1}}$ and prove Theorem 1.

Lemma 2. *Let $G(y)$ be a differential polynomial and $H(y) = G(c\,x^\mu + y)$. Let $P = (t, h)$ be the point of highest ordinate in $L(G;\mu) \cap \mathcal{N}(G)$. Let $A = \{(t', h') \in \mathbb{R}^2 \mid h' \geq h\}$. Then $\mathcal{N}(G) \cap A = \mathcal{N}(H) \cap A$. Moreover, $\mathcal{N}(H)$ is contained in the right half-plane defined by $L(G;\mu)$. Finally, if $\Phi_{(G,\mu)}(c) = 0$, then the intersection point of $L(G,\mu)$ with the x-axis is not a vertex of $\mathcal{N}(H)$.*

Proof. Let $M(y) = a\,x^\alpha\, y_0^{\rho_0} \cdots y_n^{\rho_n}$ be a differential monomial. Let $P_{\alpha,\underline{\rho}} = (t, h)$ its corresponding point. By simple computation, $M(c\,x^\mu + y) = M(y) + V(y)$, where $V(y)$ is a sum of differential monomials whose corresponding points have ordinate less than h and lie in the line passing through $P_{\alpha,\underline{\rho}}$ with inclination μ. This implies the first two statements. In order to prove the last one, we remark that if $T(y)$ is the sum of monomials of $G(y)$ whose corresponding points lie in $L(G;\mu)$, then $T(c\,x^\mu)$ is the coefficient of the monomial whose corresponding point is the intersection of $L(G,\mu)$ with the x-axis. Since $T(c\,x^\mu) = \Phi_{(G;\mu)}(c)$, we are done. □

Lemma 3. *Let τ be a node of $\mathcal{T}$ with $\mu_\tau \neq \infty$. Then either $y = 0$ is a solution of $F_\tau(y) = 0$ or $\mathcal{N}(F_\tau)$ has a side of inclination $\mu > \mu_\tau$.*

Proof. It is a consequence of Lemma 2 and the construction of $\mathcal{T}$.

Definition 1 (The Pivot Point). *For any non leaf node τ of $\mathcal{T}$, we will denote p_τ to be the point of least ordinate in $L(F_\tau;\mu_\tau) \cap \mathcal{N}(F_\tau)$.*

Let $(\tau_i)_{i\geq 0}$ be an infinite branch of $\mathcal{T}$. Let $p_{\tau_i} = (\alpha_i, \beta_i)$. We have that $\beta_i \in \mathbb{N}$. By Lemma 2, $\beta_i \geq \beta_{i+1} \geq 1$ for $i \geq 1$. Hence, there exists i_0 such that, if $i \geq i_0$, then $\beta_{i_0} = \beta_i$. We have also that $p_{\tau_i} = p_{\tau_{i_0}}$ for $i \geq i_0$. We call point $p = p_{\tau_{i_0}}$ the pivot point *of the branch.*

We have that p is a vertex of $\mathcal{N}(F_{\tau_i})$ for $i \geq i_0$. Hence, the monomials of $F_{\tau_{i+1}}(y)$ and those of $F_{\tau_i}(y)$ corresponding to p are exactly the same for $i \geq i_0$.

In particular, the indicial polynomial $\Psi_{(F_{\tau_i};p)}(\mu)$ *is the same for* $i \geq i_0$*; denote it by* $\Psi(\mu)$*. We say that the branch stabilizes at step* $i_1 \geq i_0$ *if* μ_{τ_i} *is not a root of* $\Psi(\mu)$ *for* $i \geq i_1$*.*

Example 3. Following with the notation of example 2, let $(\tau_i)_{i\geq 0}$ be a branch of $\mathcal{T}$ such that $\tau_1 = \sigma_{1,1}$. We have that $p_{\tau_2} = (-2,1)$. Since its ordinate is already equal to 1, the pivot point p of the branch is p_{τ_2}. The indicial polynomial $\Psi_{(F_{\tau_2};p)}(\mu) = \mu(1+3\mu)$. For $j \geq 2$ we have that $\mu_{\tau_j} \geq \mu_{\tau_2} = 2$, and so μ_{τ_j} is not a root of $\Psi(\mu)$. Hence, the branch $(\tau_i)_{i\geq 0}$ stabilizes at step 2.

By a simple use of the chain rule, we can prove (see Sect. 4 in [4]) the following

Lemma 4. *Let* h *be the ordinate of the pivot point* p *of the branch* $(\tau_i)_{i\geq 0}$*. Assume that* $h \geq 2$ *and that the differential variable* $y^{(k)}$ *actually appears in at least one of the monomials of* $F_{\tau_{i_0}}(y)$ *corresponding to the pivot point* p*. Then* $\phi(x) = \sum c_{\tau_i}\, x^{\tau_i}$ *is also a solution of* $\frac{\partial F}{\partial y^{(k)}}(y) = 0$*, and the pivot point of* $\phi(x)$ *with respect to* $\frac{\partial F}{\partial y^{(k)}}(y)$ *has ordinate* $h-1$*.*

Remark 1. Let $(\tau_i)_{i\geq 0}$ be a branch of the tree $\mathcal{T}(F)$ associated with $F=0$. By a successive application of the above lemma, there exists $G(y) = \frac{\partial^{|a|} F}{\partial y_0^{a_0} \cdots \partial y_n^{a_n}}(y) \neq 0$ such that the tree $\mathcal{T}(G)$ associated with $G(y)=0$ has a branch $(\tau'_i)_{i\geq 0}$ with $c_{\tau'_i} = c_{\tau_i}$ and $\mu_{\tau'_i} = \mu_{\tau_i}$ for $i \geq 0$, and whose corresponding pivot point p has ordinate equal to 1. Since p has ordinate equal to 1, we have that $\Psi_{(G_{\tau'_i};p)}(\mu)$ is a nonzero polynomial, hence it has a finite number of real roots. This implies that the branch $(\tau'_i)_{i\geq 0}$ stabilizes at some step i_1.

Definition 2. *Let* $F(y)$ *be a differential polynomial. We say that* $F=0$ *has* quasi-linear solved form *if* $p=(0,1)$ *is a vertex of* $\mathcal{N}(F)$*,* $\mathcal{N}(F) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$*, and the indicial polynomial* $\Psi_{(F;p)}(\mu)$ *has no positive real roots.*

Proposition 2. *Let* $F(y) \in R((x))^g[y_0,\ldots,y_n]$*, where* R *is a ring,* $\mathbb{C} \subseteq R \subseteq K$*, and* K *is a field. Assume that* $F=0$ *has quasi-linear solved form. Let us write*

$$F(y) = \sum_{\alpha,\underline{\rho}} a_{\alpha,\underline{\rho}}\, x^{\alpha+\rho_1+2\rho_2+\cdots+n\rho_n} y_0^{\rho_0} y_1^{\rho_1} \cdots y_n^{\rho_n} \ ,$$

where the exponents α *lie in a finitely generated semi-group* Γ *of* $\mathbb{R}_{\geq 0}$*. Then there exists a unique series solutions* $\phi(x) \in K((x))^g$ *of* $F=0$ *with positive order. Moreover, let us write* $\Gamma = \{\gamma_i\}_{i\geq 0}$*, where* $\gamma_i < \gamma_{i+1}$*, for all* i*. Then, for each* $i \geq 1$*, there exists a polynomial* $Q_i(\{A_{\alpha,\underline{\rho}}\}, T_1, \ldots, T_{i-1})$*, which only depends on* Γ*, and with coefficients in* R*, such that, if we write* $\phi(x) = \sum_{i=1}^{\infty} d_i x_i^{\gamma}$*, we have that*

$$d_i = -\frac{Q_i(\{a_{\alpha,\underline{\rho}}\}, d_1, \ldots, d_{i-1})}{\Psi_{(F;(0,1))}(\gamma_i)}, \quad i \geq 1 \ . \qquad (4)$$

Proof. Consider $F(y) = M(y)+G(y)$, where $M(y)$ is the sum of those monomials of $F(y)$ whose corresponding point is $(0,1)$. For any series $\phi(x) = \sum_{i\geq 1} d_i\, x^{\gamma_i}$, we have that $M(\phi(x)) = \sum_{i\geq 1} d_i \Psi_{(F;(0,1))}(\gamma_i)\, x^{\gamma_i}$ and $G(\phi(x)) = \sum_{i\geq 1} g_i\, x^{\gamma_i}$, where each g_i is a polynomial expression on $a_{\alpha,\underline{\rho}}$ and $d_1,\ldots,d_{i-1}$. Since $\Psi_{(F;(0,1))}(\gamma_i) \neq 0$, the relations $d_i\Psi_{(F;(0,1))}(\gamma_i) + g_i = 0$, $i \geq 1$, determine uniquely $\phi(x)$. □

Lemma 5. *Let $F(y) \in K((x))^g[y_0,\ldots,y_n]$ and $(\tau_i)_{i\geq 0}$ be an infinite branch of $\mathcal{T}$ with pivot point $p = (\alpha, 1)$ and which stabilizes at step i_1. Let ξ be a rational, $\mu_{\tau_i} < \xi < \mu_{\tau_{i+1}}$. Consider $H(y) = x^{-\alpha}\, F_{\tau_{i_1}}(x^{\xi}\, y)$. Then $H(y) \in K((x))^g[y, y']$ and has quasi-linear solved form.*

Proof. Let $G(y) = F_{\tau_{i_1}}(y)$. The affine map $(a,t) \mapsto (a - \alpha_0 + \xi\, t, t)$ is a bijection between the clouds of points of $G(y)$ and $H(y)$; hence it is a bijection between their Newton polygons, sending a side of inclination μ to a side of inclination $\mu - \xi$. Moreover, we have that $\Psi_{(H;(0,1))}(\mu) = \Psi_{(G;q)}(\mu + \xi)$. This proves that H has quasi-linear solved form. The fact that the series coefficients of $H(y)$ are grid-based is a consequence of the following fact: let Γ be a finitely generated semi-group and $\gamma \in \mathbb{R}$, then $A = (\gamma + \Gamma) \cap \mathbb{R}_{\geq 0}$ is contained in a finitely generated semi-group. To see this, let Γ be generated by $s_1,\ldots,s_k$. Let Σ be the set of $(n_1,\ldots,n_k) \in \mathbb{N}^k$ such that $\gamma + \sum n_i s_i > 0$ and for any $(n'_1,\ldots,n'_k) \neq (n_1,\ldots,n_k)$, with $n'_i \leq n_i$ for $1 \leq i \leq k$, one has that $\gamma + \sum n'_i s_i < 0$. Then Σ is finite, and A is generated by $1, s_1,\ldots,s_k$ and $\gamma + \sum n_i s_i$, where $(n_1,\ldots,n_k) \in \Sigma$.

Proof of theorem 1. By Remark 1, we may assume that the pivot point p of $(\tau_i)_{i\geq 0}$ has ordinate 1. Now apply Lemma 5 and Proposition 2. □

3 Formal Power Series Solutions of First Order ODE

Definition 3. *Let $F(y) \in \mathbb{C}((x))^g[y,y']$ and $\mathcal{T}$ be the tree constructed in the previous section. Let σ be a descendant node of τ which is not a leaf. We say that the node σ is of type (I)–(IV) if (c_σ, μ_σ) is a necessary condition of F_τ of the corresponding type (I)–(IV).*

Let τ be a node of $\mathcal{T}$. We call τ irrationally dicritical *if it is of type (II) or (III). We say that τ is* dicritical *if the path $\tau_0,\ldots,\tau_{k+1} = \tau$ from τ_0 to τ satisfies that each τ_i is of type (I) for $1 \leq i \leq k$, τ is of type (II) or (III), and $\mu_\tau \in \mathbb{Q}$.*

A branch of $\mathcal{T}$ is called dicritical *(resp.* irrationally dicritical*) if it contains a dicritical (resp. irrationally dicritical) node. We will say that $F = 0$ is* dicritical *(resp.* irrationally dicritical*) at the origin if its tree has at least a dicritical (resp. irrationally dicritical) branch $(\tau_i)_{i\geq 0}$ with $\mu_{\tau_1} > 0$.*

We remark that the tree $\mathcal{T}$ is discrete if and only if $F = 0$ has no irrationally dicritical branches.

This section is devoted to proving the following

Theorem 2. *Let $F(y) \in \mathbb{C}((x))^g[y, y']$ be irrationally dicritical. Then there exists a one-parameter family of grid-bases series solutions of $F(y) = 0$ as follows*

$$y_c(x) = \left(\sum_{i=1}^{k-1} b_i\, x^{\mu_i}\right) + c\, x^{\mu_k} + \left(\sum_{i=k+1}^{s} c_i\, x^{\mu_i}\right) + \left(\sum_{i=s+1}^{\infty} d_i\, x^{\mu_i}\right) \qquad (5)$$

where

- *each b_i is a fixed constant,*
- *the parameter $c \in \mathbb{C} \setminus E$, where E is a finite set,*
- *each c_i satisfies a polynomial equation $Q_i(c, c_{k+1}, \ldots, c_i) = 0$, and*
- *each $d_i = R_i(c_, c_{k+1}, \ldots, c_s, d_{s+1}, \ldots, d_{i-1})/\mu_i\, g(c, c_{k+1}, \ldots, c_s)$, where R_i and g are polynomials with coefficients in $\mathbb{C}$.*
- *The exponents $\{\mu_i\}_{i\geq 1}$ do not depend on the parameter c (we allow zero coefficients in (5)).*

Moreover, if $F(y) \in \mathbb{C}((x))^*[y, y']$ and $F = 0$ is dicritical, then there exists a one-parameter family as (5) with exponents $\{\mu_i\}_{i\geq 1} \subseteq \frac{1}{q}\mathbb{Z}$. If coefficients of $F(y)$ are convergent Puiseux series, then each $y_c(x)$ is also a convergent Puiseux series, hence it corresponds to the parametrization of an analytic branch curve.

Lemma 6. *Let $F(x, y, y')$ be a differential polynomial with coefficients in $\mathbb{C}((x))^g$. Let C be an indeterminate over $\mathbb{C}$, and let L be an extension field of $\mathbb{C}(C)$. Let $C_1 \ldots, C_t \in L$. Consider the differential polynomial*

$$G(x, y, y') = F(x,\, \phi + y,\, \phi_x + y') \in L((x))^g[y, y'],$$

where $\phi(x) = C + C_1\, x^{\mu_1} + \cdots + C_t\, x^{\mu_t}$, $0 < \mu_1 < \cdots < \mu_t$, and $\phi_x = \frac{d\,\phi}{d\,x}$. Let $p = (a, h)$ be any vertex of the bottom part of the Newton polygon of $G(x, y, y')$. (The bottom part of $\mathcal{N}$ is constituted by the sides of $\mathcal{N}$ with positive slope.)

Then there is only one monomial of $G(x, y, y')$ whose corresponding point is p and this monomial has the form $g\, x^a\, (x\, y')^h$, where g is a non-zero element of L. In particular, the characteristic polynomial associated to any side of the bottom part of $\mathcal{N}(G)$ has nonzero roots.

Proof. Multiplying F by a convenient x^α, we may assume that

$$F = \sum_{r\in\Gamma} \sum_{s\in\mathbb{N}} \varphi_{r,s}(y)\, x^r\, (xy')^s, \quad \varphi_{r,s}(y) \in \mathbb{C}[y]\ ,$$

where Γ is a finitely generated semi-group of $\mathbb{R}_{\geq 0}$ containing $1, \mu_1, \ldots, \mu_t$. We have that

$$G = \sum_{r\in\Gamma, s, l\geq 0, k\geq 0} \binom{s}{k} \frac{1}{l!}\, \varphi_{r,s}^{(l)}(\phi)\, x^r\, (x\phi_x)^{s-k}\, y^l\, (xy')^k\ . \qquad (6)$$

Let $\frac{d}{dC}$ be the derivative with respect to C in $\mathbb{C}(C)$. We choose an extension of this derivation operator to L, and we extend it trivially to $L((x))^*[y,y']$. Hence,

$$\frac{d}{dC}\mathrm{Coeff}_{x^a y^l (xy')^k}(G) = \mathrm{Coeff}_{x^a}\left(\sum_{r,s}\binom{s}{k}\frac{1}{l!}\frac{d}{dC}\left\{\varphi^{(l)}_{r,s}(\phi)\, x^r\, (x\phi_x)^{s-k}\right\}\right) = A+B,$$

where

$$\begin{aligned}
A &= \mathrm{Coeff}_{x^a}\left(\frac{d\phi}{dC}\sum_{r,s}\binom{s}{k}\frac{1}{l!}\,\varphi^{(l+1)}_{r,s}(\phi)\, x^r\,(x\phi_x)^{s-k}\right)\\
&= \sum_{0\le t\le a}\mathrm{Coeff}_{x^{a-t}}\left(\frac{d\phi}{dC}\right)\mathrm{Coeff}_{x^t}\left(\sum_{r,s}\binom{s}{k}\frac{1}{l!}\varphi^{(l+1)}_{r,s}(\phi)\, x^r\,(x\phi_x)^{s-k}\right)\\
&= \sum_{0\le t\le a}\mathrm{Coeff}_{x^{a-t}}\left(\frac{d\phi}{dC}\right)(l+1)\mathrm{Coeff}_{x^t y^{l+1}(xy')^k}(G),\\
B &= \sum_{0\le t\le a}\mathrm{Coeff}_{x^{a-t}}\left(x\frac{\partial\phi}{\partial x\partial C}\right)\mathrm{Coeff}_{x^t}\left(\sum_{r,s}\binom{s}{k}\frac{s-k}{l!}\varphi^{(l)}_{r,s}(\phi)\, x^r\,(x\phi_x)^{s-k-1}\right)\\
&= (k+1)\sum_{0\le t<a}\mathrm{Coeff}_{x^{a-t}}\left(x\frac{\partial\phi}{\partial x\partial C}\right)\mathrm{Coeff}_{x^t y^l (xy')^{k+1}}(G)\ .
\end{aligned}$$

We remark that $t\in\Gamma$ and all the above sums are finite. The above equalities hold by (6) and because $\mathrm{Coeff}_{x^0}\left(x\frac{\partial\phi}{\partial x\partial C}\right)=0$.

Let $p=(a,h)$ be a vertex of the bottom part of $\mathcal{N}(G)$. Let $l'+k=h$. Assume that $k<h$, so that $l'\ge 1$. Let $l=l'-1$. Then

$$\mathrm{Coeff}_{x^t y^{l+1}(xy')^k}(G) = \mathrm{Coeff}_{x^t y^l (xy')^{k+1}}(G) = 0, \quad \text{for } t<a\ .$$

From this, in the above computation, $A=\mathrm{Coeff}_{x^a y^{l'}(xy')^h}(G)$ and $B=0$. Hence

$$\mathrm{Coeff}_{x^a y^{l'}(xy')^h}(G) = \frac{d}{dC}\mathrm{Coeff}_{x^a y^l (xy')^k}(G) = 0\ .$$

The last equality holds because the point $(a,h-1)$ does not belong to $\mathcal{N}(N)$. So, the only coefficient different from zero is that of $x^a\,y^0\,(xy')^h$. Now let S be a side of the bottom part of $\mathcal{N}(G)$ with inclination $\mu_0>0$. Let (a,h) and $(a+k\mu_0,h-k)$ be the vertices of S. We have that

$$\Phi_{(G;\mu_0)}(T) = g_h\,(\mu_0\,T)^h+\cdots+g_{h-k}\,(\mu_0\,T)^{h-k}\ .$$

Since g_h and g_{h-k} are different from zero, $\Phi_{(G;\mu_0)}(T)$ has non-zero roots. □

Theorem 3. *Let $F(x,y,y')\in\mathbb{C}((x))^g[y,y']$. Assume that $F(y)=0$ has a necessary initial condition (c,μ_0) of type (II) or (III). Let C be an indeterminate over $\mathbb{C}$ and $L=\mathbb{C}(C)$. Then $F=0$ has a solution as follows*

$$y(x) = C\,x^{\mu_0}+\sum_{i=1}^{\infty}C_i\,x^{\mu_i}\in\bar{L}((x))^g\ .$$

Moreover, there exist i_1, a polynomial $g(T, T_1, \ldots, T_{i_1})$ with coefficients in $\mathbb{C}$ such that $\bar{g} = g(C, C_1, \ldots, C_{i_1}) \neq 0 \in \bar{L}$, and

$$C_i \in \mathbb{C}[C, C_1, \ldots, C_{i_1}, \frac{1}{\bar{g}}], \quad \text{for all } \ i \geq 1 \ .$$

Proof. Consider $G(y) = F(x^{\mu_0}\, y)$. Then $(c, 0)$ is a necessary initial condition for $G = 0$ of type (II) or (III). It suffices to prove the statement for $G(y)$ and $\mu_0 = 0$. We have that $\Phi_{(G;0)}(C) \equiv 0$. We consider $G(y) \in L((x))^g[y, y']$, hence $(C, 0)$ is a necessary initial condition for $G = 0$. Let $\mathcal{T} = \mathcal{T}(G; L)$ be the tree of G constructed in the previous section. Let τ_1 be the node of $\mathcal{T}$ such that $c_{\tau_1} = C$ and $\mu_{\tau_1} = 0$. Let us prove that there exists a branch $(\tau_i)_{i\geq 0}$ of $\mathcal{T}$, passing through τ_1, which corresponds to a solution of $G(y) = 0$. Let $(\tau_i)_{i=0}^k$ be a path of $\mathcal{T}$. If $y = 0$ is a solution of $F_{\tau_k}(y) = 0$, we are done. If $y = 0$ is not a solution, by lemma 3 there exists a side S of $\mathcal{N}(F_{\tau_k})$ with inclination $\mu_{k+1} > \mu_k$. In particular, S is a side of the bottom part of $\mathcal{N}(F_{\tau_k})$. By Lemma 6, the characteristic polynomial $\Phi_{(F_{\tau_k};\mu_{k+1})}(T)$ has a non zero root C_{k+1} in $\bar{L}$. Hence, the path $(\tau_i)_{i=0}^k$ can be continued to a branch $(\tau_i)_{i\geq 0}$ which corresponds to a solution $\phi(x) = C + \sum_{i\geq 1} C_i\, x^{\mu_i} \in \bar{L}((x))^g$ of $G(y) = 0$, where $C_i = c_{\tau_i}$ and $\mu_i = \mu_{\tau_i}$, for all $i \geq 1$.

It remains to prove the second part. If $\phi(x)$ is a finite sum of monomials we are done. Let $p = (\alpha, h)$ be the pivot point of $\phi(x)$ with respect to $G(y)$. If $h \geq 2$ then, by Lemmas 4 and 6, $\phi(x)$ is a solution of $\frac{\partial^{h-1} G}{\partial y'^{h-1}}$. Hence, we may assume that the pivot point is $p = (\alpha, 1)$ and it is reached at step i_1. By Lemma 6, we have that $\Psi_{(G;p)}(\mu) = g\,\mu$, where $g\, x^{\alpha}(xy')$ is the only monomial of $G_{\tau_{i_1}}(y)$ whose corresponding point is p. The element g is a polynomial expression on $C, C_1, \ldots, C_{i_1}$ with coefficients in $\mathbb{C}$. Let ξ be a rational number such that $\mu_{i_1} < \xi < \mu_{i_1+1}$. Consider $H(y) = x^{-\alpha}\, G_{\tau_1}(x^x i\, y)$. By Lemma 5 $H(y)$ has quasi-linear solved form. ¿From proposition 2, $\psi(x) = \sum_{i>i_1} C_i\, x^{\mu_i - \xi}$ is the only solution of $H(y) = 0$ with positive order and the coefficients C_i satisfy the recurrent equations (4). Let $R = \mathbb{C}[C, C_1, \ldots, C_{i_1}] \subseteq \bar{L}$, so that $H(y) \in R((x))^g[y, y']$. We have $\Psi_{(H;(0,1))}(\mu) = (\mu - \xi)\, g$. Using the recurrent equations (4), one sees that $C_i \in R[\frac{1}{g}] \subseteq \bar{L}$. □

Corollary 1. *Let $\eta : \mathbb{C}[C, C_1, \ldots, C_{i_1}, \frac{1}{\bar{g}}] \to \mathbb{C}$ be a ring homomorphism. Let $c = \eta(C)$ and $c_i = \eta(C_i)$, for $i \geq 1$. Then $\eta(y(x)) = c\, x^{\mu_0} + \sum_{i=1}^{\infty} c_i\, x^{\mu_i}$ is a solution of $F(y) = 0$.*

Proof. Set $C_0 = C$ and $c_0 = c$. Let $F_k = F(\sum_{i=0}^k C_i\, X^{\mu_i})$ and $G_k = F(\sum_{i=0}^k c_i\, X^{\mu_i})$. We have that $\mathrm{ord}(G_k) \geq \mathrm{ord}(F_k)$, for all $k \geq 1$. □

Remark 2. Since $C_1, \ldots, C_{i_1}$ are algebraic over $\mathbb{C}(C)$, there exist polynomials $Q_i \in \mathbb{C}[T, T_1, \ldots, T_i]$, $1 \leq i \leq i_1$, and $g \in \mathbb{C}[T, \ldots, T_1, \ldots, T_{i_1}]$ such that if $\mathcal{C} = \{\underline{c} = (c, c_1, \ldots, c_{i_1}) \in \mathbb{C}^{i_1+1} \mid Q_i(\underline{c}) = 0,\ 1 \leq i \leq i_1,\ g(\underline{c}) \neq 0\}$ then there exists a homomorphism $\eta : \mathbb{C}[C, C_1, \ldots, C_{i_1}, \frac{1}{\bar{g}}] \to \mathbb{C}$ with $\eta(C_i) = c_i$ for $0 \leq i \leq i_1$ if and only if $\underline{c} \in \mathcal{C}$. Moreover, there exists a finite set $E \subseteq \mathbb{C}$ such that the projection $\pi : \mathcal{C} \to \mathbb{C} \setminus E$ over the first coordinate is onto. Then we obtain a one-parameter family of solutions of $F(y) = 0$ as described in Theorem 2.

Proof of Theorem 2. Let τ be an irrationally dicritical node of $\mathcal{T}(F; \mathbb{C})$. Let $(\tau_i)_{i=0}^{k+1}$ be the path from τ_0 to τ. Then $(c_{\tau_{k+1}}, \mu_{\tau_{k+1}})$ is a necessary initial condition of $F_{\tau_k}(y) = 0$ of type (II) or (III). Apply Theorem 3 and the above remark to $F_{\tau_k}(y) = 0$. If $F(y) \in \mathbb{C}((x))^*[y, y']$, consider a dicritical node τ and the path $(\tau_i)_{i=0}^{k+1}$, such that τ_i are of type (I) for $1 \leq 1 \leq \tau_k$. Hence $F_{\tau_{k+1}}$ has rational exponents. The solution constructed in Theorem 3 has also rational exponents because the necessary initial conditions used there correspond to sides. Hence, the family of solutions have rational exponents. Finally, by Lemma 6 the pivot point of all them has a corresponding monomial of type $\bar{g}\, x^{\alpha}\, (xy')$, with $\bar{g} \neq 0$. This guarantee the convergence of the solutions by a direct application of Theorem 2 of [4] or by the main theorem of [9]. □

References

1. Briot, C., Bouquet, J., Propriétés des fonctions définies par des équations différentielles. Journal de l'Ecole Polytechnique **36** (1856) 133–198
2. Fine, H., On the functions defined by differential equations, with an extension of the Puiseux Polygon construction to these equations. Amer. Jour. of Math. **XI** (1889) 317–328
3. Grigoriev, D.Y., Singer, M., Solving Ordinary Differential Equations in Terms of Series with Real Exponents. Trans A.M.S. **327** (1991) 329–351
4. Cano, J., On the series defined by differential equations, with an extension of the Puiseux Polygon construction to these equations. Analysis, Inter. Jour. Anal. its Appli. **13** (1993) 103–119
5. Cano, J., An extension of the Newton-Puiseux Polygon construction to give solutions of pfaffian forms. Ann. Inst. Fourier **43** (1993) 125–142
6. Della Dora, J., Jung, F., About the Newton Polygon Algorithm for Non Linear Ordinary Differential Equations. Proceedings of ISSAC'97 (1997)
7. van der Hoeven, J., Asymptotique automatique. PhD thesis, École Polytechnique (1997)
8. Cano, F., Moussu, R., Rolin, J.P., Non-oscillating integral curves and valuations. Crelle's journal (to appear)
9. Malgrange, B., Sur le théorème de Maillet. Asymptotic Anal. **2** (1989) 1–4